International Journal of Mathematical Sciences
Vol. 17, Nos. 3-4, July-December 2018, pp. 301-314

CERTAIN INTEGRAL PROPERTIES OF GENERALIZED CLASS OF POLYNOMIALS AND GENERALIZED *k*-BESSEL FUNCTION ASSOCIATED WITH FEYMANN INTEGRALS

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Abstract: The objective of the present paper is to study certain integral properties of general class of polynomials and generalized k-Bessel function with certain class of Feymann integrals. We establish certain new double integral relations pertaining to a product involving general class of polynomials and generalized k-Bessel function.

I. INTRODUCTION AND DEFINITIONS

In 2006, Diaz and Pariguan [5] introduced the *k*-Pochhemmer symbol defined as follows:

$$(\gamma)_{n,k} := \begin{cases} \dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & \left(n\in\mathbb{N}; k\in\mathbb{R}^+; \gamma\in\mathbb{C}\setminus\{0\}\right) \\ \gamma(\gamma+k)\ldots(\gamma+(n-1)k) & \left(n\in\mathbb{N}; \gamma\in\mathbb{C}\right), \end{cases} \tag{1.1}$$

and Γ_k is the *k*-gamma function having the relation with the classical Euler's gamma function as : (see [9])

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right), \quad \left(\gamma\in\mathbb{C}, k\in\mathbb{R}^+; n\in\mathbb{N}\right), \tag{1.2}$$

When $k = 1$, equation (1.1) reduces to

$$(\gamma)_n := \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1 & \left(n\in\mathbb{N}; \gamma\in\mathbb{C}\setminus\{0\}\right) \\ \gamma(\gamma+1)\ldots(\gamma+(n-1)) & \left(n\in\mathbb{N}; \gamma\in\mathbb{C}\right), \end{cases} \tag{1.3}$$

The *k*-Bessel function of first kind [8] is defined as:

$$j_{\kappa,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+1)} \frac{(z\setminus 2)^n}{(n!)^2} \tag{1.4}$$

$$k \in \mathbb{R}; z, \lambda, \gamma, \nu \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0,$$

where $(\gamma)_{n,k}$ is well known k-Pochhammer symbol defined in eqn. (1.1).

In [4], Agarwal *et al.* considered the following generalized k-Bessel function defined as: (see also, [1, 2, 3])

$$\omega^{\gamma,\lambda}_{k,\nu,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k\left(\lambda n + \nu + \frac{b+1}{2}\right)} \frac{\left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2}, \tag{1.5}$$
$$k \in \mathbb{R}; z, \lambda, \gamma, \nu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0,$$

The general class of polynomials $s^{m_1,\ldots m_r}_{n_1,\ldots n_r}[x]$ is defined as [10] is defined as

$$s^{m_1,\ldots m_r}_{n_1,\ldots n_r}[x] = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} x^{l_i} \tag{1.6}$$

where $n_1 \ldots n_r = 0,1,2,\ldots; m_1,\ldots m_r$ are arbitrary positive integers, the coefficients $A_{n_i l_i}$ $(n_i, l_i \geq 0)$ are arbitrary constants, real or complex. $S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}[x]$ yields a number of known polynomials as its special cases. These includes the Jacobi polynomials, the Bessel polynomials, the Hermite polynomials, the Lagurre polynomials and several others [11].

Fox [7] and Wright [12] introduced and investigated the generalized Fox-Wright function ${}_p\Psi_q(z)$ $(p, q \in \mathbb{N}_0)$ with p numerator and q denominator parameters defined for $a_1, \ldots, a_p \in \mathbb{C}$ and $b_1, \ldots, b_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by

$${}_p\Psi_q\left[\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p); \\ (b_1, \beta_1), \ldots, (b_q, \beta_q);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 n) \ldots \Gamma(a_p + \alpha_p n)}{\Gamma(b_1 + \beta_1 n) \ldots \Gamma(b_q + \beta_q n)} \frac{z^n}{n!} \tag{1.7}$$

where the coefficients $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{R}^+$ are such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0 \tag{1.8}$$

For $\alpha_i = \beta_j = 1$ ($i = 1, \ldots, p; j = 1, \ldots, q$), Eq. (1.7) reduces immediately to the generalized hypergeometric function ${}_pF_q\ (p, q \in \mathbb{N}_0)$ (see [9], Section 1.5):

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z\right] = \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)}\, {}_p\Psi_q\left[\begin{matrix} (a_1,1),\ldots,(a_p,1); \\ (b_1,1),\ldots,(b_q,1); \end{matrix} z\right] \tag{1.9} $$

The following formulas (see, e.g.[6], p. 145, p. 177, p. 243) will be required in our present study:

$$ \int_0^1\int_0^1 \frac{(1-x)^{\alpha-1}y^{\alpha}(1-y)^{\beta-1}}{(1-xy)^{\alpha+\beta-1}}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.10} $$

$$ \int_0^{\infty}\int_0^{\infty} \phi(x+y)x^{\alpha}y^{\beta}dxdy = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\int_0^{\infty}\phi(z)z^{\alpha+\beta+1}dz. \tag{1.11} $$

$$ \int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}y^{\alpha}(1-y)^{\beta-1}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^1 f(z)(1-z)^{\alpha+\beta-1}dz. \tag{1.12} $$

II. MAIN RESULTS

We establish the following results:

Theorem 1. *Let* $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, n_1 \ldots n_r$ $= 0,1,2,\ldots; m_1,\ldots m_r \in \mathbb{Z}^+ \, and\, A_{n_i,l_i}\ (n_i, l_i \geq 0)$ are arbitrary (real or complex) constants, then

$$ \int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\alpha}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1-xy}{(1-x)(1-y)}\right] S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[\frac{(1-x)ty}{1-xy}\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy \tag{2.1} $$

$$ = \sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_il_i}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}}\times {}_2\Psi_3\left[\begin{matrix} \left(\frac{\gamma}{k},1\right),(\beta+\nu,2); \\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l_i+\nu,2),(1,1); \end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right] $$

Proof. Let L.H.S. of eqn. (2.1) is denoted by *I*. By putting value of $s_{n_{1,\ldots}n_r}^{m_{1,\ldots}m_r}[x]$ and $\omega_{\kappa,\mu,b,c}^{\gamma,\delta}[x]$ from (1.6) and (1.5) and interchanging order of integration and summation, we have

$$I=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i\,!}A_{n_il_i}t^{l_i}\sum_{n=0}^{\infty}\frac{(-1)^n c^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)}\frac{(t/2)^{\nu+2n}}{(n!)^2} \tag{2.2}$$

$$\times\int_0^1\int_0^1\frac{(1-x)^{\alpha+l_i-1}(y)^{\alpha+l_i}(1-y)^{\beta+\nu+2n-1}}{(1-xy)^{(\alpha+\beta+l_i+\nu+2n-1)}}dxdy$$

By using result (1.10), we have

$$I=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i\,!}A_{n_il_i}t^{l_i}\times\sum_{n=0}^{\infty}\frac{(-1)^n c^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)}\frac{\Gamma(\alpha+l_i)\Gamma(\beta+\nu+2n)}{\Gamma(\alpha+\beta+l_i+\nu+2n)}\frac{(t/2)^{\nu+2n}}{(n!)^2} \tag{2.3}$$

After simplification, we have

$$I=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i\,!}A_{n_il_i}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma(\beta+\nu+2n)}{\Gamma\left(\frac{\mu}{k}+\frac{b+1}{2k}+\frac{\lambda n}{k}\right)\Gamma(\alpha+\beta+l_i+\nu+2n)\Gamma(1+n)}\left[\frac{(-c)k^{1-\frac{\lambda}{k}}t^2}{4}\right]^n\frac{1}{n!} \tag{2.4}$$

Interpreting eqn. (2.4), in view of result (1.7), we get the required result (2.1).

Theorem 2. *Let* $k\in\mathbb{R};z,\lambda,\gamma,\mu,b,c\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,n_1...n_r$ $=0,1,2,...;m_1,...m_r\in\mathbb{Z}^+$ *and* $A_{n_i,l_i}(n_i,l_i\geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^\infty\int_0^\infty\phi(x+y)x^{\alpha-1}y^{\beta-1}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[tx\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[ty\right]=dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i\,!}A_{n_il_i}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}}$$

$$\times {}_2\Psi_3\left[\begin{array}{c|c} \left(\frac{\gamma}{k},1\right),(\beta+\nu,2) & \frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4} \\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l_i+\nu,2),(1,1); & \end{array}\right] \quad (2.5)$$

$$\times \int_0^\infty \phi(z) z^{\alpha+\beta+l_i+\nu+2n-1} dz.$$

Proof. Let L.H.S. of eqn. (2.5) is denoted by *I*. By putting value of $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x]$ and $\omega_{k,\mu,b,c}^{\gamma,\delta}[x]$ from (1.6) and (1.5) and interchanging order of integration and summation, we have

$$I = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} t^{l_i} \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)} \frac{(t/2)^{\nu+2n}}{(n!)^2} \quad (2.6)$$

$$\times \int_0^\infty \int_0^\infty \phi(x+y) x^{\alpha+l_i-1} y^{\beta+\nu+2n-1} dx dy$$

By using result (1.11), we have

$$I = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} t^{l_i} \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)} \frac{(t/2)^{\nu+2n}}{(n!)^2} \quad (2.7)$$

$$\times \frac{\Gamma(\alpha+l_i)\Gamma(\beta+\nu+2n)}{\Gamma(\alpha+\beta+l_i+\nu+2n)} \int_0^\infty \phi(z) z^{\alpha+\beta+l_i+\nu+2n-1} dz$$

After simplification, we have

$$I = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} \frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)} \frac{t^{l_i+\nu}}{2^\nu}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma(\beta+\nu+2n)}{\Gamma\left(\frac{\mu}{k}+\frac{b+1}{2k}+\frac{\lambda n}{k}\right)\Gamma(\alpha+\beta+l_i+\nu+2n)\Gamma(1+n)} \left[\frac{(-c)k^{1-\frac{\lambda}{k}}t^2}{4}\right]^n \frac{1}{n!} \quad (2.8)$$

$$\times \int_0^\infty \phi(z) z^{\alpha+\beta+l_i+\nu+2n-1} dz$$

Interpreting eqn. (2.8), in view of result (1.7), we get the required result (2.5).

Theorem 3. *Let* $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, n_1 ... n_r$ $= 0, 1, 2, ...; m_1, ... m_r \in \mathbb{Z}^+$ *and* A_{n_i, l_i} $(n_i, l_i \geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^1 \int_0^1 f(xy)(1-x)^{\alpha-1}(1-y)^{\beta-1} y^\alpha S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[ty(1-x)\right] \omega_{k,\nu,b,c}^{\gamma,\lambda}\left[t(1-y)\right] dxdy$$

$$= \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} \frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}} \Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)} \frac{t^{l_i+\nu}}{2^\nu} \tag{2.9}$$

$$\times {}_2\Psi_3\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (\beta+\nu, 2); \\ \left(\frac{\nu}{k} + \frac{b+1}{2k}, \frac{\lambda}{k}\right), (\alpha+\beta+l_i+\nu, 2), (1,1); \end{matrix} \middle| \frac{(-c)k^{1-\frac{\lambda}{t}} t^2}{4}\right] \times \int_0^1 f(z)(1-z)^{\alpha+\beta+l_i+\nu+2n-1} dz.$$

Proof. Let L.H.S. of eqn. (2.9) is denoted by *I*. By putting value of $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x]$ and $\omega_{k,\mu,b,c}^{\gamma,\delta}[x]$ From (1.6) and (1.5) and interchanging order of integration and summation, we have

$$I = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} t^{l_i} \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k\left(\lambda n + \nu + \frac{b+1}{2}\right)} \frac{(t/2)^{\nu+2n}}{(n!)^2}$$

$$\times \int_0^1 \int_0^1 f(xy)(1-x)^{\alpha+l_i-1} y^{\alpha+l_i} (1-y)^{\beta+\nu+2n-1} dxdy \tag{2.10}$$

By using result (1.12), we have

$$I = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i l_i} t^{l_i} \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k\left(\lambda n + \nu + \frac{b+1}{2}\right)} \frac{(t/2)^{\nu+2n}}{(n!)^2} \tag{2.11}$$

$$\times \frac{\Gamma(\alpha+l_i)\Gamma(\beta+\nu+2n)}{\Gamma(\alpha+\beta+l_i+\nu+2n)}\int_0^1 f(z)(1-z)^{\alpha+\beta+l_i+\nu+2n-1}dz$$

After simplification, we have

$$I=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_il_i}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma(\beta+\nu+2n)}{\Gamma\left(\frac{\mu}{k}+\frac{b+1}{2k}+\frac{\lambda n}{k}\right)\Gamma(\alpha+\beta+l_i+\nu+2n)\Gamma(1+n)}\left[\frac{(-c)k^{1-\frac{\lambda}{k}}t^2}{4}\right]^n\frac{1}{n!} \tag{2.12}$$

$$\times\int_0^1 f(z)(1-z)^{\alpha+\beta+l_i+\nu+2n-1}dz$$

Interpreting eqn. (2.12), in view of result (1.7), we get the required result (2.9).

Theorem 4. *Let* $k\in\mathbb{R}; z,\lambda,\gamma,\mu,b,c\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,n_1...n_r$ $=0,1,2,...;m_1,...m_r\in\mathbb{Z}^+$ *and* $A_{n_i,l_i}(n_i,l_i\geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\alpha+\sigma}\left[\frac{1-y}{1-xy}\right]^{\beta}\frac{1}{(1-x)}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[\frac{(1-x)ty}{1-xy}\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_il_i}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+\sigma+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}} \tag{2.13}$$

$$\times\,{}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu+1,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\sigma+\beta+l_i+\nu+1,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right].$$

Proof. Proof of Theorem 3 is similar to Theorem 1, so we skip the details.

III. SPECIAL CASES

By applying the our results in eqn. (2.1), (2.5), (2.9), and (2.13) to the case of Hermite polynomials (1.6) by setting $S_n^2(x) \to x^{n/2} H_n\left[\frac{1}{2\sqrt{x}}\right]$, in which $m_1,...m_r = 2; n_1,...,n_r = n; r = 1; A_{n_i,l_i} = (-1)^l$, we have the following interesting results:

Corollary 1. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0,$ then

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\alpha}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1-xy}{(1-x)(1-y)}\right]\left[\frac{(1-x)ty}{1-xy}\right]^{n/2} H_n\left[\frac{1}{2\sqrt{\frac{(1-x)ty}{1-xy}}}\right].$$

$$\times\ \omega_{k,\nu,b,c}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy = \sum_{l=0}^{[n/2]}\frac{(-n)_{2l}(-1)^l}{l!}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.1}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l+\nu,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right]$$

Corollary 2. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, then$

$$\int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}(tx)^{n/2}H_n\left[\frac{1}{2\sqrt{tx}}\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}[ty]dxdy$$

$$= \sum_{l=0}^{[n/2]}\frac{(-n)_{2l}(-1)^l}{l!}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.2}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l+\nu,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right]\times\int_0^\infty \phi(z)z^{\alpha+\beta+l+\nu+2n-1}dz.$$

Corollary 3. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0,$ *then*

$$\int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}(1-y)^{\beta-1}y^{\alpha}\left[t(1-x)\right]^{n/2} H_n\left[\frac{1}{2\sqrt{t(1-x)^{n/2}}}\right]$$

$$\times\ \omega_{k,\nu,b,c}^{\gamma,\lambda}\left[t(1-y)\right]dxdy = \sum_{l=0}^{[n/2]}\frac{(-n)_{2l}(-1)^l}{l!}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.3}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l+\nu-1,2),(1,1);\end{matrix}\left|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right.\right]\times\int_0^1 f(z)(1-z)^{\alpha+\beta+l+\nu+2n-1}dz.$$

Corollary 4. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0,$ *then*

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\alpha+\sigma}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1}{(1-x)}\right]\left[\frac{(1-x)ty}{1-xy}\right]^{n/2} H_n\left[\frac{1}{2\sqrt{\frac{(1-x)ty}{1-xy}}}\right]$$

$$\times\ \omega_{k,\nu,b,c}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy = \sum_{l=0}^{[n/2]}\frac{(-n)_{2l}(-1)^l}{l!}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+\sigma+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.4}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu+1,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\sigma+\beta+l+\nu+1,2),(1,1);\end{matrix}\left|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right.\right].$$

By applying the our results in eqn. (2.1), (2.5), (2.9), and (2.13) to the case of Laguerre polynomials (1.6) by setting $S_n^2(x) \to L_n^{(\rho)}[x],$ in which $m_1,...m_r = 1; n_1,...,n_r = n; r = 1; A_{n_i,l_i} = \binom{n+\rho'}{n}\frac{1}{\rho'+1},$ we have the following interesting results:

Corollary 5. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, then$

$$\int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\alpha}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1-xy}{(1-x)(1-y)}\right] L_n^{(\rho)}\left[\frac{(1-x)ty}{1-xy}\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\rho'}{n}\frac{1}{\rho'+1}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.5}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l+\nu,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right].$$

Corollary 6. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, then$

$$\int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}L_n^{(\rho)}(tx)\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[ty\right]dxdy$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\rho'}{n}\frac{1}{\rho'+1}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.6}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l+\nu,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right]\times\int_0^\infty \phi(z)z^{\alpha+\beta+l+\nu+2n-1}dz.$$

Corollary 7. Let $k \in \mathbb{R}; z, \lambda, \gamma, \mu, b, c \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, then$

$$\int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}(1-y)^{\beta-1}y^{\alpha}L_n^{(\rho)}\left[t(1-x)\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[t(1-y)\right]dxdy$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\rho'}{n}\frac{1}{\rho'+1}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.7}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\beta+l+\nu-1,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right]\times\int_0^1 f(z)(1-z)^{\alpha+\beta+l+\nu+2n-1}dz.$$

Corollary 8. Let $k\in\mathbb{R};z,\lambda,\gamma,\mu,b,c\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,then$

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\alpha+\sigma}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1}{(1-x)}\right]L_n^{(\rho)}\left[\frac{(1-x)ty}{1-xy}\right]\omega_{k,\nu,b,c}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\rho'}{n}\frac{1}{\rho'+1}\frac{k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}\Gamma(\alpha+\sigma+l)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l+\nu}}{2^{\nu}} \tag{3.8}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu+1,2);\\ \left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\alpha+\sigma+\beta+l+\nu+1,2),(1,1);\end{matrix}\middle|\frac{(-c)k^{1-\frac{\lambda}{t}}t^2}{4}\right].$$

If we put b = c = 1 in eqn. (2.1), (2.5), (2.9), and (2.13), we have the following interesting results:

Corollary 9 Let $k\in\mathbb{R};z,\lambda,\gamma,\mu\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,n_1...n_r=0,1,2,...;m_1,...m_r$ $\in\mathbb{Z}^+ \,and\, A_{n_i,l_i}\ (n_i,l_i\geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\alpha}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1-xy}{(1-x)(1-y)}\right]S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{(1-x)ty}{1-xy}\right]$$

$$\times\left[\frac{(1-y)t}{2(1-xy)}\right]^{\nu}J_{k,\nu}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy \tag{3.9}$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_il_i}\frac{k^{1-\frac{\nu-1}{k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}}$$

$$\times\ {}_2\Psi_3\left[\begin{array}{c|c} \left(\frac{\gamma}{k},1\right),(\beta+\nu,2); & \frac{-k^{1-\frac{\lambda}{t}}t^2}{4} \\ \left(\frac{\nu+1}{k},\frac{\lambda}{k}\right),(\alpha+\beta+l_i+\nu,2),(1,1); & \end{array}\right].$$

Corollary 10 Let $k\in\mathbb{R};z,\lambda,\gamma,\mu\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,n_1...n_r=0,1,2,...;m_1,...m_r$ $\in\mathbb{Z}^+$ *and* $A_{n_i,l_i}\ (n_i,l_i\geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[tx]\times\left(\frac{ty}{2}\right)^{\nu}J_{k,\nu}^{\gamma,\lambda}[ty]dxdy = \sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_il_i}\frac{k^{1-\frac{\nu-1}{k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}} \tag{3.10}$$

$$\times\ {}_2\Psi_3\left[\begin{array}{c|c} \left(\frac{\gamma}{k},1\right),(\beta+\nu,2); & \frac{-k^{1-\frac{\lambda}{t}}t^2}{4} \\ \left(\frac{\nu+1}{k},\frac{\lambda}{k}\right),(\alpha+\beta+l_i+\nu,2),(1,1); & \end{array}\right]\times\int_0^\infty \phi(z)z^{\alpha+\beta+l_i+\nu+2n-1}dz.$$

Corollary 11 Let $k\in\mathbb{R};z,\lambda,\gamma,\mu\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,n_1...n_r=0,1,2,...;m_1,...m_r$ $\in\mathbb{Z}^+$ *and* $A_{n_i,l_i}\ (n_i,l_i\geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^1\int_0^1 f(xy)(1-x)^{\alpha-1}(1-y)^{\beta-1}y^{\alpha}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[ty(1-x)]$$

$$\times\left(\frac{t(1-y)}{2}\right)^{\nu}J_{k,\nu}^{\gamma,\lambda}[t(1-y)]dxdy = \sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_il_i}\frac{k^{1-\frac{\nu-1}{k}}\Gamma(\alpha+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}} \tag{3.11}$$

$$\times\ {}_2\Psi_3\left[\begin{array}{c|c} \left(\frac{\gamma}{k},1\right),(\beta+\nu,2); & \frac{-k^{1-\frac{\lambda}{t}}t^2}{4} \\ \left(\frac{\nu+1}{k},\frac{\lambda}{k}\right),(\alpha+\beta+l_i+\nu,2),(1,1); & \end{array}\right]\times\int_0^1 f(z)(1-z)^{\alpha+\beta+l_i+\nu+2n-1}dz.$$

Corollary 12 Let $k\in\mathbb{R};z,\lambda,\gamma,\mu\in\mathbb{C},\Re(\lambda)>0,\Re(\nu)>0,n_1...n_r=0,1,2,...;m_1,...m_r$ $\in\mathbb{Z}^+$ *and* $A_{n_i,l_i}\ (n_i,l_i\geq 0)$ are arbitrary (real or complex) constants, then

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\alpha+\sigma}\left[\frac{1-y}{1-xy}\right]^{\beta}\left[\frac{1}{(1-x)}\right]S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[\frac{(1-x)ty}{1-xy}\right]\left[\frac{(1-y)t}{2(1-xy)}\right]^{\nu}J_{k,\nu}^{\gamma,\lambda}\left[\frac{(1-y)t}{1-xy}\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i l_i}\frac{k^{1-\frac{\nu-1}{k}}\Gamma(\alpha+\sigma+l_i)}{\Gamma\left(\frac{\gamma}{k}\right)}\frac{t^{l_i+\nu}}{2^{\nu}} \tag{3.12}$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\beta+\nu+1,2);\\ \left(\frac{\nu+1}{k},\frac{\lambda}{k}\right),(\alpha+\sigma+\beta+l_i+\nu+1,2),(1,1);\end{matrix}\left|\frac{-k^{1-\frac{\lambda}{t}}t^2}{4}\right.\right].$$

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