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VERTEX POLYNOMIALS DERIVED THROUGH VARIOUS GRAPH THEORETICAL OPERATIONS

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Abstract: *Binary operations on graphs and its properties have been a fascinating area in the graph theory ever since its evolvement. Graph theoretic polynomials play a vital role in describing the properties associated with a graph. The polynomials such as chromatic polynomial gives idea on the colouring requirements of the vertices of graph, matching polynomials gives highlights on the matching properties of graph, vertex polynomial gives clear idea about the vertex degree, edge polynomial on weight of edge and so on.*

In this paper, we study, vertex polynomials of certain classes of graphs resulting from graph operations like cartesian product, conjunction, join and symmetric difference.

Key Words: *graph operations, vertex polynomials.*

Classification Code: *05C.*

1. INTRODUCTION

1.1. Graph Operations

Certain binary operations are applied on two distinct graphs G_1 and G_2, to generate a new one G, with some specific properties as per defined for the particular operation. For the operations like cartesian product, conjunction and symmetric difference, the vertex set V of the resultant graph G, is $V = V_1 \times V_2$, and edge set E depends on the incidence relation between the vertices (u_i, v_j) and (u_k, v_l) as per the concerned operation (where u_i's $\in V_1$ and v_i's $\in V_2$ of graphs G_1 and G_2 respectively). When the operation join is considered, the vertex set $V = V_1 \cup V_2$ and edge set corresponds to that of $E_1 \cup E_2$ together with the edges joining every vertex of G_1 to that of G_2. For all other terminologies and notations we follow, [2].

1.2. Vertex Polynomial

In [3], S. Sedghi, N. Shobe and M.A. Salahshoor defined the vertex polynomial $S_G(x)$ of a graph G defined as $S_G(x) = \Sigma_{j=0}^{\Delta(G)} a_j x^j$, where a_j is the number of vertices with degree j, $\Delta(G)$ refers to the maximum degree of the graph G.

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Vertex polynomials of some standard graphs are as follows:

(i) $S_{K_n}(x) = nx^{n-1}$

(ii) $S_{C_n}(x) = nx^2$

(iii) $S_{P_n}(x) = 2x + (n-2)x^2$

(iv) $S_{G_k}(x) = nx^k$

(v) $S_{K_{m,n}}(x) = mx^n + nx^m$

Next we obtain the vertex polynomials of the graphs planar grids, ladder graphs, torus grids, wheels and fans. Also we study some properties of the vertex polynomials.

2. MAIN RESULTS

Observation 2.1: Vertex polynomial of a planar grid is $4x^2 + 2(m+n-4)\, x^3 + (m-2)\,(n-2)\, x^4$.

Proof. The planar grid is obtained by the cartesian product of P_m and P_n. It is clear that $P_m \times P_n$ has got mn vertices and $(m-1)n + (n-1)\, m$ edges.

Now first consider the planar grid $P_5 \times P_6$.

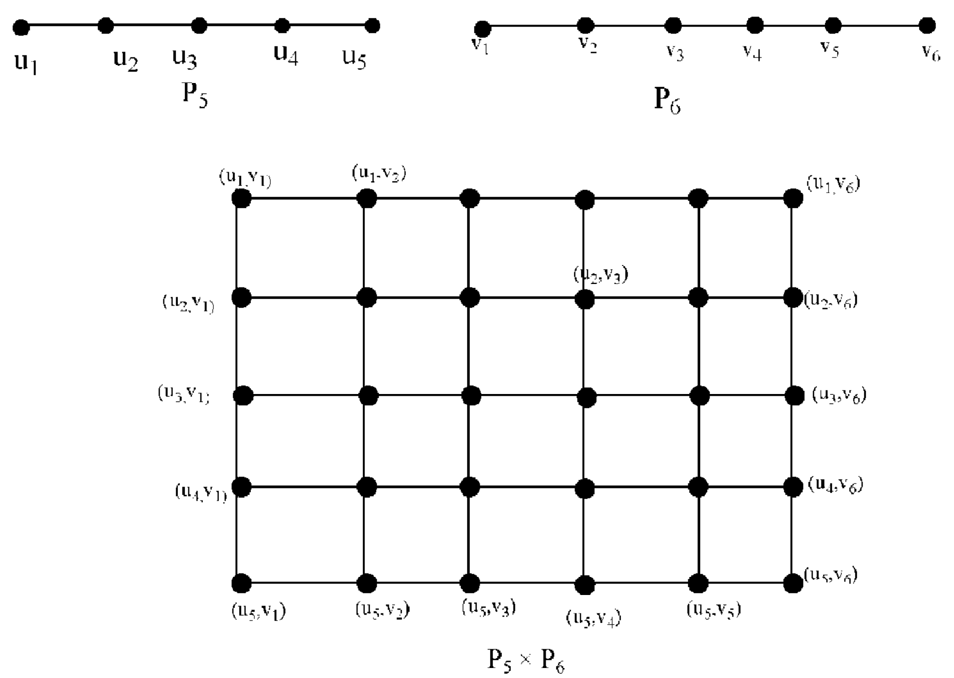

Figure 1: Cartesian product of P_5 and P_6

From the definition of vertex polynomial, it is clear that $S_{P_5 \times P_6}(x) = 4x^2 + 14x^3 + 12x^4$.

In general, for any m, $n > 2$, the graphical structure remains the same, but differs only in the number of grids formed. On observing the figure it is understood that there are only vertex degrees 2, 3, and 4 and whose cardinality is 4, $2(m-2) + 2(n-2)$, $(m-2)(n-2)$ respectively. Therefore, for a planar grid $G = P_m \times P_n$, we have

$$S_G(x) = 4x^2 + 2(m + n - 4)\, x^3 + (m-2)(n-2)\, x^4$$

Observation 2.2: Vertex polynomial of a ladder graph is $4x^2 + (m-2)2x^3$.

Proof. Ladder graph $L_n = P_m \times K_2$. Put $n = 2$ in $P_m \times P_n$. Then from observation 2.1, the result follows. Thus, $S_{L_n}(x) = 4x^2 + 2(m-2)\, x^3$

Observation 2.3 Vertex polynomial of a torus grid is mnx^4.

Proof. Torus grid is obtained by the cartesian product of two cycles C_m and C_n.

Now Consider, the torus grid $C_4 \times C_3$.

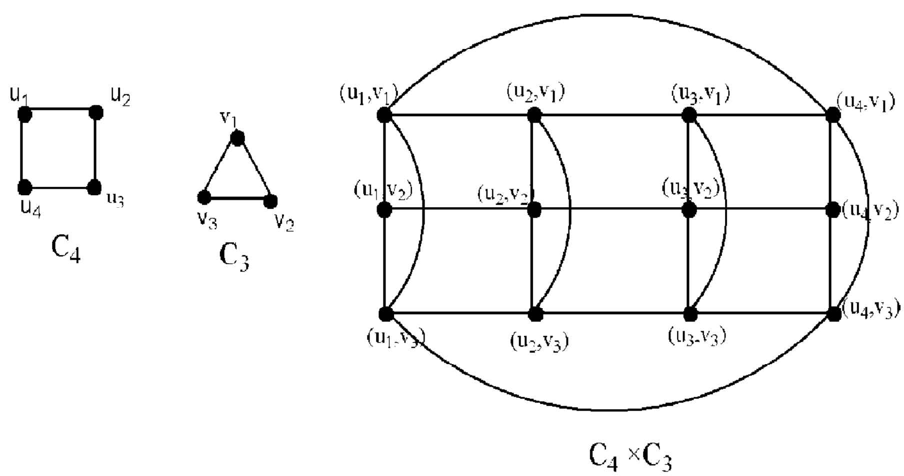

Figure 2: Cartesian product of C_4 and C_3

By the definition of vertex polynomial we have $S_{C_4 \times C_3}(x) = 12x^4$

In general if we fix m and change n, or fix n and change m, the change occurs only in the number of vertices.

Therefore the graph $G = C_m \times C_n$ has mn vertices and each of which is of degree 4. It follows that for torus grid,

$S_G(x) = mnx^4$.

Observation 2.4 Vertex polynomial of a prism is $2mx^3 + m(n–2)\, x^4$.

Proof. The prism is obtained by the cartesian product of P_n and C_m, $m \geq 3$.

Consider the graph $P_4 \times C_5$

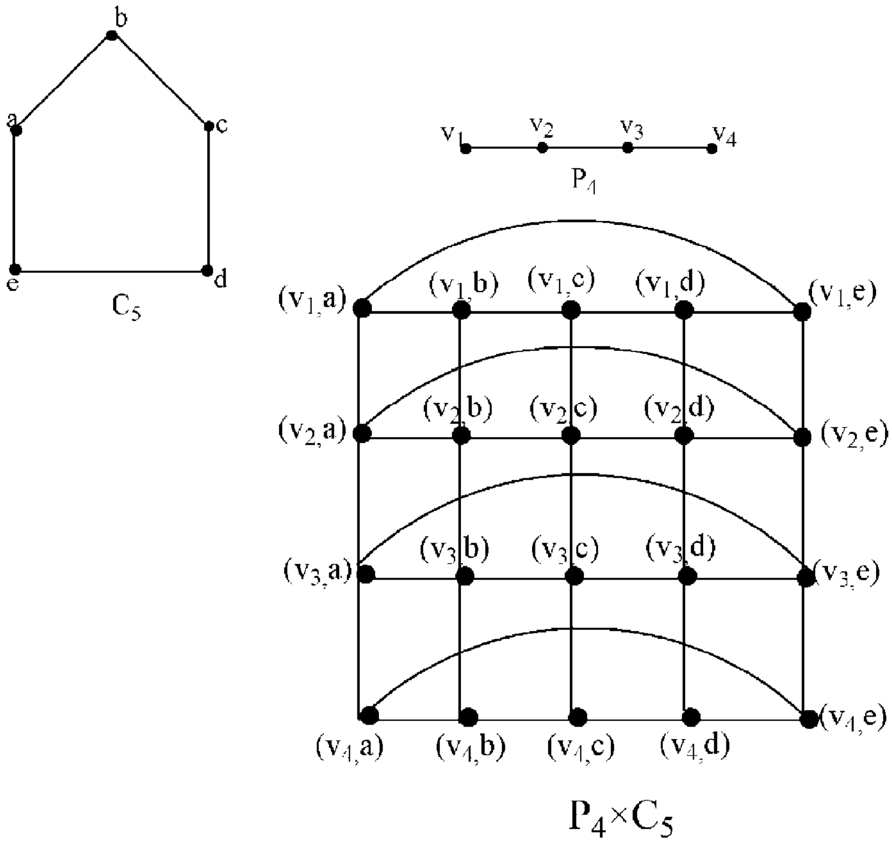

Figure 3: Cartesian product of P_4 and C_5

From the definition of vertex polynomial, $S_{P_4 \times C_5}(x) = 10x^3 + 10x^4$.

In general, for any graph $G = P_n \times C_m$ there are only $2m$ vertices of degree 3 and $(mn–2m)$ vertices of degree 4.

Hence in a prism, $S_G\,(x) = 2mx^3 + m(n–2)\, x^4$.

Corollary 2.5 Vertex polynomial of $P_2 \times C_m$, $m \geq 3$ is $2mx^3$

Proof. The result follows directly from observation 2.4.

Theorem 2.6 Let G_1 and G_2 be two graphs of order m and n respectively and let,

$G = G_1 \times G_2$. Then, $\deg(S_G(x)) = \Delta(G_1) + \Delta(G_2)$.

Proof. Let G_1 and G_2 be two graphs of order m and n respectively and let,

$V(G_1) = \{u_1, u_2, u_3, \ldots u_m\}$ and $V(G_2) = \{v_1, v_2, v_3, \ldots v_n\}$, then following the definition of $G = G_1 \times G_2$, the vertices (u_i, v_j), (u_k, v_l) are adjacent if either $u_i = u_k$ and v_j adjacent to v_l in G_2 or $v_j = v_l$ and u_i adjacent to u_k in G_1.

That is, degree of vertex $w_k = (u_i, v_j) = d(u_i) + d(v_j)$ [$d(u_i)$ represents the degree of the vertex u_i, where $u_i \in V(G_1)$].

There exists at least one vertex $w_g = (u_k, v_i)$ of G with maximum degree. Since, $d(w_g) = d(u_k) + d(v_i)$, the sum is maximum only if, $d(u_k) = \Delta(G_1)$ and $d(v_i) = \Delta(G_2)$.

That is, $d(w_g) = \Delta(G_1) + \Delta(G_2)$.

We have, $S_G(x) = \Sigma_{j=0}^{\Delta(G)} a_j x^j$. The degree of $S_G(x)$ corresponds to the maximum degree of the vertex of graph G. Since in cartesian product, $\Delta(G) = \Delta(G_1) + \Delta(G_2)$, it follows $\deg(S_G(x)) = \Delta(G_1) + \Delta(G_2)$.

Example

Consider the figure 4

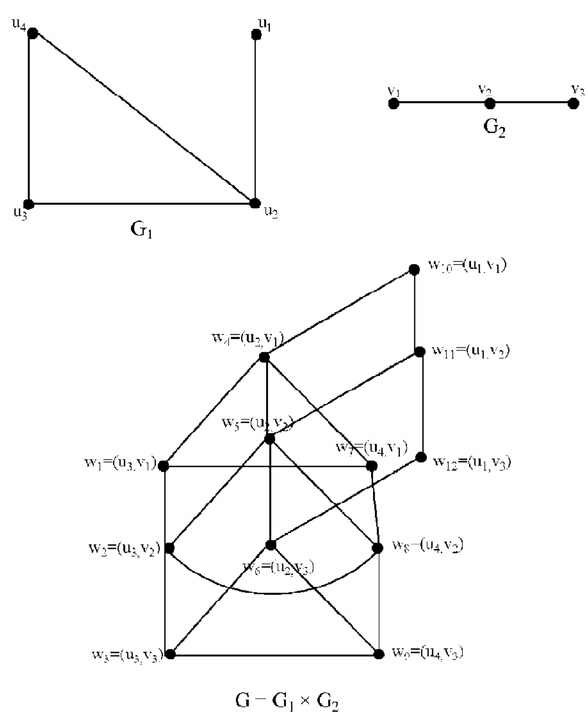

Figure 4: Cartesian product of G_1 and G_2

From the figure,

$d(u_2) = 3 = \Delta(G_1)$,

$d(v_2) = 2 = \Delta(G_2)$,

$d(w_5 = (u_2, v_2)) = 5 = \Delta(G_1) + \Delta(G_2) = \Delta(G)$

Also, $S_G(x) = 2x^2 + 5x^3 + 4x^4 + x^5$.

$\deg(S_G(x)) = 5 = \Delta(G_1) + \Delta(G_2)$

Theorem 2.7 Let $= G_1 \times G_2$. Then, $S_G(x) = S_{G_1}(x) \times S_{G_2}(x)$.

Proof. We have $S_G(x) = \Sigma_{j=0}^{\Delta(G)} a_j x^j$.

Now, let $a_1, a_2, \ldots a_m$ be the degrees of the vertices of G_1 and $b_1, b_2, \ldots b_n$ be that of G_2, where a_i's and b_j's need not be all distinct.

Then the degree of vertices of $G_1 \times G_2$ can be observed as,

$a_1 + b_1, a_1 + b_2, \ldots a_1 + b_n, a_2 + b_1, a_2 + b_2, \ldots a_2 + b_n, \ldots, a_m + b_1, a_m + b_2, \ldots a_m + b_n$.

Now,

$$S_{G_1}(x) = x^{a_1} + x^{a_2} + \ldots + x^{a_m}$$

and

$$S_{G_2}(x) = x^{b_1} + x^{b_2} + \ldots + x^{b_n}$$

But for $G = G_1 \times G_2$,

$$\begin{aligned} S_G(x) &= x^{a_1+b_1} + x^{a_1+b_2} + \ldots + x^{a_1+b_n} + \ldots + x^{a_m+b_1} + x^{a_m+b_2} + \ldots + x^{a_m+b_n} \\ &= x^{a_1}(x^{b_1} + x^{b_2} + \ldots x^{b_n}) + \ldots + x^{a_m}(x^{b_1} + x^{b_2} + \ldots x^{b_n}) \\ &= (x^{a_1} + x^{a_2} + \ldots x^{a_m})(x^{b_1} + x^{b_2} + \ldots x^{b_n}) \\ &= S_{G_1}(x) \times S_{G_2}(x) \end{aligned}$$

Example

In figure 4, $S_{G_1}(x) = x + 2x + x^3$, $S_{G_2}(x) = 2x + x^2$ and $S_G(x) = 2x^2 + 5x^3 + 4x^4 + x^5$.

$$\begin{aligned} S_{G_1}(x) \times S_{G_2}(x) &= (x + 2x + x^3)(2x + x^2) \\ &= 2x^2 + 5x^3 + 4x^4 + x^5 \\ &= S_G(x). \end{aligned}$$

Lemma 2.8 Vertex polynomial of mesh $P_m \wedge P_n$ is

$$4x + 2(m + n - 4)\, x^2 + (m - 2)\,(n - 2)\, x^4.$$

Proof. Let $G = P_m \wedge P_n$. As per the definition of conjunction, $|V_G| = mn$ and two vertices $u = (u_i, v_j)$, $v = (u_k, v_l)$ are adjacent if u_i adjacent to u_k in P_m and v_j adjacent to v_l in P_n, which in turn implies that the vertex degree of $u = (u_i, v_j)$ corresponds to the product of the adjacencies of vertices u_i in P_m and v_j in P_n. Since in P_m and P_n, the only vertex degrees are 1 and 2, in the resultant graph, vertex degrees must be 1 (= 1×1), 2(=1×2) and 4(=2×2).

In P_m and P_n, respectively $(m - 2)$ and $(n - 2)$ vertices are of degree 2. Also there are 2 vertices are of degree 1 in both P_m and P_n. Hence in $P_m \wedge P_n$, 4 vertices are of degree 1, $2(m + n–4)$ vertices are of degree 2 and $(m - 2)\,(n - 2)$ vertices are of degree 4.

Therefore, $S_G(x) = 4x + 2\,(m + n - 4)x^2 + (m - 2)\,(n - 2)\, x^4$.

Corollary 2.9 Vertex polynomial of $P_m \wedge K_2$ is $4x + 2(m - 2)\, x^2$.

Proof. Putting $n = 2$ in lemma 2.8, the result follows.

Observation 2.10 Vertex polynomial of $C_m \wedge C_n$ is mnx^4.

Proof. $G = C_m \wedge C_n$ has mn vertices, each vertex of C_m and C_n is of degree 2. Following the definition of conjunction, each vertex of G has degree 4(=2 × 2).

Hence $S_G(x) = mnx^4$.

Observation 2.11 Vertex polynomial of $P_m \wedge C_n$ is $2nx^2 + (m - 2)nx^4$.

Proof. $G = P_m \wedge C_n$, has mn vertices, with 2 vertices of P_m of degree 1 and $(m - 2)$ vertices of P_m as well as n vertices of C_n are of degree 2. Hence in $G = P_m \wedge C_n$, $2n$ vertices are of degree 2 and $(m - 2)n$ vertices are of degree 4.

Hence, $S_G(x) = 2nx^2 + (m - 2)nx^4$.

Corollary 2.12 Vertex polynomial of $K_2 \wedge C_n$ is $2nx^2$.

Proof. Directly follows from lemma 2.11, by considering $m = 2$.

Theorem 2.13 If $G = G_1 \wedge G_2$. Then $deg(S_G(x)) = \Delta(G_1).\ \Delta(G_2)$.

Proof. Let d_i' s be the vertex degrees of G_1 and f_i' s be that of G_2, where all d_i's and f_i's need not be distinct. From the definition of conjunction, for any vertex $u = (u_i, v_j)$

$$d(u) = d(u_i).\ d(v_j) \text{ and}$$

u has maximum degree in G, if $d(u_i) = \Delta(G_1)$ and $d(v_j) = \Delta(G_2)$.

Also, from the definition of $S_G(x)$, the degree of $S_G(x)$ corresponds to the maximum degree of the vertex of the graph G, considered.

Hence deg (*SG* (x))=?(G_1).?(G_2)

Theorem 2.14

Let $G = G_1 \wedge G_2$, where G_1 and G_2 are graphs with orders *n* and *m* respectively. Then $S_G(x) = \Sigma_{j=1}^{m} \Sigma_{i=1}^{n} x^{d_i f_j}$; where $d_i's$ and $f_j's$ corresponds to the vertex degrees of G_1 and G_2 respectively.

Proof: Let the vertex degrees of G_1 and G_2 be $d_1, d_2, d_3 \ldots d_n$ and $f_1, f_2, f_3 \ldots f_m$ respectively (where all $d_i's$ and $f_j's$ need not be distinct).

Then vertex degrees of mn vertices of $G = G_1 \wedge G_2$, are as:

$d_1 f_1, d_1 f_2, \ldots d_1 f_m, d_2 f_1, d_2 f_2, \ldots d_2 f_m, \ldots d_n f_1, d_n f_2, \ldots d_n f_m$

For any graph *G* we have $S_G(x) = \Sigma_{j=0}^{\Delta(G)} a_j x^j$.

Thus $S_G(x) = x^{d_1 f_1} + x^{d_1 f_2} + \ldots + x^{d_2 f_1} + \ldots + x^{d_2 f_m} + \ldots + x^{d_n f_1} + \ldots + x^{d_n f_m}$.

$$= \Sigma_{j=1}^{m} \Sigma_{i=1}^{n} x^{d_i f_j}$$

Hence the theorem.

Corollary 2.15 If G_1 is an *r* – regular graph of order *n* and G_2 is *k*–regular of order *m*. If $G = G_1 \wedge G_2$ then, $S_G(x) = mnx^{rk}$.

Proof. Follows from theorem 2.14.

Observation 2.16 The vertex polynomial of a Wheel is $nx^3 + x^n$

Proof. A Wheel W_n is obtained by the join of K_1 and C_n.

Consider the wheel $W_6 = K_1 \vee C_6$,

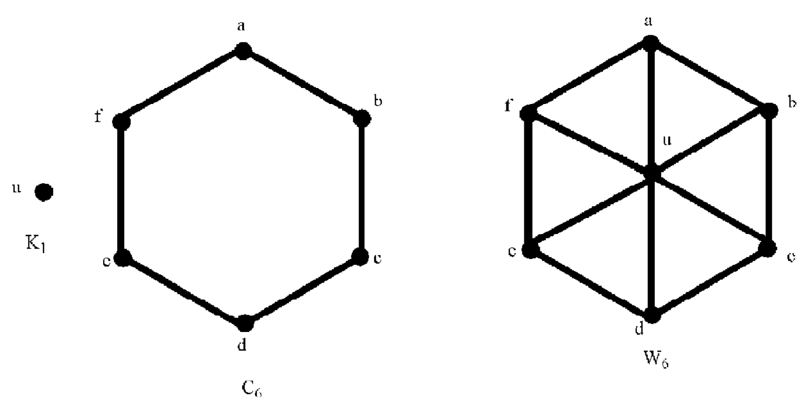

Figure 5: Wheel (W_6)

From the figure it is clear that,

$$S_{W_6}(x) = 6x^3 + x^6.$$

In general, $K_1 \vee C_n$ consists of n vertices of degree 3 and one vertex of degree n.

Therefore, if $W_n = K_1 \vee C_n$,

$$S_{W_n}(x) = nx^3 + x^n.$$

Observation 2.17 Vertex polynomial of a Fan graph is $2x^2 + (n - 2)\, x^3 + x^n$.

Proof. $K_1 \vee P_n$ is considered as the fan F_n. F_n consists of P_n along with all edges joining every vertex of P_n to K_1.

Consider $F_5 = K_1 \vee P5$

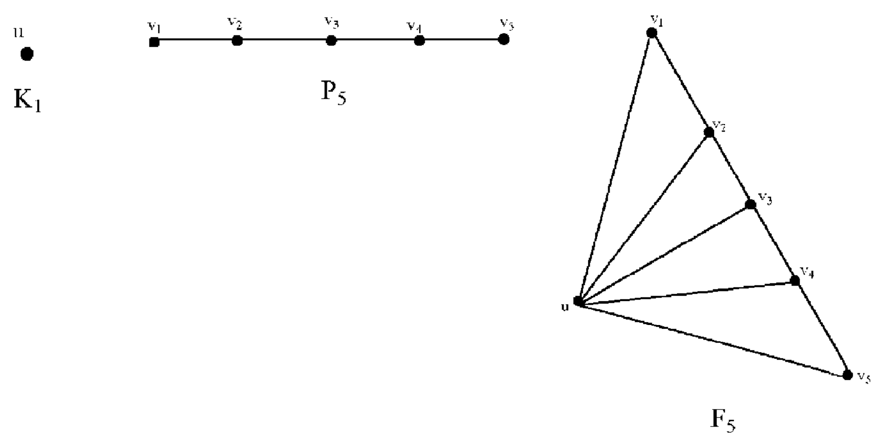

Figure 6: Fan *F*$_5$

F_5 has 2 vertices of degree 2, 3 vertices of degree 3 and 1 vertex of degree 5.

So $S_{F_5}(x) = 2x^2 + 3x^3 + x^5$.

In P_n, $(n - 2)$ vertices are of degree 2 and two vertices are of degree 1. Hence in $K_1 \vee P_n$, there exists $(n - 2)$ vertices of degree 3, two vertices of degree 2 and a single vertex of degree n.

Thus, for $F_n = K_1 \vee P_n$,

$$S_{F_n}(x) = 2x^2 + (n-2)x^3 + x^n$$

Theorem 2.18 Let $G = G_1 \vee G_2$, where G_1 and G_2 are of order m and n respectively. Then, $deg(S_G(x)) = m + n - 1$.

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Proof. Since order of $G_1 = n$ and order of $G_2 = m$, the maximum degree of a vertex in G_1 is $(n - 1)$ and that of G_2 is $(m - 1)$. Let $d(u_i) = n - 1$ and, $d(v_j) = m-1$. In $G = G_1 \vee G_2$, each vertex u_i of G_1 is joined to every vertex v_j of G_2, in addition to the edges of G_1 and G_2. Therefore, $d_G(u_i) = n-1+m$, $[d_G(u_i)$ represents the degree of vertex u_i in $G = G_1 \vee G_2]$. Similarly, $d_G(v_j) = m - 1 + n$.

That is, maximum degree of a vertex in G is $m + n - 1$

Hence, $deg(S_G(x)) = m + n - 1$.

Theorem 2.19 Vertex polynomial of the symmetric difference $(\oplus)$ [1], of any graph G_1 of order n and K_2 is $2nx^n$.

Proof: Let G_1 is a graph of order n and G_2 of order m. In $G = G_1 \oplus G_2$, two vertices $u = (u_i, v_j)$ and $v = (u_k, v_l)$ are adjacent if either u_i adjacent to u_k in G_1 or v_j adjacent to v_l in G_2, but not the both. Hence the degree of a vertex $u = (u_i, v_j)$ in G is,

$$d(u) = d(u_i)\,(m - d(v_j)) + d(v_j)\,(n - d(u_i))$$
$$= m\,d(u_i) + n\,d(v_j) - 2d(u_i)\,d(v_j).$$

Consider the graph drawn below,

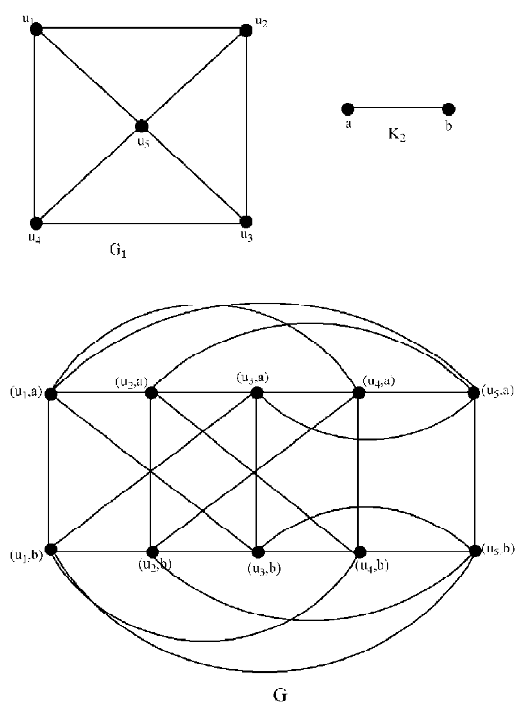

Figure 6: Symmetric difference of G_1 and K_2

Here G_1 is of order 5, and the vertex polynomial of $G = G_1 \oplus K_2$ is $S_G(x) = 10x^5$.

In general, if G_1 is any arbitrary graph of order n and G_2 is K_2, then in G, $|V| = 2n$. Let $d_1, d_2, d_3 \ldots d_n$ be the degrees of the vertices of G_1. Then the degrees of $2n$ vertices of G must be n [say for vertex with degree d_1 in G_1 and for either of the two vertices of K_2, $(d_1 \times 2) + (1 \times n) - 2 \times d_1 \times 1 = n$].

Thus $S_G(x) = 2nx^n$.

Acknowledgement

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