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Theory and Applications of Extended *q*-Difference Operator

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Abstract : In this paper, we define the extended *q*-difference operator $\Delta_{q(\ell)}$ and present the discrete version of the Leibnitz theorem according to $\Delta_{q(\ell)}$. Also, we define the inverse of extended *q*-difference operator $\Delta_{q(\ell)}^{-1}$, and to obtain the formula for the sum of higher powers of arithmetic progressions in the field of Numerical Methods. Suitable examples are provided to illustrate the main results.
Keywords : *q*-difference operator, inverse operator, arithmetic progressions.

1. INTRODUCTION

The theory of *q*-difference equations is based on the operator $D_q f(x)$ is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Also, studies on linear *q*-difference equations were started at the begining of the last century with the intensive works by C.R.Adams [1], Carmichael [3], Jackson [4], Mason [10], Trjitzinsky [11] and some others such as, Picared and Ramanujan. However, from 1930's upto the begining of 1980's, the theory of linear *q*-difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. Since 1980's an extensive and somewhat surprising interest in the subject reappeared in many areas of mathematics, physics and applications including new finite difference calculus and orthogonal polynomials, *q*-Combinatories, *q*-arithmetics, integrable systems and variational *q*-Calculus [2,12,13].

In 2006, M.S.Manuel, et. al., extended from the difference operator to generalized difference operator is denoted by Δ_ℓ is defined on the real valued function $u(k)$ by

$$\Delta_\ell u(k) = u(k+\ell) - u(k), k \in [0,\infty), \ell \in (0,\infty)$$

and developed the theory of difference equations in a different direction. Also, they defined the inverse of generalized difference operator and obtained the formulae for sum of higher powers of arithmetic progressions, sum of consecutive terms of arithmetic progressions and sum of arithmetic-geometric progressions using the Stirling numbers of first kind and second kind respectively in the field of Numerical methods.

By extending, the study for sequences of Complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and weblike were studies for the solutions of difference equations

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involving Δ_ℓ. Also, a method to find a formula for the sum of n^{th} power of arithmetico-geometric progression using the generalized Bernoulli polynomial $B_{n+1}(k,-\ell)$, and solution to the generalized difference equation $u(k-\ell)-u(k)=(n+1)k^n, n\in\mathbb{N}(1)$. The results obtained can be found [5-9].

With this background, in this paper, we define the extended *q*-difference operator and develop the basic theory for the extended *q*-difference operator $\Delta_{q(\ell)}$ and obtain the relation between $\Delta_{q(\ell)}$ and the *q*-shift operator $\mathrm{E}^{q(\ell)}$, the basic properties of $\Delta_{q(\ell)}$ and generalized version of Leibnitz theorem according to $\Delta_{q(\ell)}$. Also, we define the inverse of extended *q*-difference operator $\Delta_{q(\ell)}^{-1}$ and obtain the formula for finding the sum of the higher powers of an arithmetic progression using Striling numbers of second kind. Suitable examples are presented to establish the results in the field of Numerical Methods.

2. DEFINITIONS AND PRELIMINARIES

In this section, we define the extended *q*-difference operator and obtaining relation between the shift operator and extended *q*-difference operator.

Definition 2.1 If $u(k)$ is real valued function, then we define the extended *q*-difference operator $\Delta_{q(\ell)}$ as

$$\Delta_{q(\ell)}u(k) \;=\; u\big((k+\ell)q\big)-u(k), \ell\in(0,\infty), q\in(0,\infty). \tag{1}$$

Remark 2.2. (*i*) When $q=1$, $\Delta_{q(\ell)}$ becomes Δ_ℓ, the generalized difference operator.

(*ii*) When $q=1$ and $\ell=1$, $\Delta_{q(\ell)}$ becomes Δ, the difference operator.

The following are the immediate consequences and extensions.

Lemma 2.3. The relation between $\Delta_{q(\ell)}$ and $\mathrm{E}^{q(\ell)}$ is

$$\mathrm{E}^{q(\ell)} \;=\; \Delta_{q(\ell)}+1. \tag{2}$$

Proof. The shift operator $\mathrm{E}^{q(\ell)}$ is defined by

$$\mathrm{E}^{q(\ell)}u(k) \;=\; u\big((k+\ell)q\big), k\in[0,\infty). \tag{3}$$

The proof follows from (1) and (3).

Lemma 2.4. If *q* and ℓ are positive integers, then

$$1+\Delta_{q(\ell)} \;=\; \left(1+\Delta\right)^{q(\ell)}. \tag{4}$$

Lemma 2.5. If *a* and *b* are any two non-zero scalars, $u(k)$ and $v(k)$ are any two real valued functions, then

$$\Delta_{q(\ell)}[au(k)+bv(k)] \;=\; a\Delta_{q(\ell)}u(k)+b\Delta_{q(\ell)}v(k).$$

Lemma 2.6. Let $u(k)$ and $v(k)$ be any two real valued functions. Then

$$\Delta_{q(\ell)}[u(k)v(k)] \;=\; v\big((k+\ell)q\big)\Delta_{q(\ell)}u(k)+u(k)\Delta_{q(\ell)}v(k).$$

Lemma 2.7. If $u(k)$ and $v(k)\neq 0$ are any two real valued functions, then

$$\Delta_{q(\ell)}\left[\frac{u(k)}{v(k)}\right] \;=\; \frac{v(k)\Delta_{q(\ell)}u(k)-u(k)\Delta_{q(\ell)}v(k)}{v(k)v((k+\ell)q)}.$$

3. HIGHER ORDER OF EXTENDED *Q*-DIFFERENCE OPERATOR

In this section, we define the higher order of $\Delta_{q(\ell)}$ and establish the generalized version of Leibnitz theorem according to $\Delta_{q(\ell)}$.

Definition 3.1. The second order extended q-difference operator denoted by $\Delta^2_{q(\ell)}$ is defined as $\Delta^2_{q(\ell)} = \Delta_{q(\ell)}\left(\Delta_{q(\ell)}\right)$.

In general, the n^{th} order extended q-difference operator denoted by $\Delta^n_{q(\ell)}$ is defined as $\Delta^n_{q(\ell)} = \Delta_{q(\ell)}\left(\Delta^{n-1}_{q(\ell)}\right)$.

We present the following remarks which can be easily established.

Remark 3.2. If q and ℓ are positive reals, m and n are any two positive integers, then

$$\Delta^m_{q(\ell)}\Delta^n_{q(\ell)} = \Delta^n_{q(\ell)}\Delta^m_{q(\ell)}.$$

Remark 3.3. If c is a constant and $u(k)$ is any positive real valued function, then

$$\Delta^m_{q(\ell)}\left[c\,u(k)\right] = c\,\Delta^m_{q(\ell)}\left[u(k)\right].$$

Lemma 3.4. If m and n are any two positive integers, then

$$\Delta^m_{q(\ell)}k^n = \sum_{t=0}^{m}(-1)^t\left[(k+\ell)q^{m-t} + \sum_{r=1}^{m-1-t}\ell q^r\right]^n. \tag{5}$$

Proof. The proof follows by induction method on m and n.

Lemma 3.5. If q and ℓ are positive reals and n is positive integer, then

$$\Delta^n_{q(\ell)} = \sum_{r=0}^{n}(-1)^r\, n\mathrm{C}_r\,\mathrm{E}^{(n-r)q(\ell)} \tag{6}$$

and hence

$$\Delta^n_{q(\ell)}u(k) = \sum_{r=0}^{n}(-1)^r\, n\mathrm{C}_r\left[u\left((k+\ell)q^{n-r} + \sum_{t=1}^{n-r-1}\ell q^{n-r-t}\right)\right]. \tag{7}$$

Proof. From (2) and Binomial theorem, we find

$$\Delta^n_{q(\ell)} = \mathrm{E}^{nq(\ell)} - n\mathrm{C}_1\mathrm{E}^{(n-1)q(\ell)} + \cdots + 1. \tag{8}$$

(6) follows from (8) and operating bothsides on $u(k)$ in (8) and simplifying, we get (7).

Lemma 3.6. If $q(\ell_i),\ i = 1, 2, \cdots, n$ are positive reals, then

$$1 + \Delta_{\left[\sum_{i=1}^{n} q(\ell_i)\right]} = \prod_{i=1}^{n}\left(1 + \Delta_{q(\ell_i)}\right). \tag{9}$$

Proof. The proof follows by (2).

Lemma 3.7. If q and ℓ are positive reals and n is positive integer, then

$$\Delta_{nq(\ell)} = \sum_{r=1}^{n} n\mathrm{C}_r\Delta^r_{q(\ell)}. \tag{10}$$

Proof. From (2), we have

$$\Delta_{nq(\ell)} = \mathrm{E}^{nq(\ell)} - 1. \tag{11}$$

The proof follows from (2), (11) and binomial theorem.

Lemma 3.8. If q and ℓ are positive reals and n is positive integer, then

$$\Delta^n_{q(\ell)} = \sum_{r=0}^{n-1}(-1)^r\, n\mathrm{C}_r\Delta_{q(\ell)(n-r)}. \tag{12}$$

Proof. The proof follows from (2) and binomial theorem.

The discrete version of the Leibnitz's theorem according to $\Delta_{q(\ell)}$ is given below.

Theorem 3.9. If $u(k)$ and $v(k)$ are any two positive real valued functions, then

$$\Delta^n_{q(\ell)}\left[u(k)v(k)\right] = \sum_{t=0}^{n-1} n\mathrm{C}_t\Delta^t_{q(\ell)}u(k)\Delta^{n-t}_{q(\ell)}v\left((k+\ell)q^t + \sum_{r=1}^{t}\ell q^r\right). \tag{13}$$

Proof. Define the operators $E_1^{q(\ell)}$ and $E_2^{q(\ell)}$ as

$$E_1^{q(\ell)}\left[u(k)v(k)\right] = u\big((k+\ell)q\big)v(k)$$

and
$$E_2^{q(\ell)}\left[u(k)v(k)\right] = u(k)v\big((k+\ell)q\big). \tag{14}$$

Hence, we get
$$E^{q(\ell)} = E_1^{q(\ell)}E_2^{q(\ell)}. \tag{15}$$

Also, we define
$$\left[\Delta_{q(\ell)}\right]_1 = E_1^{q(\ell)}-1 \text{ and } \left[\Delta_{q(\ell)}\right]_2 = E_2^{q(\ell)}-1. \tag{16}$$

This implies
$$\Delta_{q(\ell)} = E_1^{q(\ell)}E_2^{q(\ell)}-1.$$

From (16), we get
$$\Delta_{q(\ell)} = \left[\Delta_{q(\ell)}\right]_2+\left[\Delta_{q(\ell)}\right]_1 E_2^{q(\ell)}. \tag{17}$$

The proof follows by using Binomial theorem and (17).

Lemma 3.10. If q and ℓ are positive reals and n is positive integer, then

$$E^{nq(\ell)} = \sum_{r=0}^{n} nCr\Delta_{q(\ell)}^{r}. \tag{18}$$

Proof. Equation (18) follows by (2).

Lemma 3.11. If $a(k)$ is a real valued function, and x is an positive integer, then

$$\sum_{j=0}^{\infty}\frac{x^{jq(\ell)}}{j!(q(\ell))^{j}}a\left(q(\ell)\left(\sum_{r=0}^{j-1}q^{r}\right)\right) = \left(e^{\frac{x^{q(\ell)}E^{q(\ell)}}{q(\ell)}}\right)a(0) = \left(e^{\frac{x^{q(\ell)}}{q(\ell)}}e^{\frac{x^{q(\ell)}\Delta_{q(\ell)}}{q(\ell)}}\right)a(0). \tag{19}$$

Proof. The proof follows from exponential function and (2).

4. INVERSE OF EXTENDED *q*-DIFFERENCE OPERATOR AND ITS APPLICATIONS

In this section, we define the inverse of the extended q-difference operator $\Delta_{q(\ell)}^{-1}$ and obtain the formula for finding sum of higher powers of an arithmetic progression using Striling numbers of second kind. Suitable examples are presented to illustrate the results.

Definition 4.1. The inverse of extended q-difference operator denoted by $\Delta_{q(\ell)}^{-1}$ is defined as if

$$\Delta_{q(\ell)}v(k) = u(k) \textit{ then } v(k) = \Delta_{q(\ell)}^{-1}u(k)+c \tag{20}$$

and the n^{th} order inverse operator denoted by $\Delta_{q(\ell)}^{-n}$ is defined as

if
$$\Delta_{q(\ell)}^{n}v(k) = u(k) \text{ then } v(k) = \Delta_{q(\ell)}^{-n}u(k)+c,$$

where c is a constant depend upon k.

Remark 4.2 Let $u(k)$ be a real valued function. Then

$$\Delta_{q(\ell)}\left(\Delta_{q(\ell)}^{-1}u(k)\right) \neq \Delta_{q(\ell)}^{-1}\left(\Delta_{q(\ell)}u(k)\right).$$

Theorem 4.3. If q and ℓ are positive reals and $k\in N_{\ell}(j)=\{j, j+\ell, j+2\ell,\cdots\}$, then

$$\Delta_{q(\ell)}^{-1}u(k)\Big|_{j_{q(\ell)}}^{k} = \Delta_{q(\ell)}^{-1}u(k)-\Delta_{q(\ell)}^{-1}u\left(\frac{k-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]}q^{t}}{q^{\left[\frac{k}{\ell}\right]}}\right) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]}u\left(\frac{k-\ell\sum_{t=1}^{r}q^{t}}{q^{r}}\right), \tag{21}$$

where $$j_{q(\ell)} = \frac{k-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]} q^t}{q^{\left[\frac{k}{\ell}\right]}}$$ and $\left[\frac{k}{\ell}\right]$ is the integer part of $\frac{k}{\ell}$.

Proof. The proof follows from (20) and the relation

$$\Delta_{q(\ell)}\left(\sum_{r=1}^{\left[\frac{k}{\ell}\right]} u\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)\right) = u(k).$$

Theorem 4.4. If k, ℓ and q are positive real values, then

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}+\frac{\ell q}{q-1}\right) = \left(\frac{k}{q-1}\right)-\left(\frac{k-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]} q^t}{q^{\left[\frac{k}{\ell}\right]}(q-1)}\right). \tag{22}$$

Proof. From (20), we have $$\frac{k}{q-1} = \Delta_{q(\ell)}^{-1}\left(k+\frac{\ell q}{q-1}\right). \tag{23}$$

The proof follows from (21) and (23).

The following example is illustration of Theorem 4.4.

Example 4.5. In (22), by taking $k = 32$ and $\ell = 3$, we get

$$\sum_{r=1}^{10}\left[\frac{32-3\sum_{t=1}^{r} q^t}{q^r}+\frac{3q}{q-1}\right] = \left(\frac{32}{q-1}\right)-\left(\frac{32-3\sum_{t=1}^{10} q^t}{q^{10}(q-1)}\right). \tag{24}$$

In particular, when $q = 2$, we have

$$\sum_{r=1}^{10}\left[\frac{32-3\sum_{t=1}^{r} 2^t}{2^r}+6\right] = 32-\left(\frac{32-3\sum_{t=1}^{10} 2^t}{2^{10}}\right) = 32+\left(\frac{6106}{1024}\right) = 37.962890625.$$

Theorem 4.6. Let $k^{(n)} = k(k-\ell)...(k-(n-1)\ell)$ be the generalized polynomial factorial, then

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\frac{1}{\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)_{\ell}^{(2)}} = \frac{1}{\ell^2}\left(\frac{k-2\ell}{k-\ell}\right)-\frac{1}{\ell^2}\left(\frac{k-2\ell-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]} q^t}{k-\ell-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]} q^t}\right). \tag{25}$$

Proof. In Lemma 2.7 by taking $u(k) = k-2\ell$ and $v(k) = k-\ell$, we find

$$\Delta_{q(\ell)}\left(\frac{k-2\ell}{k-\ell}\right) = \frac{\ell^2}{k_{\ell}^{(2)}}. \tag{26}$$

The proof follows from (20), (21) and (26).

Example 4.7. Substituting $k = 46, \ell = 3$ and $q = 2$ in (25), we get

$$\sum_{r=1}^{15} \frac{1}{\left(\frac{46-3\sum_{t=1}^{r} 2^t}{2^r}\right)_3^{(2)}} = \frac{1}{9}\left(\frac{40}{43}\right) - \frac{1}{9}\left(\frac{40-3\sum_{t=1}^{15} 2^t}{43-3\sum_{t=1}^{15} 2^t}\right) = -0.472472161.$$

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