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SOME COMMON FIXED POINT RESULTS IN COMPLEX VALUED b-METRIC SPACES

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Abstract: In this paper we prove some common fixed point theorems for two self-mappings in complex valued b-metric spaces. Our results generalize the results of A. Azam, B. Fisher and M. Khan [3]. Also we give some similar new results.

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1. INTRODUCTION AND PRELIMINARIES

The notion of complex valued metric space was introduced by A. Azam, B. Fisher and M. Khan [3] in 2011. The concept of b-metric space was introduced by Bakhtin [4] in 1989. Rao et al. [8] introduced complex valued b-metric space which is more general than well-known complex valued metric space. There are many fixed point results in complex valued metric spaces [see [2], [5], [6], [7], [9], [10]] also in complex valued b-metric spaces [see [1], [8]]. In this paper we present some come fixed point results of two self-mappings satisfying a rational inequality in complex valued b-metric space. Our results generalize the results obtained by A. Azam, B. Fisher and M. Khan [3] and Aiman A. Mukheimer [1], also we get some similar new results.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim$ bz, $\forall$ z $\in \mathbb{C}$.

(ii) $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam *et al.* [3] defined the complex valued metric space in the following way:

Definition 1 ([3]): Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1([6]): Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = i\,|z_1 - z_2|\ \forall\, z_1, z_2 \leq \mathbb{C}.$$

One can easily verify that (X, d) is a complex valued metric space.

Definition 2([8]): Let X be a nonempty set and let $s \geq 1$ be given real number. A function d: $X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) \precsim s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called complex valued b-metric space.

Example 2([8]): Let $X = [0,1]$. Define the mapping d: $X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i|x - y|^2, \text{ for all } x, y \in X.$$

Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 3([8])): Let (X, d) be a complex valued b-metric space. Then

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X: d(x, y) \prec r\} \subseteq A.$$

A subset $A \subseteq X$ is called open if each element of A is an interior point of A.

(ii) A point $x \in X$ is called a limit point of $A \subseteq X$ if for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A - \{x\}) \neq \phi.$$

A subset $A \subseteq X$ is called closed if each element of $X - A$ is not a limit point of A.

(iii) The family

$$F = \{B(x, r): x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ "on" X.

Definition 4([8]): Let (X, d) be a complex valued b-metric space. Then

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec r$, $\forall\, n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec r$ for all $n > N$, $m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X, d).

(iii) If every Cauchy sequence in X is convergent in X then (X, d) is called a complete complex valued b-metric space.

Lemma 1 ([8]): Let (X, d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2 ([8]): Let (X, d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.

2. RESULTS

In this section we present the main results of the paper.

Theorem 1: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $S, T: X \to X$ be self-mappings satisfying the condition:

$$d(Sx, Ty) \precsim \alpha d(x, y) + d(x, Sx) d(y, Ty) \left[\frac{\beta}{1 + d(x, y)} + \frac{\gamma}{1 + d(Sx, Ty)} \right]$$

for all $x, y \in X$ and α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Then S and T have unique common fixed point in X.

Proof : Let $x_0 \in X$ be arbitrary.

We define a sequence $\{x_n\}$ in X as

$$x_{2k+1} = Sx_{2k}, \qquad k = 0, 1, 2, \ldots$$

$$x_{2k+2} = Tx_{2k+1}, \qquad k = 0, 1, 2, \ldots$$

Then

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\precsim \alpha d(x_{2k}, x_{2k+1}) + d(x_{2k}, Sx_{2k})\, d(x_{2k+1}, Tx_{2k+1}) \\ &\quad \times \left[\frac{\beta}{1+d(x_{2k}, x_{2k+1})} + \frac{\gamma}{1+d(Sx_{2k}, Tx_{2k+1})}\right] \\ &= \alpha d(x_{2k}, x_{2k+1}) + d(x_{2k}, x_{2k+1})\, d(x_{2k+1}, x_{2k+2}) \\ &\quad \times \left[\frac{\beta}{1+d(x_{2k}, x_{2k+1})} + \frac{\gamma}{1+d(x_{2k+1}, x_{2k+2})}\right]. \end{aligned}$$

Thus

$$\begin{aligned} |\, d(x_{2k+1}, x_{2k+2}) \,| &\le \alpha\, |\, d(x_{2k}, x_{2k+1})| + |d(x_{2k}, x_{2k+1})|\, |d(x_{2k+1}, x_{2k+2})| \\ &\quad \times \left[\left|\frac{\beta}{1+d(x_{2k}, x_{2k+1})}\right| + \left|\frac{\gamma}{1+d(x_{2k+1}, x_{2k+2})}\right|\right] \\ &\le \alpha\, |\, d(x_{2k}, x_{2k+1})\,| + \beta\, |\, d(x_{2k+1}, x_{2k+2})\,| + \gamma\, |\, d(x_{2k}, x_{2k+1})\,|, \end{aligned}$$

since $|d(x_{2k}, x_{2k+1})| \le |1 + d(x_{2k}, x_{2k+1})|$ and

$|d(x_{2k+1}, x_{2k+2})| \le |1 + d(x_{2k+1}, x_{2k+2})|$

Thus

$$|\, d(x_{2k+1}, x_{2k+2})\,| \le \frac{\alpha+\gamma}{1-\beta}\, |\, d(x_{2k}, x_{2k+1})\,|. \tag{1}$$

Similarly

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\precsim \alpha d(x_{2k+2}, x_{2k+1}) + d(x_{2k+2}, Sx_{2k+2})\, d(x_{2k+1}, Tx_{2k+1}) \end{aligned}$$

$$\times\left[\frac{\beta}{1+d(x_{2k+2}, x_{2k+1})}+\frac{\gamma}{1+d(Sx_{2k+2}, Tx_{2k+1})}\right]$$

$$= \alpha d(x_{2k+2}, x_{2k+1}) + d(x_{2k+2}, x_{2k+3}) d(x_{2k+1}, x_{2k+2})$$

$$\times\left[\frac{\beta}{1+d(x_{2k+2}, x_{2k+1})}+\frac{\gamma}{1+d(x_{2k+3}, x_{2k+2})}\right].$$

Thus

$$\begin{aligned}|d(x_{2k+2}, x_{2k+3})| &\le \alpha|d(x_{2k+2}, x_{2k+1})| + |d(x_{2k+2}, x_{2k+3})||d(x_{2k+1}, x_{2k+2})| \\ &\times \left[\left|\frac{\beta}{1+d(x_{2k+2}, x_{2k+1})}\right| + \left|\frac{\gamma}{1+d(x_{2k+3}, x_{2k+2})}\right|\right] \\ &\le \alpha|d(x_{2k+2}, x_{2k+1})| + \beta|d(x_{2k+2}, x_{2k+3})| + \gamma|d(x_{2k+1}, x_{2k+2})|.\end{aligned}$$

Thus

$$|d(x_{2k+2}, x_{2k+3})| \le \frac{\alpha+\gamma}{1-\beta}|d(x_{2k+1}, x_{2k+2})|. \tag{2}$$

Put $h = \dfrac{\alpha+\gamma}{1-\beta}$.

Since $0 \le \alpha + \beta + \gamma < 1$, we have $0 \le h < 1$.

Therefore from (1) and (2) for $n \in \mathbb{N}$ we have

$$|d(x_{n+1}, x_{n+2})| \le h|d(x_n, x_{n+1})| \le h^2|d(x_{n-1}, x_n)| \le \cdots \le h^{n+1}|d(x_0, x_1)|.$$

So for $m, n \in \mathbb{N}$,

$$\begin{aligned}|d(x_n, x_{m+n})| &\le s[|d(x_n, x_{n+1})| + |d(x_{n+1}, x_{m+n})|] \\ &\le s|d(x_n, x_{n+1})| + s^2[|d(x_{n+1}, x_{n+2})| + |d(x_{n+2}, x_{m+n})|] \\ &\le s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| \\ &+ \ldots + s^{m-1}|d(x_{m+n-2}, x_{m+n-1})| + s^{m-1}|d(x_{m+n-1}, x_{m+n})| \\ &\le s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})|\end{aligned}$$

$$+\ldots+ s^{m-1}\,|d(x_{m+n-2}, x_{m+n-1})\,| + s^{m}\,|d(x_{m+n-1}, x_{m+n})\,|$$

$$\leq sh^{n}\,|d(x_0, x_1)\,| + s^{2}h^{n+1}\,|d(x_0, x_1)\,| +\ldots+ s^{m}h^{m+n-1}\,|d(x_0, x_1)|$$

$$\leq sh^{n}\,(1 + sh + (sh)^{2} + \ldots +(sh)^{m-1})|d(x_0, x_1)|$$

$$\to 0 \text{ as } n \to \infty \text{ where } m \in \mathbb{N}.$$

Therefore from Lemma 2 we see that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete $\exists\, z \in X$ such that $x_n \to z$ as $n \to \infty$.

Thus

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z. \tag{3}$$

Now

$$d(Sz, z) \precsim s[d(Sz, Tx_{2n+1}) + d(Tx_{2n+1}, z)]$$

$$\precsim s\alpha d(z, x_{2n+1}) + sd(z, Sz)d(x_{2n+1}, Tx_{2n+1})$$

$$\times \left[\frac{\beta}{1+d(z, x_{2n+1})} + \frac{\gamma}{1+d(Sz, Tx_{2n+1})}\right] + sd(Tx_{2n+1}, z)$$

$$= s\alpha d(z, x_{2n+1}) + sd(z, Sz)d\,(x_{2n+1}, x_{2n+2})$$

$$\times \left[\frac{\beta}{1+d(z, x_{2n+1})} + \frac{\gamma}{1+d(Sz, Tx_{2n+1})}\right] + sd(Tx_{2n+1}, z).$$

Letting $n \to \infty$ and using the equation (3) we get that

$$d(Sz, z) \precsim 0.$$

Thus $d(Sz, z) = 0$ and so $Sz = z$.

Similarly,

$$d(z, Tz) = d(Sz, Tz)$$

$$\precsim \alpha d(z, z) + d(z, Sz)d(z, Tz) \left[\frac{\beta}{1+d(z, z)} + \frac{\gamma}{1+d(Sz, Tz)}\right]$$

$$= 0.$$

Thus $d(z, Tz) = 0$ and so $Tz = z$.

Therefore z is a common fixed point of S and T.

Now for the uniqueness part, let us suppose that $Sz^* = Tz^* = z^*$ for some $z^* \in X$.

Then

$$\begin{aligned} d(z, z^*) &= d(Sz, Tz^*) \\ &\precsim \alpha d(z, z^*) + d(z, Sz)d(z^*, Tz^*) \left[\frac{\beta}{1+d(z,z^*)} + \frac{\gamma}{1+d(Sz,Tz^*)}\right] \\ &= \alpha d(z, z^*) \end{aligned}$$

Since $0 \le \alpha < 1$, we must have $d(z, z^*) = 0$ and so $z = z^*$.

This completes the proof.

Corollary 1: Let (X, d) be a complete complex valued metric space and $S, T: X \to X$ be self-mappings satisfying the condition:

$$d(Sx,Ty) \precsim \alpha d(x,y) + d(x,Sx)d(y,Ty)\left[\frac{\beta}{1+d(x,y)} + \frac{\gamma}{1+d(Sx,Ty)}\right]$$

for all $x, y \in X$ where α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Then S and T have a unique common fixed point in X.

Proof : This result follows from Theorem 1 by setting $s = 1$.

Corollary 2: Let (X, d) be a complete complex valued b-metric space with coefficient $s \ge 1$ and $S, T: X \to X$ be self-mappings satisfying the condition:

$$d(Sx,Ty) \precsim \alpha d(x,y) + \frac{\beta . d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x, y \in X$ where α, β are nonnegative reals with $\alpha + \beta < 1$. Then S and T have unique common fixed point in X.

Proof : This result follows from Theorem 1 by setting $\gamma = 0$.

Note : The above sufficient condition is better than that of theorem 15 of [1].

Corollary 3 (Theorem 4, [3]): Let (X, d) be a complete complex valued metric space and let the mappings $S, T: X \to X$ satisfy the following condition:

$$d(Sx,Ty) \precsim \alpha d(x,y) + \frac{\beta . d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x, y \in X$ where α, β are nonnegative reals with $\alpha + \beta < 1$. Then S and T have a unique common fixed point.

Proof : This result follows from Corollary 2 by setting $s = 1$.

By setting $\beta = 0$ in Theorem 1, we get similar sufficient conditions like Corollary 2 for existence of unique common fixed point of two self-mappings of a complete complex valued b- metric space with coefficient $s \geq 1$.

Corollary 4: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $S, T: X \to X$ be self-mappings satisfying the condition:

$$d(Sx,Ty) \precsim \alpha d(x,y) + \frac{\gamma . d(x,Sx)d(y,Ty)}{1+d(Sx,Ty)}$$

for all $x, y \in X$ where α, γ are nonnegative reals with $\alpha + \gamma < 1$. Then S and T have unique common fixed point in X.

Corollary 5: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and let the mapping $T: X \to X$ satisfy the following condition:

$$d(Tx,Ty) \precsim \alpha d(x,y) + d(x,Tx)d(y,Ty)\left[\frac{\beta}{1+d(x,y)} + \frac{\gamma}{1+d(Tx,Ty)}\right]$$

for all $x, y \in X$ where α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point.

Proof : This result follows from Theorem 1 by setting $S = T$.

Corollary 6 (Corollary 5, [3]): Let (X, d) be a complete complex valued metric space and let the mapping $T: X \to X$ satisfy the following condition:

$$d(Tx,Ty) \precsim \alpha d(x,y) + \frac{\beta . d(x,Tx)d(y,Ty)}{1+d(x,y)}$$

for all $x, y \in X$ where α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof : This follows from corollary 5 by setting $\gamma = 0$ and $s = 1$.

We get similar result like Corollary 6, by setting $\beta = 0$, $s = 1$ in Corollary 5.

Corollary 7: Let (X, d) be a complete complex valued metric space and let the mapping $T: X \to X$ satisfy the following condition:

$$d(Tx,Ty) \precsim \alpha d(x,y) + \frac{\gamma . d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}$$

for all $x, y \in X$ where α, γ are nonnegative reals with $\alpha + \gamma < 1$. Then T has a unique fixed point.

Theorem 2: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and let the mapping $T: X \to X$ satisfy the following condition:

$$d(T^n x, T^n y) \precsim \alpha d(x,y) + d(x, T^n x) d(y, T^n y)\left[\frac{\beta}{1+d(x,y)} + \frac{\gamma}{1+d(T^n x, T^n y)}\right]$$

for all $x, y \in X$ where α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point.

Proof : Applying Corollary 5 we get a unique fixed point z of T^n.

Now note that $T^n\, Tz = TT^n z = Tz$.

Thus Tz is also a fixed point of T and uniqueness of z we have $Tz = z$ and hence z is a fixed point of T.

Since any fixed point of T is also a fixed point of T^n, z is the unique fixed point of T and this completes the proof of the theorem.

Corollary 8: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and let the mapping $T: X \to X$ satisfy the following condition:

$$d(T^n x, T^n y) \precsim \alpha d(x,y) + \frac{\beta . d(x, T^n x) d(y, T^n y)}{1+d(x,y)}$$

for all $x, y \in X$ where α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof : In Theorem 2, if we take $\gamma = 0$ then we get the above result.

Corollary 9 (Corollary 6, [3]): Let (X, d) be a complete complex valued metric space and let the mapping $T: X \to X$ satisfy the following condition:

$$d(T^n x, T^n y) \precsim \alpha d(x,y) + \frac{\beta . d(x, T^n x) d(y, T^n y)}{1+d(x,y)}$$

for all $x, y \in X$ where α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof : It follows from corollary 8 by setting $s = 1$.

In Theorem 2, taking $\beta = 0$, $s = 1$ we get the following result like Corollary 9.

Corollary 10 : Let (X, d) be a complete complex valued metric space and $T: X \to X$ satisfy

$$d(T^n x, T^n y) \precsim \alpha d(x, y) + \frac{\gamma . d(x, T^n x) d(y, T^n y)}{1 + d(T^n x, T^n y)}$$

for all $x, y \in X$ where α, γ are nonnegative reals with $\alpha + \gamma < 1$. Then T has a unique fixed point.

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