Communications on Stochastic Analysis
Vol. 12, No. 2 (2018) 197-213

Serials Publications
www.serialspublications.com

A STOCHASTIC INTEGRAL BY A NEAR-MARTINGALE

SHINYA HIBINO, HUI-HSIUNG KUO, AND KIMIAKI SAITÔ*

Abstract. In this paper we discuss the new stochastic integral in [1] in terms of the Itô isometry. We prove the Doob-Meyer decomposition theorem for near-submartingales in the classes (D) and (DL). Moreover, we introduce a stochastic integral by a near-martingale as an application of the decomposition theorem.

1. Introduction

A new stochastic integral was introduced in [1]. The Itô isometry based on the new integral for special processes was discussed in [10]. The Doob-Meyer decomposition theorem for continuous near-submartingales was also discussed in [3]. This stochastic integral has been studied from different points of view [2, 4, 7, 8, 9] and references cited therein.

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)_{a\leq t\leq b}$ be a basic probability space with a filtration $\{\mathcal{F}_t\}_{a\leq t\leq b}$, and $B = \{B(t);\ a \leq t \leq b\}$ a $\{\mathcal{F}_t\}$-Brownian motion on $(\Omega, \mathcal{F}, P)$. A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be instantly independent of $\{\mathcal{F}_t\}$ if $g(t)$ is independent of $\mathcal{F}_t$ for all $t \in [a, b]$. A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be in $L^2_{\text{ind}}([a, b] \times \Omega)$ if the process satisfies the following conditions:

(1) $g = \{g(t); a \leq t \leq b\}$ is instantly independent of $\{\mathcal{F}_t\}$.
(2) $\int_a^b E[|g(t)|^2]dt < \infty$.
(3) g is right-continuous in t.

A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be in $\mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$ if the process satisfies the following conditions:

(1) $g = \{g(t); a \leq t \leq b\}$ is instantly independent of $\{\mathcal{F}_t\}$.
(2) $\int_a^b |g(t)|^2 dt < \infty$, a. e.

In this article we discuss the new stochastic integral through the Itô isometry. In Section 2 we discuss the stochastic integral by the Brownian motion B for processes in $L^2_{\text{ind}}([a, b] \times \Omega)$ through the Itô isometry with its properties. In Section 3 we extend the stochastic integral to that on a class $\mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$ which is larger than the space in Section 2. In Section 4 we give the proof of the Doob-Meyer decomposition theorem for near-submartingales in the classes (D) and (DL). This theorem is important to discuss the new integral in [1] for its extension. In the last

---

Received 2018-8-9; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60H05; Secondary 60H20.
Key words and phrases. Near-martingale, near-submartingale, stochastic integral, the Doob-Meyer decomposition.
* Corresponding author.

section we introduce a stochastic integral by a near-martingale as an application of the decomposition theorem. This is a formulation of the new integral in [1] from the point of view of the stochastic integral by the near-martingale.

2. Stochastic Integrals on $L^2_{\rm ind}([a,b]\times\Omega)$

Let g be in $L^2_{\rm ind}([a,b]\times\Omega)$. Then g is called to be an instantly independent step process if there exist a partition $a=t_0<t_1<\cdots<t_n=b$ and instantly independent random variables $\eta_i,\ i=1,2,\ldots,n$ with $E[\eta_i^2]<\infty$ such that

$$g(t,\omega)=\sum_{i=1}^{n}\eta_i(\omega)1_{[t_{i-1},t_i)}(t),\quad \omega\in\Omega, t\in[a,b]. \tag{2.1}$$

We denote the set of all instantly independent step processes by $\mathrm{Step}_{\rm ind}([a,b]\times\Omega)$.

For any $g\in\mathrm{Step}_{\rm ind}([a,b]\times\Omega)$ given as (2.1), we define $\mathcal{J}(g)$ by

$$\mathcal{J}(g):=\sum_{i=1}^{n}\eta_i(B(t_i)-B(t_{i-1})).$$

Then we have the following.

Lemma 2.1. *For any $g,h\in\mathrm{Step}_{\rm ind}([a,b]\times\Omega)$ and $a,b\in\mathbb{R}$, it holds that*

$$\mathcal{J}(ag+bh)=aJ(g)+bJ(h).$$

Lemma 2.2. *For any $g\in\mathrm{Step}_{\rm ind}([a,b]\times\Omega)$, the following equalities hold.*

(1) $E[\mathcal{J}(g)]=0$.

(2) $E[|\mathcal{J}(g)|^2]=\int_a^b E[|g(t)|^2]dt$.

Proof. Let g be a function in $\mathrm{Step}_{\rm ind}([a,b]\times\Omega)$ given as (2.1).

(1): Since, for any $1\le i\le n$, η_i is independent to $B(t_i)-B(t_{i-1})$, we have

$$E[\eta_i(B(t_i)-B(t_{i-1}))]=E[\eta_i]E[B(t_i)-B(t_{i-1})]=0.$$

Therefore, $E[\mathcal{J}(g)]=0$.

(2): If $i<j$, we have

$$\begin{aligned}&E[\eta_i\eta_j(B(t_i)-B(t_{i-1}))(B(t_j)-B(t_{j-1}))]\\&=E[(B(t_i)-B(t_{i-1}))]E[\eta_i\eta_j(B(t_j)-B(t_{j-1}))]=0.\end{aligned}$$

If $i=j$, we have

$$\begin{aligned}E[\eta_i^2(B(t_i)-B(t_{i-1}))^2]&=E[(B(t_i)-B(t_{i-1}))^2]E[\eta_i^2]\\&=(t_i-t_{i-1})E[\eta_i^2].\end{aligned}$$

Therefore, we obtain

$$E[|\mathcal{J}(g)|^2]=\sum_{i,j=1}^{n}E[\eta_i\eta_j(B(t_i)-B(t_{i-1}))(B(t_j)-B(t_{j-1}))]=\int_a^b E[|g(t)|^2]dt.$$

□

Lemma 2.3. *For any* $g \in L^2_{\mathrm{ind}}([a,b]\times\Omega)$, *there exists* $\{g_n\}_{n=1}^{\infty} \subset \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$ *such that*

$$\lim_{n\to\infty} \int_a^b E[|g(t)-g_n(t)|^2]dt = 0$$

holds.

Let $g \in L^2_{\mathrm{ind}}([a,b]\times\Omega)$. Then there exists $\{g_n\}_{n=1}^{\infty} \subset \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty} \int_a^b E[|g(t)-g_n(t)|^2]dt = 0.$$

By Lemmas 2.1, 2.2 and 2.3, we have

$$E[|\mathcal{J}(g_n)-\mathcal{J}(g_m)|^2] = \int_a^b E[|g_n(t)-g_m(t)|^2]dt \xrightarrow[n,m\to\infty]{} 0.$$

Therefore, $\{\mathcal{J}(g_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. By the completeness of $L^2(\Omega)$, there exists $\mathcal{J}(g) \in L^2(\Omega)$ such that

$$\mathcal{J}(g) = \lim_{n\to\infty} \mathcal{J}(g_n), \quad \text{in } L^2(\Omega).$$

Thus we can define the stochastic integral $\int_a^b g(t)dB(t)$ by

$$\int_a^b g(t)dB(t) := \mathcal{J}(g)$$

as an element of $L^2(\Omega)$. This is well-defined. In fact, assume that there exist $\{g_n(t)\}_{n=0}^{\infty}, \{h_n(t)\}_{n=0}^{\infty} \subset \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty} \int_a^b E[|g(t)-g_n(t)|^2]dt = 0, \lim_{n\to\infty} \int_a^b E[|g(t)-h_n(t)|^2]dt = 0.$$

Then we can see that

$$\begin{aligned} E[|\mathcal{J}(g_n)-\mathcal{J}(h_n)|^2] &= \int_a^b E[|g_n(t)-h_n(t)|^2]dt \\ &= \int_a^b E[|g(t)-g_n(t)|^2]dt + \int_a^b E[|g(t)-h_n(t)|^2]dt \\ &\xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} \mathcal{J}(g_n) = \lim\limits_{n\to\infty} \mathcal{J}(h_n)$ in $L^2(\Omega)$.

Theorem 2.4. *For any* $g \in L^2_{\mathrm{ind}}([a,b]\times\Omega)$, $\mathcal{J}(g)$ *has the following properties:*

(1) $E[\mathcal{J}(g)] = 0$.

(2) $E[|\mathcal{J}(g)|^2] = \int_a^b E[|g(t)|^2]dt$.

Proof. (1) follows from $E[\mathcal{J}(g)] = \lim\limits_{n\to\infty} E[\mathcal{J}(g_n)] = 0$. (2) follows from

$$E[|\mathcal{J}(g)|^2] = \lim_{n\to\infty} E[|\mathcal{J}(g_n)|^2] = \lim_{n\to\infty} \int_a^b E[|g_n(t)|^2]dt = \int_a^b E[|g(t)|^2]dt.$$

□

Corollary 2.5. *For any* $g, h \in L^2_{\mathrm{ind}}([a,b]\times\Omega)$, *the equality*

$$E\left[\int_a^b g(t)dB(t)\int_a^b h(t)dB(t)\right] = \int_a^b E[g(t)h(t)]dt$$

holds.

Proof. By Theorem 2.4, we have

$$E\left[\Big|\int_a^b g(t)dB(t) + \int_a^b h(t)dB(t)\Big|^2\right] = \int_a^b E[|g(t)+h(t)|^2]dt.$$

Then we can see that

$$\begin{aligned}
&E\left[\Big|\int_a^b g(t)dB(t) + \int_a^b h(t)dB(t)\Big|^2\right] \\
&= E\left[\left(\int_a^b g(t)dB(t)\right)^2\right. \\
&\quad \left. +2\left(\int_a^b g(t)dB(t)\right)\left(\int_a^b h(t)dB(t)\right) + \left(\int_a^b h(t)dB(t)\right)^2\right] \\
&= \int_a^b E[|g(t)|^2]dB(t) \\
&\quad + 2E\left[\int_a^b g(t)dB(t)\int_a^b h(t)dB(t)\right] + \int_a^b E[|h(t)|^2]dB(t).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&\int_a^b E[|g(t)+h(t)|^2]dt \\
&= \int_a^b E[|g(t)|^2]dB(t) + 2\int_a^b E[g(t)h(t)]dt + \int_a^b E[|h(t)|^2]dB(t).
\end{aligned}$$

Consequently, we obtain

$$E\left[\int_a^b g(t)dB(t)\int_a^b h(t)dB(t)\right] = \int_a^b E[f(t)g(t)]dt.$$

□

Example 2.6. For any $g \in L^2_{\mathrm{ind}}([a,b]\times\Omega)$, the stochastic process

$$\left\{\int_t^b g(s)dB(s);\ a \le t \le b\right\}$$

is an instantly independent process of $\{\mathcal{F}_t\}$.

3. Stochastic Integrals on $\mathcal{L}_{\mathrm{ind}}(\Omega, L^2[a,b])$

Lemma 3.1. *For any $g \in \mathcal{L}_{\mathrm{ind}}(\Omega, L^2[a,b])$, there exists a sequence $\{g_n\}_{n=0}^{\infty} \subset L^2_{\mathrm{ind}}([a,b]\times\Omega)$ such that*

$$\lim_{n\to\infty}\int_a^b |g_n(t)-g(t)|^2dt = 0, \quad \text{a. e.}$$

Proof. For any $n \in \mathbb{N}$, we set

$$g_n(t,\omega) = \begin{cases} g(t,\omega), & \int_t^b |g(s,\omega)|^2 ds \le n, \\ 0, & \int_t^b |g(s,\omega)|^2 ds > n. \end{cases}$$

Then $\{g_n(t);\ a \le t \le b\}$ is instantly independent of $\{\mathcal{F}_t\}$ and

$$\int_a^b |g_n(t,\omega)|^2 dt = \int_{\tau_n(\omega)}^b |g(t,\omega)|^2 dt, \quad \text{a. e. } \omega$$

holds, where $\tau_n(\omega) = \inf\left\{t; \int_t^b |g(s,\omega)|^2 ds \le n\right\}$. Therefore, we have

$$\int_a^b |g_n(t)|^2 dt \le n, \quad \text{a. e. } \omega.$$

Since $\int_a^b E[|g_n(t)|^2]dt \le n$ and $g \in \mathcal{L}_{\mathrm{ind}}(\Omega, L^2[a,b])$, it holds that

$$\int_a^b |g(t,\omega)|^2 dt \le n, \quad \text{a. e. } \omega \in \Omega$$

for a large n. Then we have $g(t,\omega) = g_n(t,\omega)$ for all $t \in [a,b]$. Consequently, we obtain

$$\lim_{n\to\infty}\int_a^b |g_n(t,\omega)-g(t,\omega)|^2 dt = 0, \quad \text{a. e. } \omega.$$

□

Lemma 3.2. *Let $g \in \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$. Then for any $\epsilon > 0$, there exists $c > 0$ such that*

$$P\left(\left|\int_a^b g(t)dB(t)\right| > \epsilon\right) \le \frac{c}{\epsilon^2} + P\left(\int_a^b |g(t)|^2 dt > c\right).$$

Proof. For any $c > 0$, we define $g_c(t,\omega)$ by

$$g_c(t,\omega) = \begin{cases} g(t,\omega), & \int_t^b |g(s,\omega)|^2 ds \le c, \\ 0, & \int_t^b |g(s,\omega)|^2 ds > c. \end{cases}$$

Since

$$\begin{aligned} &\left\{\left|\int_a^b g(t)dB(t)\right| > \varepsilon\right\} \\ &\subset \left\{\left|\int_a^b g_c(t)dB(t)\right| > \varepsilon\right\} \cup \left\{\int_a^b g(t)dB(t) \ne \int_a^b g_c(t)dB(t)\right\}, \end{aligned}$$

for any $\epsilon > 0$ and $c > 0$, we have

$$\begin{aligned} &P\left(\left|\int_a^b g(t)dB(t)\right| > \varepsilon\right) \\ &\leq P\left(\left|\int_a^b g_c(t)dB(t)\right| > \varepsilon\right) + P\left(\int_a^b g(t)dB(t) \neq \int_a^b g_c(t)dB(t)\right). \end{aligned}$$

Then since $g \in \mathrm{Step_{ind}}([a,b]\times\Omega)$, we have

$$\left\{\int_a^b g(t)dB(t) \neq \int_a^b g_c(t)dB(t)\right\} \subset \left\{\int_a^b |g(t)|^2 dt > c\right\}$$

Therefore,

$$P\left(\left|\int_a^b g(t)dB(t)\right| > \varepsilon\right) \leq P\left(\left|\int_a^b g_c(t)dB(t)\right| > \varepsilon\right) + P\left(\int_a^b |g(t)|^2 dt > c\right).$$

By the Chebyshev inequality, we obtain

$$\begin{aligned} &P\left(\left|\int_a^b g(t)dB(t)\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} E\left[\left|\int_a^b g_c(t)dB(t)\right|^2\right] + P\left(\int_a^b |g(t)|^2 dt > c\right) \\ &= \frac{1}{\varepsilon^2}\int_a^b E[|g_c(t)|^2]dt + P\left(\int_a^b |g(t)|^2 dt > c\right) \\ &\leq \frac{c}{\varepsilon^2} + P\left(\int_a^b |g(t)|^2 dt > c\right). \end{aligned}$$

□

Lemma 3.3. *For any* $g \in \mathcal{L}_{\mathrm{ind}}(\Omega, L^2[a,b])$*, there exists a sequence* $\{g_n\}_{n=1}^\infty \subset \mathrm{Step_{ind}}([a,b]\times\Omega)$ *such that*

$$\lim_{n\to\infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}$$

Proof. By Lemma 3.1, for any $g \in \mathcal{L}_{\mathrm{ind}}(\Omega, L^2[a,b])$, we can take $\{h_n\}_{n=1}^\infty \subset L^2_{\mathrm{ind}}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty} \int_a^b |h_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}$$

For any n, applying Lemma 2.3 to h_n, there exists $\{g_n\}_{n=1}^\infty \subset \mathrm{Step_{ind}}([a,b]\times\Omega)$ such that

$$E\left[\int_a^b |g_n(t) - h_n(t)|^2 dt\right] < \frac{1}{n}.$$

Then we have

$$\left\{\int_a^b |g_n(t)-g(t)|^2dt > \varepsilon\right\}$$
$$\subset \left\{\int_a^b |g_n(t)-h_n(t)|^2dt > \frac{\varepsilon}{4}\right\} \cup \left\{\int_a^b |h_n(t)-g(t)|^2dt > \frac{\varepsilon}{4}\right\}$$

for all $\varepsilon > 0$. Hence, for all $\varepsilon > 0$,

$$P\left(\int_a^b |g_n(t)-g(t)|^2dt > \varepsilon\right)$$
$$\leq P\left(\int_a^b |g_n(t)-h_n(t)|^2dt > \frac{\varepsilon}{4}\right) + P\left(\int_a^b |h_n(t)-g(t)|^2dt > \frac{\varepsilon}{4}\right).$$

Therefore, by the Chebyshev inequality,

$$P\left(\int_a^b |g_n(t)-g(t)|^2dt > \varepsilon\right)$$
$$\leq \frac{4}{\varepsilon}E\left[\int_a^b |g_n(t)-h_n(t)|^2dt\right] + P\left(\int_a^b |h_n(t)-g(t)|^2dt > \frac{\varepsilon}{4}\right)$$
$$\leq \frac{4}{n\varepsilon} + P\left(\int_a^b |h_n(t)-g(t)|^2dt > \frac{\varepsilon}{4}\right)$$

for all $\varepsilon > 0$. Consequently, we obtain

$$\lim_{n\to\infty} P\left(\int_a^b |g_n(t)-g(t)|^2dt > \varepsilon\right) = 0$$

for all $\varepsilon > 0$. This means the assertion:

$$\lim_{n\to\infty} \int_a^b |g_n(t)-g(t)|^2dt = 0, \quad \text{in probability.}$$

□

By Lemma 3.3, for any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a,b])$, there exists $\{g_n\}_{n=1}^\infty \subset \text{Step}_{\text{ind}}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty} \int_a^b |g_n(t)-g(t)|^2dt = 0, \quad \text{in probability.}$$

Then by Lemma 3.2, for any $\varepsilon > 0$,

$$P(|\mathcal{J}(g_n)-\mathcal{J}(g_m)| > \varepsilon) \leq \frac{\varepsilon}{2} + P\left(\int_a^b |g_n(t)-g_m(t)|^2dt > \frac{\varepsilon^3}{2}\right).$$

Since

$$\left\{\int_a^b |g_n(t)-g_m(t)|^2 dt > \frac{\varepsilon^3}{2}\right\}$$
$$\subset \left\{\int_a^b |g_n(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right\} \cup \left\{\int_a^b |g_m(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right\},$$

we have

$$P\left(\int_a^b |g_n(t)-g_m(t)|^2 dt > \frac{\varepsilon^3}{2}\right)$$
$$\leq P\left(\int_a^b |g_n(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right) + P\left(\int_a^b |g_m(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right).$$

Hence, since there exists $N \in \mathbb{N}$ such that

$$P\left(\int_a^b |g_n(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right) < \frac{\varepsilon}{4}$$

for all $n \geq N$ by Lemma 3.3, it holds that

$$P\left(\int_a^b |g_n(t)-g_m(t)|^2 dt > \frac{\varepsilon^3}{2}\right) < \frac{\varepsilon}{2}$$

for all $n, m \geq N$. Consequently, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$P\left(|\mathcal{J}(g_n) - \mathcal{J}(g_m)| > \varepsilon\right) < \varepsilon$$

for all $n, m \geq N$. This implies that $\{\mathcal{J}(g_n)\}$ converges in probability. Thus we define the stochastic integral $\int_a^b g(t)dB(t)$ by

$$\int_a^b g(t)dB(t) = \lim_{n\to\infty} \mathcal{J}(g_n), \quad \text{in probability.}$$

This is well-defined. In fact, suppose that there exist sequences $\{g_n(t)\}_{n=0}^\infty$ and $\{h_n(t)\}_{n=0}^\infty \subset \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty} \int_a^b |g(t)-g_n(t)|^2 dt = 0, \lim_{n\to\infty} \int_a^b |g(t)-h_n(t)|^2 dt = 0 \quad \text{in probability.}$$

Then by Lemma 3.2, for any $\varepsilon > 0$, we have

$$P\left(|\mathcal{J}(g_n) - \mathcal{J}(h_n)| > \varepsilon\right) \leq \frac{\varepsilon}{2} + P\left(\int_a^b |g_n(t)-h_n(t)|^2 dt > \frac{\varepsilon^3}{2}\right).$$

Since

$$\left\{\int_a^b |g_n(t)-h_n(t)|^2 dt > \frac{\varepsilon^3}{2}\right\}$$
$$\subset \left\{\int_a^b |g_n(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right\} \cup \left\{\int_a^b |h_n(t)-g(t)|^2 dt > \frac{\varepsilon^3}{8}\right\},$$

we have

$$P\left(\int_a^b |g_n(t)-h_n(t)|^2dt > \frac{\varepsilon^3}{2}\right)$$
$$\leq P\left(\int_a^b |g_n(t)-g(t)|^2dt > \frac{\varepsilon^3}{8}\right) + P\left(\int_a^b |h_n(t)-g(t)|^2dt > \frac{\varepsilon^3}{8}\right).$$

By Lemma 3.3, for any $\epsilon > 0$, there exist $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

$$P\left(\int_a^b |g_n(t)-g(t)|^2dt > \frac{\varepsilon^3}{8}\right) < \frac{\varepsilon}{4}, \quad \text{for all } n \geq N_1,$$

and

$$P\left(\int_a^b |h_n(t)-g(t)|^2dt > \frac{\varepsilon^3}{8}\right) < \frac{\varepsilon}{4}, \quad \text{for all } n \geq N_2.$$

Therefore, putting $N = \max\{N_1, N_2\}$, we have

$$P\left(\int_a^b |g_n(t)-h_n(t)|^2dt > \frac{\varepsilon^3}{2}\right) < \frac{\varepsilon}{2}$$

for all $n, m \geq N$. Consequently,

$$P\left(|\mathcal{J}(g_n) - \mathcal{J}(h_n)| > \varepsilon\right) < \varepsilon$$

holds for all $n \geq N$. Thus, we obtain $\lim_{n\to\infty} \mathcal{J}(g_n) = \lim_{n\to\infty} \mathcal{J}(h_n)$ in probability.

4. The Doob-Meyer Deomposition by the Near-martingale

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)_{a\leq t\leq b}$ be a basic probability space with a filtration $\{\mathcal{F}_t\}_{a\leq t\leq b}$. A stochastic process $\{X(t); a \leq t \leq b\}$ is called to be the near-martingale with filtration $\{\mathcal{F}_t\}_{a\leq t\leq b}$ if it satisfies the following conditions:

(1) $E[|X(t)|] < \infty$ for all $a \leq t \leq b$,
(2) $E[X(t)|\mathcal{F}_s] = E[X(s)|\mathcal{F}_s]$ for all $s < t$.

If the condition

(3) $E[X(t)|\mathcal{F}_s] \geq E[X(s)|\mathcal{F}_s]$ for all $s < t$

holds instead of the condition (2), the stochastic process $\{X(t); a \leq t \leq b\}$ is called to be the near-submartingale with the filtration $\{\mathcal{F}_t\}_{a\leq t\leq b}$.

Theorem 4.1. ([5]) *Let $X = \{X(t);\ n \in \mathbb{N}\}$ be a near-submaritngale. Then, there exist a near-martingale $N = \{N(n);\ n \in \mathbb{N}\}$ and an increasing process $A = \{A(n); n \in \mathbb{N}\}$ such that*

$$X(n) = N(n) + A(n), n \in \mathbb{N},$$

where A is called to be the increasing process if it satisfies the following conditions:

(1) *$A(1) = 0$,*
(2) *for each $n \geq 2$, $A(n)$ is $\mathcal{F}_{n-1}$-measurable,*
(3) *for any $m \leq n$, $A(m) \leq A(n)$,* a. e.

Theorem 4.2. *Let $X(t) = \int_t^b g(s)dB(s)$ for any $a \leq t \leq b$ and $g \in L^2_{\text{ind}}([a,b]\times\Omega)$. Then the stochastic process $\{X(t); a \leq t \leq b\}$is a near-martingale with $\{\mathcal{F}_t\}_{a\leq t\leq b}$.*

Proof. Let $g \in \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$. Then g has the form

$$g(u,\omega)=\sum_{i=1}^{n}\eta_i(\omega)1_{[t_{i-1},t_i)}(u),\ \ s=t_0<t_1<\cdots<t_j=t<\cdots<t_n=b,$$

where $\eta_i, i=0,1,2,\cdots,n$, are random variables which independent to $\mathcal{F}_{t_i}$ safisfying $E[\eta_i^2]<\infty$. Then we obtain

$$\begin{aligned}
&E\left[\int_t^b g(u)dB(u)\Big|\mathcal{F}_s\right]\\
&=E\left[\sum_{i=j+1}^{n}\eta_i(B(t_i)-B(t_{i-1}))\Big|\mathcal{F}_s\right]\\
&=E\left[\sum_{i=j+1}^{n}\eta_i(B(t_i)-B(t_{i-1}))\Big|\mathcal{F}_s\right]+\sum_{i=1}^{j}E\left[\eta_i\right]E[(B(t_i)-B(t_{i-1}))]\\
&=E\left[\sum_{i=j+1}^{n}\eta_i(B(t_i)-B(t_{i-1}))\Big|\mathcal{F}_s\right]+E\left[\sum_{i=1}^{j}\eta_i(B(t_i)-B(t_{i-1}))\Big|\mathcal{F}_s\right]\\
&=E\left[\int_s^b g(u)dB(u)\Big|\mathcal{F}_s\right].
\end{aligned}$$

Next we prove the theorem in the case of $g\in L^2_{\mathrm{ind}}([a,b]\times\Omega)$. By Lemma 2.3, there exists $\{g_n\}_{n=1}^{\infty}\subset \mathrm{Step}_{\mathrm{ind}}([a,b]\times\Omega)$ such that $\lim_{n\to\infty}\int_a^b E[|g(t)-g_n(t)|^2]dt=0$. Let

$$X^{(n)}(t)=\int_t^b g_n(u)dB(u),\ \ n\in\mathbb{N}.$$

Then $\{X^{(n)}(t);\ a\le t\le b\}$ is a near-martingale for each $n\in\mathbb{N}$ from above argument. For any $s<t$, we have

$$E[X(t)-X(s)|\mathcal{F}_s]=E[X(t)-X^{(n)}(t)|\mathcal{F}_s]+E[X^{(n)}(s)-X(s)|\mathcal{F}_s].$$

Since

$$\begin{aligned}
E[|E[X(t)-X^{(n)}(t)|\mathcal{F}_s]|^2]&\le E[E[|X(t)-X^{(n)}(t)|^2|\mathcal{F}_s]]\\
&=E[|X(t)-X^{(n)}(t)|^2]\\
&=\int_t^b E[|g(u)-g_n(u)|^2]du\\
&\le\int_a^b E[|g(u)-g_n(u)|^2]du\xrightarrow[n\to\infty]{}0,
\end{aligned}$$

and by taking subsequence of $\{X^{(n)}(t)\}$, we get

$$E[X(t)-X^{(n)}(t)|\mathcal{F}_s]\xrightarrow[n\to\infty]{}0,\quad \text{a. e.}$$

Similarly, we have

$$E[X(s) - X^{(n)}(s)|\mathcal{F}_s] \xrightarrow[n\to\infty]{} 0, \quad \text{a. e.}$$

Consequently, we obtain

$$E[X(t) - X(s)|\mathcal{F}_s] = 0, \quad \text{a. e.}$$

This implies

$$E[X(t)|\mathcal{F}_s] = E[X(s)|\mathcal{F}_s], \quad \text{a. e.}$$

□

From now on, we assume that the submartingale and the near-submartingale are right-continuous. Let $\{\mathcal{F}_t; t \geq 0\}$ be a right-continuous filtration and set

$$\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t.$$

The Doob decomposition theorem for the near-submartingale is proved in [11]. In [3] the Doob-Meyer decomposition theorem is proved for the continuous near-submartingale. In this section we prove the Doob-Meyer decomposition theorem for the right-continuous near-submartingale.

Definition 4.3. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale (respectively, near-martingale). Suppose there exists an $\mathcal{F}_\infty$-measurable and integrable random variable $X(\infty)$ such that

$$E[X(t)|\mathcal{F}_t] \leq E[X(\infty)|\mathcal{F}_t], \quad (\text{respectively, } E[X(t)|\mathcal{F}_t] = E[X(\infty)|\mathcal{F}_t])$$

for all $t \in \mathbb{R}_+(\equiv [0,\infty))$. Then we call $\{X(t), t \in \overline{\mathbb{R}}_+(\equiv [0,\infty])\}$ a *closed near-submartingale* (respectively, *closed near-martingale*).

Definition 4.4. An $(\mathcal{F}_t)$-adapted right-continuous process $A = \{A(t); t \in \mathbb{R}_+\}$ is called an *increasing process* if $A(t)$ is an increasing function in t and $A(0) = 0$ almost surely.

Definition 4.5. An integrable increasing process A is called a *natural increasing process* if it satisfies the equality

$$E\left[\int_0^t X(s)dA(s)\right] = E\left[\int_0^t X(s-)dA(s)\right], \quad \forall t \in \mathbb{R}_+$$

for all bounded martingales X.

Let $X = \{X(\lambda); \lambda \in \Lambda\}$be a system of integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$. If X satisfies

$$\sup_{\lambda\in\Lambda} \int_{|X(\lambda)|>c} |X(\lambda)|dP \xrightarrow[c\to\infty]{} 0,$$

then X is called to be uniformly integrable. A near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ is called to have the Doob-Meyer decomposition if X is expressed in the form

$$X(t) = N(t) + A(t), \forall t \in \mathbb{R}_+$$

for some near-martingale N and natural increasing process A.

Lemma 4.6. *Let A, B be natual increasing processes. Then, if $A - B$ is a near-martingale, for any bounded $(\mathcal{F}_t)$-adapted process $f = \{f(t);\ t \geq 0\}$, the equality*

$$E\left[\int_0^t f(s)dA(s)\right] = E\left[\int_0^t f(s)dB(s)\right]$$

holds.

Proof. Let $N(t) = A(t) - B(t)$ for all $t \in \mathbb{R}_+$. Take a partition of $[0, t]$:

$$\delta := \{0 = t_0 < \cdots < t_n = t\}.$$

Then since N is a near-martingale, we get

$$\begin{aligned} &E\left[\sum_{k=1}^n f(t_{k-1})(N(t_k) - N(t_{k-1}))\right] \\ &= E\left[\sum_{k=1}^n E[f(t_{k-1})(N(t_k) - N(t_{k-1}))|\mathcal{F}_{t_{k-1}}]\right] \\ &= E\left[\sum_{k=1}^n f(t_{k-1})(E[N(t_k)|\mathcal{F}_{t_{k-1}}] - E[N(t_{k-1})|\mathcal{F}_{t_{k-1}}])\right] = 0. \end{aligned}$$

Therefore,

$$E\left[\sum_{k=1}^n f(t_{k-1})(A(t_k) - A(t_{k-1}))\right] = E\left[\sum_{k=1}^n f(t_{k-1})(B(t_k) - B(t_{k-1}))\right]$$

holds. Here, setting $f^\delta(s) = f(t_k), t_k < s \leq t_{k+1};\ k = 0, 1, \ldots, n-1$, we have

$$E\left[\int_0^t f^\delta(s)dA(s)\right] = E\left[\int_0^t f^\delta(s)dB(s)\right].$$

Consequently, by $|\delta| \to 0$ and the left-continuity, we obtain

$$E\left[\int_0^t f(s)dA(s)\right] = E\left[\int_0^t f(s)dB(s)\right].$$

□

Lemma 4.7. (cf.[5]) *Let A be an integrable increasing process. Then A is natural if and only if*

$$E[X(t)A(t)] = E\left[\int_0^t X(s-)dA(s)\right]$$

holds for any bounded martingale X.

Theorem 4.8. *The Doob-Meyer decomposition of a near-submartingale is uniquely determined if it exists.*

Proof. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale. Suppose that both of $X = M + A$ and $X = N + B$ are the Doob-Meyer decompositions. Then since $A - B$ is a near-martingale and by Lemma 4.6, for any bounded martingale $\{Y(t); t \in \mathbb{R}_+\}$, we have

$$E\left[\int_0^t Y(s-)dA(s)\right] = E\left[\int_0^t Y(s-)dB(s)\right].$$

Since A, B is natural increasing and by Lemma 4.7, we have

$$E[Y(t)A(t)] = E[Y(t)B(t)].$$

For any bounded random variable Y, we define $\mathbf{Y} = \{Y(t); t \in \mathbb{R}_+\}$ by $Y(t) := E[Y|\mathcal{F}_t]$ for all $t \in \mathbb{R}_+$. Then, $\mathbf{Y}$ is a $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-martingale, and therefore, we have

$$\begin{aligned} E[YA(t)] &= E[E[YA(t)|\mathcal{F}_t]] = E[Y(t)A(t)] \\ &= E[Y(t)B(t)] = E[E[YB(t)|\mathcal{F}_t]] = E[YB(t)]. \end{aligned}$$

Consequently, putting $Y = 1_\Lambda$ for all $\Lambda \in \mathcal{F}$, we obtain $P(A(t) = B(t)) = 1$ for each $t \in \mathbb{R}_+$. This implies

$$P(\forall t \in \mathbb{R}_+; A(t) = B(t)) = 1$$

by the right-continuity of $A(t)$ and $B(t)$. □

Let $\mathcal{T}$ be the set of stopping times and set $\mathcal{T}_a := \{\tau \in \mathcal{T}; \tau(\omega) \le a, \quad \forall\omega \in \Omega\}$. A closed near-submartingale $X = \{X(t), t \in \overline{\mathbb{R}}_+\}$ is called to be in the class (D) if $X(\tau)$ is uniformly integrable for any $\tau \in \mathcal{T}$. A near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ is called to be in the class (DL) if $X(\tau)$ is uniformly integrable for any $a > 0$ and $\tau \in \mathcal{T}_a$.

Lemma 4.9. (cf. [5]) *$\{A^n_\infty; n \in \mathbb{N}\}$ is uniformly integrable.*

Theorem 4.10. *Let X be a near-submartingle in the class (DL). If $X(t) \to X(\infty)$* a. e. *and there exists an integrable random variable Y such that $|X_t| \le Y$ for all $t \ge 0$, then X has the Doob-Meyer decomposition $X = N + A$. Moreover, if X is in the class (D), then N and A in the decomposition of X are uniformly integrable.*

Proof. It is enough to prove the theorem in the case of a near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ in the class (D). Let $Y(t)$ be $Y(t) = X(t) - E[X(\infty)|\mathcal{F}_t]$ for all $t \in \mathbb{R}_+$. Then, $\{Y(t), t \in \mathbb{R}_+\}$ is a near-submartingale, and hence $\lim_{t\to\infty} Y(t) = 0$, a. e. Let $\{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale satisfying $\lim_{t\to\infty} X(t) = 0$, a. e. Take a sequence $\delta_n = \{t_j^{(n)} = \frac{j}{2^n}, j \in \mathbb{N}\}$, $n = 1, 2, 3, \dots$ of partitions of $[0, \infty)$. For an arbitrarily fixed δ_n, we denote $t_j^{(n)}$by t_j simply. For each n, we define an increasing process $A^n(t), t \in \delta_n$ by

$$A^n(t_k) = \sum_{i=1}^{k-1} \{E[X(t_{j+1})|\mathcal{F}_{t_j}] - E[X(t_j)|\mathcal{F}_{t_j}]\}, \quad t_j \in \delta_n.$$

Then by Lemma 4.9, $A^n(\infty)$ is uniformly integrable. Therefore, there exist some subsequence $A^{n_\ell}(\infty), \ell = 1, 2, \cdots$ and an integrable random variable $A(\infty)$ such that $A^{n_l}(\infty) \to A(\infty)$ in L^1. For any $t \in \mathbb{R}_+$, we define $A(t)$ by

$$A(t) = E[X(t)|\mathcal{F}_t] + E[A(\infty)|\mathcal{F}_t]. \tag{4.1}$$

Then A is a $(\mathcal{F}_t)$-adapted process. Since

$$\begin{aligned}E[A^{n_\ell}(\infty)|\mathcal{F}_0] &= \lim_{k\to\infty} E\left[\sum_{j=0}^{k-1}\big\{E[X(t_{j+1})|\mathcal{F}_{t_j}] - E[X(t_j)|\mathcal{F}_{t_j}]\big\}\Big|\mathcal{F}_0\right]\\ &= \lim_{k\to\infty}\big\{E[X(t_k)|\mathcal{F}_0] - E[X(0)|\mathcal{F}_0]\big\}\\ &= -E[X(0)|\mathcal{F}_0],\ t_k\in\delta_{n_l}\end{aligned}$$

for any $\ell = 1, 2, \cdots$, we have

$$A(0) = E[X(0)|\mathcal{F}_0] + \lim_{\ell\to\infty} E[A^{n_\ell}(\infty)|\mathcal{F}_0] = 0.$$

We next prove that A is a natural increasing process. Take s and t with $s < t$ in $\bigcup_n \delta_n$. Then since $s, t \in \delta_{n_\ell}$ for a large $n_\ell \in \mathbb{N}$, by Theorem 4.1, we have

$$E[X(s)|\mathcal{F}_s] + E[A^{n_\ell}(\infty)|\mathcal{F}_s] \le E[X(t)|\mathcal{F}_t] + E[A^{n_\ell}(\infty)|\mathcal{F}_t].$$

Taking $n_\ell \to \infty$, we get

$$E[X(s)|\mathcal{F}_s] + E[A(\infty)|\mathcal{F}_s] \le E[X(t)|\mathcal{F}_t] + E[A(\infty)|\mathcal{F}_t], \quad \text{a. e.}$$

Hence, $A(s) \le A(t)$. Since $\bigcup_n \delta_n$ is dense in $\mathbb{R}_+$, we obtain $A(s) \le A(t)$ for all $s < t$. This implies that A is an increasing process. For any bounded closed martingale Z, we can see that

$$\begin{aligned}E[Z(\infty)A^n(\infty)] &= \sum_k E[Z(\infty)(A^n(t_{k+1}) - A^n(t_k))]\\ &= \sum_k E[(A^n(t_{k+1}) - A^n(t_k))E[Z(\infty)|\mathcal{F}_{t_k}]]\\ &= \sum_k E[(A^n(t_{k+1}) - A^n(t_k))E[Z(t_k)|\mathcal{F}_{t_k}]]\\ &= \sum_k E[Z(t_k)(A^n(t_{k+1}) - A^n(t_k))],\ t_k\in\delta_n.\end{aligned}$$

On the other hand, since

$$E[A(t) - A(s)|\mathcal{F}_s] = E[X(t) - X(s)|\mathcal{F}_s]$$

by taking conditional expectations under $\mathcal{F}_s$ in (4.1), we have

$$\begin{aligned}&E[A(t_{k+1}) - A(t_k)|\mathcal{F}_{t_k}]\\ &= E[X(t_{k+1})|\mathcal{F}_{t_k}] - E[X(t_k)|\mathcal{F}_{t_k}]\\ &= A^n(t_{k+1}) - A^n(t_k).\end{aligned}$$

Therefore, it holds that

$$E[Z(\infty)A^n(\infty)] = \sum_k E[Z(t_k)(A(t_{k+1}) - A(t_k))].$$

Taking $n \to \infty$, we obtain

$$E[Z(\infty)A(\infty)] = E\left[\int_0^\infty Z(s-)dA(s)\right].$$

This implies that A is natural. Since

$$\begin{aligned} E[X(t)-A(t)|\mathcal{F}_s] &= E[E[X(t)-A(t)|\mathcal{F}_t]|\mathcal{F}_s] \\ &= E[-E[A(\infty)|\mathcal{F}_t]|\mathcal{F}_s] \\ &= -E[A(\infty)|\mathcal{F}_s] \\ &= E[X(s)-A(s)|\mathcal{F}_s], \end{aligned}$$

the near-maritingale part of X is given by $X - A$. □

5. A Stochastic Integral by a Near-martingale

Let $0 \le a < b$. Let $\mathcal{F}_{(t} := \sigma(B(b)-B(s);\ t < s \le b) \vee \mathcal{N}$ for any $t \in [a,b]$, and $C([a,b])$ the Banach space of all continuous functions on $[a,b]$ with norm $\|\cdot\|_\infty$ given by $\|f\|_\infty := \sup_{t\in[a,b]}|f(t)|,\ f \in C([a,b])$. Define $\mathcal{B}(C([a,b]))$ by the smallest σ-field including the family of open sets in $C(([a,b]))$, which is called the *topological Borel field*. Denote by P_W the Wiener measure on $\mathcal{B}(C([a,b]))$. For any $(\mathcal{F}_{(t})$-adapted process $g = \{g(t);\ a \le t \le b\}$ we consider

$$N(t) := \int_t^b g(u)dB(u),\ t \in [a,b]. \tag{5.1}$$

Then, g is an instantly independent process of $(\mathcal{F}_t)$ and $N = \{N(t);\ a \le t \le b\}$ is a near-martingale and also an instantly independent process of $(\mathcal{F}_t)$. Since $g(t)$ is $\mathcal{F}_{(t}$-measurable for any $t \in [a,b]$, then $g(t)$ can be expressed in the form

$$g(t) = G(B(b)-B(s);\ t < s \le b)$$

for some $\mathcal{B}(C([a,b]))$-measurable function G for any $t \in [a,b]$.

By Theorem 4.10, there exists a unique natural increasing process $A = \{A(t);\ a \le t \le b\}$ such that $-N^2 - A$ is a near-martingale. We denote A by $\langle N\rangle = \{\langle N\rangle(t);\ a \le t \le b\}$. Here, we have

$$E[(N(t)-N(s))^2|\mathcal{F}_s] = E[\langle N\rangle(t) - \langle N\rangle(s)|\mathcal{F}_s]$$

for any $s < t$. Let

$$\mathcal{L}^2(\langle N\rangle) := \left\{X;\ X \text{ is predictable and satisfies } E\left[\int_a^t |X(t)|^2 d\langle N\rangle(t)\right] < \infty\ \forall t\right\}.$$

For any X in $\mathcal{L}^2(\langle N\rangle)$, we define semi-norms $\|X\|_t(\langle N\rangle)$, $a \le t \le b$, by

$$\|X\|_t(\langle N\rangle) := E\left[\int_a^t |X|^2 d\langle N\rangle(t)\right]^{1/2}.$$

Then $\mathcal{L}^2(\langle N\rangle)$ is the complete metric space with semi-norms $\|X\|_t(\langle N\rangle)$, $a \le t \le b$.

For any $f \in C([a,b])$ and partition $\Delta : a = t_0 < t_1 < \cdots < t_n = b$, we put

$$f_\Delta = \sum_{k=1}^n f(B(t_{k-1}))1_{[t_{k-1},t_k)}$$

and define the stochastic integral $\int_a^b f_\Delta(B(t))dN(t)$ by

$$\int_a^b f_\Delta(B(t))dN(t) := \sum_{k=1}^n f(B(t_{k-1}))(N(t_k)-N(t_{k-1})), \quad \text{in } L^2(\Omega).$$

Then we have the following:

Proposition 5.1. *For any $f \in C([a,b])$ and partition*

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b,$$

the process $\int_a^{\cdot} f_\Delta dN$ is an L^2 near-martingale and satisfies

$$\left\langle \int_a^{\cdot} f_\Delta(B(\cdot))dN \right\rangle(t) = \int_a^t f_\Delta(B(t))d\langle N\rangle(t), \tag{5.2}$$

$$E\left[\left|\int_a^t f_\Delta(B(t))dN(t)\right|^2\right] = \|f_\Delta(B(\cdot))\|_t(\langle N\rangle)^2 \tag{5.3}$$

for all $a \le t \le b$.

Proof. Let $t > s > a$ and $f \in C([a,b])$. Then for any partition

$$\Delta : s = t_0 < t_1 < \cdots < t_n = b,$$

we can see that

$$\begin{aligned} &E\left[\left(\int_s^t f_\Delta(B(t))dN(t)\right)^2 \middle| \mathcal{F}_s\right] \\ &= \sum_{k=1}^n E[E[f_{k-1}^2(\Delta_k N(t))^2|\mathcal{F}_{t_{k-1}}]|\mathcal{F}_s] \\ &\quad +2\sum_{k>\ell} E[E[f_{k-1}f_{\ell-1}\Delta_k N(t)\Delta_\ell N(t)|\mathcal{F}_{t_{\ell-1}}]|\mathcal{F}_s] \\ &= \sum_{k=1}^n E[f_{k-1}^2 E[(\Delta_k N(t))^2|\mathcal{F}_{t_{k-1}}]|\mathcal{F}_s] \\ &\quad +2\sum_{k>\ell} E[E[f_{k-1}f_{\ell-1}E[(\Delta_k N(t))(\Delta_\ell N(t))|\mathcal{F}_{t_{k-1}}]|\mathcal{F}_{t_{\ell-1}}]|\mathcal{F}_s], \end{aligned}$$

where $f_{k-1} := f(B(t_{k-1}))$, and $\Delta_k N(t) := N(t_k) - N(t_{k-1})$ for $k = 1, 2, \ldots, n$. By Corollary 2.5 and Theorem 2.6, we have

$$E[\Delta_k N(t)\Delta_\ell N(t)|\mathcal{F}_{t_{k-1}}] = 0.$$

Therefore, we get

$$\begin{aligned} &E\left[\left(\int_s^t f_\Delta(B(u))dN(u)\right)^2 \middle| \mathcal{F}_s\right] \\ &= \sum_{k=1}^n E\left[f(B(t_{k-1}))^2 E\left[\langle N\rangle(t_k) - \langle N\rangle(t_{k-1})|\mathcal{F}_{t_{k-1}}\right]|\mathcal{F}_s\right] \\ &= E\left[\sum_{k=1}^n f(B(t_{k-1}))^2(\langle N\rangle(t_k) - \langle N\rangle(t_{k-1}))\middle|\mathcal{F}_s\right] \\ &= E\left[\int_s^t f_\Delta(B(u))^2 d\langle N\rangle(u)\middle|\mathcal{F}_s\right]. \end{aligned}$$

This implies (5.2), and taking the expectation of the both sides of (5.2), we obtain (5.3). □

For any $f \in C([a,b])$, we have $f_\Delta(B(t)) \to f(B(t))$ in $\mathcal{L}^2(\langle N\rangle)$ as $|\Delta| := \max\{t_k - t_{k-1};\ k = 1, 2, \ldots, n\} \to 0$. Therefore by Proposition 5.1, we can define $\int_a^b f(B(t))dN(t)$ by

$$\int_a^b f(B(t))dN(t) := \lim_{|\Delta|\to 0} \int_a^b f_\Delta(B(t))dN(t) \quad \text{in } L^2(\Omega).$$

The stochastic integral $\int_a^b f(B(t))g(t)dB(t)$ with $g(t)$ from (5.1) can be regarded as $-\int_a^b f(B(t))dN(t)$. This is a generalization of [10] and a formulation of the new integral in [1] from the point of view of the stochastic integral by the near-martingale.

References

1. Ayed, W. and Kuo, H.-H.: An extension of the Itô integral, *Communications on Stochastic Analysis* **2**, no. 3 (2008) 323–333.
2. Hwang, C.-R., Kuo, H.-H., Saitô, K., and Zhai, J.: A general Itô formula for adapted and instantly independent stochastic processes, *Communications on Stochastic Analysis* **10**, no. 3 (2016) 341–362.
3. Hwang, C.-R., Kuo, H.-H. Saitô, K. and Zhai, J.: Near-martingale property of anticipating stochastic integration, *Communications on Stochastic Analysis* **11**, no. 4 (2017) 491–504.
4. Khalifa, N., Kuo, H.-H., Ouerdiane, H. and Szozda, B.: Linear stochastic differential equations with anticipating initial conditions, *Communications on Stochastic Analysis* **7**, no. 2 (2013) 245–253.
5. Kunita, H.: *The estimations of stochastic processes* (in Japanese), Sangyo Tosho, 1976.
6. Kuo, H.-H.: *Introduction to Stochastic Integration.* Universitext (UTX), Springer, 2006.
7. Kuo, H.-H.: The Itô calculus and white noise theory: A brief survey toward general stochastic integration, *Communications on Stochastic Analysis* **8**, no. 1 (2014) 111–139.
8. Kuo, H.-H., Peng, Y. and Szozda, B.: Itô formula and Girsanov theorem for anticipating stochastic integrals, *Communications on Stochastic Analysis* **7**, no. 3 (2013) 441–458.
9. Kuo, H.-H., Peng, Y.,and Szozda, B.: Genaralization of the anticipative Girsanov theorem, *Communications on Stochastic Analysis* **7**, no. 4 (2013) 573–589.
10. Kuo, H.-H., Sae-Tang, A., amd Szozda, B.: An isometry formula for a new stochastic integral, *QP-PQ Quantum Probability and White Noise Analysis* **29**, (2013) 222–232.
11. Kuo, H.-H. and Saitô, K.: Doob's decomposition theorem for near-submartingales, *Communications on Stochastic Analysis* **9**, no. 4 (2015) 467–476.

Shinya Hibino: Graduate School of Mathematics, Meijo University, Tenpaku,Nagoya 468-8502, Japan

E-mail address: 163429005@ccalumni.meijo-u.ac.jp

Hui-Hsiung Kuo: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

E-mail address: kuo@math.lsu.edu

Kimiaki Saitô: Department of Mathematics, Meijo University, Tenpaku, Nagoya 468-8502, Japan

E-mail address: ksaito@meijo-u.ac.jp