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GENERALIZED COMMUTATIVE ASSOCIATION SCHEMES, HYPERGROUPS, AND POSITIVE PRODUCT FORMULAS

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Abstract. It is well known that finite commutative association schemes in the sense of the monograph of Bannai and Ito lead to finite commutative hypergroups with positive dual convolutions and even dual hypergroup structures. In this paper we present several discrete generalizations of association schemes which also lead to associated hypergroups. We show that discrete commutative hypergroups associated with such generalized association schemes admit dual positive convolutions at least on the support of the Plancherel measure. We hope that examples for this theory will lead to the existence of new dual positive product formulas in near future.

1. Motivation

The following setting appears quite often in the theory of Gelfand pairs, spherical functions, and associated special functions:

Let $(G_n, H_n)_{n\in\mathbb{N}}$ be a sequence of Gelfand pairs , i.e., locally compact groups G_n with compact subgroups H_n such that the Banach algebras $M_b(G_n||H_n)$ of all bounded, signed H_n-biinvariant Borel measures on G_n are commutative. Assume also that the double coset spaces $G_n//H_n := \{H_n g H_n : g \in G_n\}$ with the quotient topology are homeomorphic with some fixed locally compact space D. Then the space $M_b(D)$ carries the canonical double coset hypergroup convolutions $*_n$.

A non-trivial H_n-biinvariant continuous function $\varphi_n \in C(G_n)$ is called spherical if the product formula

$$\varphi_n(g)\varphi_n(h) = \int_{H_n} \varphi_n(gkh)\, d\omega_{H_n}(k) \quad (g, h \in G_n) \tag{1.1}$$

holds with the normalized Haar measure ω_{H_n} of H_n. Via $G_n//H_n \equiv D$, we may identify the spherical functions of (G_n, H_n) with the nontrivial continuous functions on D, which are multiplicative w.r.t. $*_n$.

For all relevant examples of such series $(G_n, H_n)_n$, the spherical functions are parameterized by some spectral parameter set $\chi(D)$ independent on n, and the associated functions $\varphi_n : \chi(D) \times D \to \mathbb{C}$ can be embedded into a family of special functions which depend analytically on n in some domain $A \subset \mathbb{C}$, where these functions are spherical for some integers n. In many cases, we can determine

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these special functions and obtain concrete versions of the product formula (1.1), in which n appears as a parameter. Based on Carleson's theorem, a principle of analytic continuation (see e.g. [32], p.186), it is often easy to extend the positive product formula for φ_n in the group cases to a continuous range of parameters. Usually, this extension leads to a continuous family of commutative hypergroups.

Classical examples of such positive product formulas are the well-known product formulas of Gegenbauer for the normalized ultraspherical polynomials

$$R_k^{(\alpha,\alpha)}(x) = {}_2F_1(2\alpha+k+1,-k,\alpha+1;(1-x)/2) \quad (k\in\mathbb{N}_0) \tag{1.2}$$

on $D=[-1,1]$ with $\chi(D)=\mathbb{N}_0$ and for the modified Bessel functions

$$\Lambda_\alpha(x,y) := j_\alpha(xy) \quad \text{with} \quad j_\alpha(z) := {}_0F_1(\alpha+1;-z^2/4) \quad (y\in\mathbb{C}) \tag{1.3}$$

on $D=[0,\infty[$ with $\chi(D)=\mathbb{C}$; see e.g. the survey [1]. In both cases, this works for $\alpha\in[-1/2,\infty[$ where $\alpha=(n-1)/2$ corresponds to the Gelfand pair (G_n,H_n) with $G_n=SO(n+2)$, $H_n=SO(n+1)$ and $G_n=SO(n+1)\ltimes\mathbb{R}^{n+1}$, $H_n=SO(n+1)$ respectively. The continuous ultraspherical product formula can be extended to Jacobi polynomials [17] which generalizes the product formulas for the spherical functions of the projective spaces over $\mathbb{F}=\mathbb{R},\mathbb{C}$ and the quaternions $\mathbb{H}$. Further prominent semisimple, rank one examples are the Gelfand pairs associated with the hyperbolic spaces over $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$ with the groups

$$\begin{aligned} \mathbb{F}=\mathbb{R}: &\qquad G=SO_o(1,k), \quad K=SO(k)\\ \mathbb{F}=\mathbb{C}: &\qquad G=SU(1,k), \quad K=S(U(1)\times U(k))\\ \mathbb{F}=\mathbb{H}: &\qquad G=Sp(1,k), \quad K=Sp(1)\times Sp(k). \end{aligned}$$

Here, $D=[0,\infty[$ with $\chi(D)=\mathbb{C}$ where the spherical functions are Jacobi functions [23]. Besides these classical rank-one examples, reductive examples as well as several examples of higher rank were studied; see e.g. [19] and references there for disk polynomials as well as [24], [28], [29], [30], [31], [38], [39] in the higher rank case, and [18] for hypergeometric functions associated with root systems. Moreover, series of discrete examples were studied in the setting of trees, graphs, buildings, association schemes, Hecke pairs, and other discrete structures; see e.g. [3], [4], [9], [8], [12], [25], [27], [37] where sometimes the connection to hypergroups is supressed. This discrete setting forms the main topic of this paper.

Before going into details, we return to the general setting. Besides positive product formulas for $\varphi_n(\lambda,.)$ on D originating from (1.1), there exist dual product formulas for the functions $\varphi_n(.,x)$ $(x\in D)$ on suitable subsets of $\chi(D)$ for the group cases. To explain this, consider the closed set $P_n(D)\subset\chi(D)$ of spectral parameters λ for which $\varphi_n(\lambda,.)\in C(D)$ corresponds to a positive definite function on G_n. For $\lambda_1,\lambda_2\in P_n(D)$, then $\varphi_n(\lambda_1,.)\cdot\varphi_n(\lambda_2,.)$ is also positive definite on G_n, which implies that $\varphi_n(\lambda_1,.)\cdot\varphi_n(\lambda_2,.)$ is positive definite on the double coset hypergroup D. Therefore, by Bochner's theorem for commutative hypergroups (see [20]), there exists a unique probability measure $\mu_{n,\lambda_1,\lambda_2}$ on $P_n(D)$ with the dual product formula

$$\varphi_n(\lambda_1,x)\cdot\varphi_n(\lambda_2,x) = \int_{P(D)} \varphi_n(\lambda,x)\,d\mu_{n,\lambda_1,\lambda_2}(\lambda) \qquad \text{for all} \quad x\in D. \tag{1.4}$$

For related results of harmonic analysis for Gelfand pairs and commutative hypergroups we refer to [7], [11], [13], [20]. The dual product formula (1.4) has an interpretation in terms of group representations of G_n of class 1, and for several classes of examples there exist explicite formulas for $\mu_{n,\lambda_1,\lambda_2}$ which again can be extended to a continuous parameter range n where usually the positivity remains available. This is for instance trivial for $G_n = SO(n+1) \ltimes \mathbb{R}^{n+1}$, $H_n = SO(n+1)$, with $P_n(D) = [0,\infty[$, where, due to the symmetry of Λ_α in x, y in (1.3), the dual product formula agrees with the original one for $\alpha \in [-1/2,\infty[$. Moreover, for $G_n = SO(n+2)$, $H_n = SO(n+1)$, it is well known that $P_n(D) = \chi(D) = \mathbb{N}_0$, and that the dual formula (1.4) corresponds to the well-known positive product linearization for the ultraspherical polynomials for $\alpha \in]-1/2,\infty[$ which is part of a famous explicit positive product linearization formula for Jacobi polynomials [16], [1]. On the other hand, for hyperbolic spaces, the positive dual product formula is more involved. Here $P_n(D)$ depends on n (see [14], [15]) with $[0,\infty[\subset P_n(D) \subset [0,\infty[\cup i\cdot[0,\infty[\subset \mathbb{C}$. Moreover, the known explicit formulas for the Lebesgue densities of the measures $\mu_{n,\lambda_1,\lambda_2}$ via triple integrals for $\lambda_1, \lambda_2 \in [0,\infty[$ e.g. in [23] are quite involved.

This picture is typical for many Gelfand pairs. We only mention the Gelfand pairs and orthogonal polynomials associated with homogeneous trees and infinite distance transitive graphs with $D = \mathbb{N}_0$ and $\chi(D) = \mathbb{C}$ in [27], [37], where dual product formulas are computed by brute force. For examples of higher rank, the existence of positive dual product formulas seems to be open except for group cases and simple self dual examples like the Bessel examples above. Such self dual examples appear e.g. as orbit spaces when compact subgroups of $U(n)$ act on $\mathbb{R}^n$ which leads e.g. to examples associated with matrix Bessel functions in [29].

In summary, there exist many continuous families of commutative hypergroup structures with explicit convolutions, for which the multiplicative functions are known special functions, and where the existence of dual product formulas is unknown except for the group parameters. The intention of this paper is to start some systematic research beyond the group cases by using more general algebraic structures behind commutative hypergroup structures.

Commutative association schemes as in [4], [3] might form such algebraic objects where these schemes are defined as finite sets $X \neq \emptyset$ with some partition D of $X \times X$ with certain intersection properties; see Section 3 for details. The most prominent examples appear as homogeneous spaces $X = G/H$ for subgroups H of finite groups G. It is known that all finite association schemes lead to hypergroup structures on D which are commutative if and only if so are the association scheme. In the group case $X = G/H$, this hypergroup is just the double coset hypergroup $G//K$. There exist commutative association schemes which do not appear as homogeneous spaces $X = G/H$ (see [4]) such that the class of commutative hypergroups associated with association schemes extends the class of commutative double coset hypergroups. Moreover, by Section 2.10 of [4], these commutative hypergroups admit positive dual product formulas. Therefore, finite commutative association schemes form a tool to establish dual positive product formulas for some finite commutative hypergroups beyond double coset hypergroups.

The aim of this paper is to show that certain generalizations of commutative association schemes also lead to commutative hypergroups with dual positive product formulas. We first study possibly infinite, commutative association schemes where the theory of [4] can be extended canonically. This obvious extension is very rigid as the use of partitions leads to a very few examples only. For a further extension, we observe that all association schemes admit $\{0,1\}$-valued adjacency matrices labeled by D which are stochastic after some renormalization. We translate the axioms of association schemes into a system of axioms for these matrices. As now the integrality conditions vanish, we obtain more examples. We show that also in this generalized case associated commutative hypergroups exist and that, under some restrictions, dual positive product formulas exist; see Section 5.

We expect that our approach may be extended to non-discrete spaces X where families of Markov kernels labeled by some locally compact space D instead of stochastic matrices are used. We shall study this non-discrete generalization in a forthcoming paper. It covers the group cases with $X = G/H$, $D = G//H$ for Gelfand pairs (G, H) as well two known continuous series of examples with $X = \mathbb{R}^2$, $D = [0, \infty[$ and $X = S^2$ the 2-sphere in $\mathbb{R}^3$, $D = [-1, 1]$ due to Kingman [22] and Bingham [6]. The associated commutative hypergroups on D with dual positive product formulas will be just the hypergroups associated with Bessel functions and ultraspherical polynomials mentioned above. Even if for these examples the dual positive product formulas are well known, these examples might be a hint that generalized continuous commutative association schemes might form a powerful tool to derive the existence of dual positive product formulas. We hope that this approach can be applied to certain Heckman-Opdam Jacobi polynomials of type BC and hypergeometric functions of type BC (see [18]), which generalize the spherical functions of compact and noncompact Grassmannianss and for which continuous families of commutative hypergroup structures exist by [28] and [30].

This paper is organized as follows. In Section 2 we recapitulate some facts about commutative hypergroups. Section 3 is then devoted to possibly infinite association schemes and associated hypergroups where the definition remains very close to the classical one in [4]. We there in particular study the relations to the double coset hypergroups $G//H$ for compact open subgroups H of G and for Hecke pairs (G, H). In Section 4 we prove that for all commutative hypergroups associated with such association schemes there exist positive dual convolutions and dual product formulas at least on the support of the Plancherel measure. In Section 5 we propose a discrete generalization of association schemes without integrality conditions. We show that under some conditions, many results of Sections 3 and 4 remain valid including the existence of dual product formulas.

2. Hypergroups

Hypergroups form an extension of locally compact groups. For this, remember that the group multiplication on a locally compact group G leads to the convolution $\delta_x * \delta_y = \delta_{xy}$ $(x, y \in G)$ of point measures. Bilinear, weakly continuous extension of this convolution then leads to a Banach-$*$-algebra structure on the Banach space $M_b(G)$ of all signed bounded regular Borel measures with the total variation norm $\|.\|_{TV}$. In the case of hypergroups we only require a convolution $*$ for measures which admits most properties of a group convolution. We here recapitulate some well-known facts; for details see [11], [20], [7].

Definition 2.1. A *hypergroup* $(D, *)$ is a locally compact Hausdorff space D with a weakly continuous, associative, bilinear convolution $*$ on the Banach space $M_b(D)$ of all bounded regular Borel measures with the following properties:

(1) For all $x, y \in D$, $\delta_x * \delta_y$ is a compactly supported probability measure on D such that the support supp $(\delta_x * \delta_y)$ depends continuously on x, y w.r.t. the so-called Michael topology on the space of all compacta in X (see [20] for details).
(2) There exists a neutral element $e \in D$ with $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for $x \in D$.
(3) There exists a continuous involution $x \mapsto \bar{x}$ on X such that for all $x, y \in D$, $e \in$ supp $(\delta_x * \delta_y)$ holds if and only if $y = \bar{x}$.
(4) If for $\mu \in M_b(D)$, μ^- denotes the image of μ under the involution, then $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$ for all $x, y \in D$.

A hypergroup is called *commutative* if the convolution $*$ is commutative. It is called *symmetric* if the involution is the identity.

Remark 2.2. (1) The identity e and the involution $.^-$ above are unique.
(2) Each symmetric hypergroup is commutative.
(3) For each hypergroup $(D, *)$, $(M_b(D), *)$ is a Banach-$*$-algebra with the involution $\mu \mapsto \mu^*$ with $\mu^*(A) := \overline{\mu(A^-)}$ for Borel sets $A \subset D$.
(4) For a second countable locally compact space D, the Michael topology agrees with the well-known Hausdorff topology; see [24].

The most prominent examples are double coset hypergroups $G//H := \{HgH : g \in G\}$ for compact subgroups H of locally compact groups G:

Example 2.3. Let H be a compact subgroup of a locally compact group G with identity e and with the unique normalized Haar measure $\omega_H \in M^1(H) \subset M^1(G)$, i.e. ω_H is a probability measure. Then the space

$$M_b(G||H) := \{\mu \in M_b(G) : \ \mu = \omega_H * \mu * \omega_H\}$$

of all H-biinvariant measures in $M_b(G)$ is a Banach-$*$-subalgebra of $M_b(G)$. With the quotient topology, $G//H$ is a locally compact space, and the canonical projection $p_{G//H} : G \to G//H$ is continuous, proper and open. Now consider the push forward (or image-measure mapping) $\tilde{p}_{G//H} : M_b(G) \to M_b(G//H)$ with $\tilde{p}_{G//H}(\mu)(A) = \mu(p_{G//H}^{-1}(A))$ for $\mu \in M_b(G)$ and Borel sets $A \subset G//H$. It is easy to see that $\tilde{p}_{G//H}$ is an isometric isomorphism between the Banach spaces $M_b(G||H)$ and $M_b(G//H)$ w.r.t. the total variation norms, and that the transfer of the convolution on $M_b(G||H)$ to $M_b(G//H)$ leads to a hypergroup $(G//H, *)$ with identity HeH and involution $HgH \mapsto Hg^{-1}H$. For details see [20].

Let us consider some typical discrete double coset hypergroups $G//H$:

Example 2.4. Let Γ be the vertex set of a locally finite, connected undirected graph with the graph metric $d : \Gamma \times \Gamma \to \mathbb{N}_0 := \{0, 1, \ldots\}$. A bijective mapping $g : \Gamma \to \Gamma$ is called an automorphism of Γ if $d(g(a), g(b)) = d(a, b)$ for all $a, b \in \Gamma$.

Clearly, the set $Aut(\Gamma)$ of all automorphisms is a topological group w.r.t. the topology of pointwise convergence, i.e., we regard $Aut(\Gamma)$ as subspace of Γ^Γ equipped with the product topology. Assume now that $Aut(\Gamma)$ acts transitively on Γ. It is well-known and easy to see that $Aut(\Gamma)$ is a totally disconnected locally compact group which contains the stabilizer subgroup $H_x \subset G$ of any $x \in \Gamma$ as a compact open subgroup. Γ can be identified with $Aut(\Gamma)/H_x$, and the discrete orbit space $\Gamma^{H_x} := \{H_x(y) :\ y \in \Gamma\}$ with the discrete double coset space $Aut(\Gamma)//H_x$.

We also consider another kind of double coset hypergroups:

Example 2.5. Let G be a discrete group with some subgroup H. (G, H) is called a Hecke pair if the so-called Hecke condition holds, i.e., if each double coset HgH $(g \in G)$ decomposes into an at most finite number of right cosets $g_1H, \ldots, g_{ind(HgH)}H$ where $ind(HgH) \in \mathbb{N}$ is called the (right-)index of HgH. Hecke pairs are studied e.g. in [25], [26] where left coset are taken.

For Hecke pairs, $G//H$ carries a discrete hypergroup structure due to [26]. To describe the associated convolution, take $a, b \in G$ and consider the disjoint decompositions $HaH = \cup_{i=1}^n a_iH$, $HbH = \cup_{j=1}^m b_jH$. If we put

$$\mu(HcH) := |\{(i, j) :\ a_ib_jH = cH\}| \in \mathbb{N}_0,$$

then $\mu(HcH)$ is independent of the representative c of HcH, and

$$\delta_{HaH} * \delta_{HbH} := \sum_{HcH \in G//H} \frac{\mu(HcH) \cdot ind(HcH)}{ind(HaH) \cdot ind(HbH)} \delta_{HcH} \qquad (a, b \in G)$$

generates a hypergroup structure on $G//H$.

The notion of Haar measures on hypergroups is similar to groups:

Definition 2.6. Let $(D, *)$ be a hypergroup, $x, y \in D$, and $f \in C_c(D)$ a continuous function with compact support. We write ${}_xf(y) := f(x * y) := \int_K f\, d(\delta_x * \delta_y)$ and $f_x(y) := f(y * x)$ where, by the hypergroup axioms, $f_x,\ {}_xf \in C_c(D)$ holds.

A non-trivial positive Radon measure $\omega \in M^+(D)$ is called a *left or right Haar measure* if

$$\int_D {}_xf\, d\omega = \int_D f\, d\omega \quad \text{or} \quad \int_D f_x\, d\omega = \int_D f\, d\omega \qquad (f \in C_c(D),\ x \in D)$$

respectively. ω is called a *Haar measure* if it is a left and right Haar measure. If $(D, *)$ admits a Haar measure, then it is called *unimodular*.

The uniqueness of left and right Haar measures and their existence for particular classes are known for a long time by Dunkl, Jewett, and Spector; see [7] for details. The general existence was settled only recently by Chapovsky [10]:

Theorem 2.7. *Each hypergroup admits a left and a right Haar measure. Both are unique up to normalization.*

Examples 2.8. (1) Let $(D, *)$ be a discrete hypergroup. Then, by [20], left and right Haar measures are given by

$$\omega_l(\{x\}) = \frac{1}{(\delta_{\bar{x}} * \delta_x)(\{e\})}, \quad \omega_r(\{x\}) = \frac{1}{(\delta_x * \delta_{\bar{x}})(\{e\})}, \qquad (x \in D).$$

(2) If $(G//H, *)$ is a double coset hypergroup and ω_G a left Haar measure of G, then its projection to $G//H$ is a left Haar measure of $(G//H, *)$.

We next recapitulate some facts on Fourier analysis on commutative hypergroups from [7], [20]. For the rest of Section 2 let $(D, *)$ be a commutative hypergroup with Haar measure ω. For $p \geq 1$ consider the L^p-spaces $L^p(D) := L^p(D, \omega)$. Moreover $C_b(D)$ and $C_o(D)$ are the Banach spaces of all bounded continuous functions on D and those which vanish at infinity respectively.

Definition 2.9. (1) The *dual space* of $(D, *)$ is defined as

$$\hat{D} := \{\alpha \in C_b(D) : \ \alpha \not\equiv 0, \ \alpha(x * \bar{y}) = \alpha(x) \cdot \overline{\alpha(y)} \text{ for all } x, y \in D\}.$$

$\hat{D}$ is a locally compact space w.r.t. the topology of compact-uniform convergence. Moreover, if D is discrete, then $\hat{D}$ is compact. All characters $\alpha \in \hat{D}$ satisfy $\|\alpha\|_\infty = 1$ with $\alpha(e) = 1$.

(2) For $f \in L^1(D)$ and $\mu \in M_b(D)$, the *Fourier transforms* are defined by

$$\hat{f}(\alpha) := \int_D f(x)\overline{\alpha(x)}\, d\omega(x), \quad \hat{\mu}(\alpha) := \int_D \overline{\alpha(x)}\, d\mu(x) \quad (\alpha \in \hat{D})$$

with $\hat{f} \in C_o(\hat{D})$, $\hat{\mu} \in C_b(\hat{D})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$, $\|\hat{\mu}\|_\infty \leq \|\mu\|_{TV}$.

(3) There exists a unique positive measure $\pi \in M^+(\hat{D})$ such that the Fourier transform $.^\wedge : L^1(D) \cap L^2(D) \to C_0(\hat{D}) \cap L^2(\hat{D}, \pi)$ is an isometry. π is called the *Plancherel measure* on $\hat{D}$.

Notice that, different from l.c.a. groups, the support $S := supp\, \pi$ may be a proper closed subset of $\hat{D}$. Quite often, we even have $\mathbf{1} \notin S$.

(4) For $f \in L^1(\hat{D}, \pi)$, $\mu \in M_b(\hat{D})$, the *inverse Fourier transforms* are given by

$$\check{f}(x) := \int_S f(\alpha)\alpha(x)\, d\pi(\alpha), \quad \check{\mu}(x) := \int_{\hat{D}} \alpha(x)\, d\mu(\alpha) \quad (x \in D)$$

with $\check{f} \in C_0(D)$, $\check{\mu} \in C_b(D)$ and $\|\check{f}\|_\infty \leq \|f\|_1$, $\|\check{\mu}\|_\infty \leq \|\mu\|_{TV}$.

(5) $f \in C_b(D)$ is called *positive definite* on the hypergroup D if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ and $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{k,l=1}^n c_k \bar{c}_l \cdot f(x_k * \bar{x}_l) \geq 0$. Obviously, all characters $\alpha \in \hat{D}$ are positive definite.

We collect some essential well-known results:

Facts 2.10. (1) (Theorem of Bochner, [20]) A function $f \in C_b(D)$ is positive definite if and only of $f = \check{\mu}$ for some $\mu \in M_b^+(\hat{D})$. In this case, μ is a probability measure if and only if $\check{\mu}(e) = 1$.

(2) For $f, g \in L^2(D)$, the convolution product $f * g(x) := \int f(x * \bar{y})g(y)\, d\omega(y)$ $(x \in D)$ satisfies $f * g \in C_0(D)$. Moreover, for $f \in L^2(D)$, $f^*(x) = \overline{f(\bar{x})}$ satisfies $f^* \in L^2(D)$, and $f * f^* \in C_0(D)$ is positive definite; see [20], [7].

(3) A function $f \in C_b(D)$ is the inverse Fourier transform $\check{\mu}$ for some $\mu \in M_b^+(\hat{D})$ with $supp\, \mu \subset S$ if and only if f is the compact-uniform limit of positive definite functions of the form $h * h^*$, $h \in C_c(D)$; see [35].

(4) Let $\alpha \in \hat{D}$. Then $\alpha \in S$ if and only if α is the compact-uniform limit of positive definite functions of the form $h * h^*$, $h \in C_c(D)$; see [35].

(5) There exists precisely one positive character $\alpha_0 \in S$ by [33], [7].
(6) If $\mu \in M^1(\hat{D})$ satisfies $\check{\mu} \geq 0$ on D, then its support $supp\,\mu$ contains at least one positive character; see [36].

In contrast to l.c.a. groups, products of positive definite functions on D are not necessarily positive definite; see e.g. Section 9.1C of [20] for an example with $|D| = 3$. However,sometimes this positive definiteness of products is available.

If for all $\alpha, \beta \in \hat{D}$ (or a subset of $\hat{D}$ like S) the products $\alpha\beta$ are positive definite, then by Bochner's theorem 2.10(1), there are probability measures $\delta_\alpha \hat{*} \delta_\beta \in M^1(\hat{D})$ with $(\delta_\alpha \hat{*} \delta_\beta)^\vee = \alpha\beta$, i.e., we obtain dual positive product formulas as claimed in Section 1. Under additional conditions, $(\hat{D}, \hat{*})$ then carries a dual hypergroup structure with $\mathbf{1}$ as identity and complex conjugation as involution. This for instance holds for all compact commutative double coset hypergroups $G//H$ by [11]. For non-compact Gelfand pairs (G, H) we have dual positive convolutions on S; see [20], [36]. These convolutions usually do not generate a dual hypergroup structure, and sometimes $\alpha\beta$ is not positive definite on D for some $\alpha, \beta \in \hat{D}$; see Theorem 4.9 for an example.

3. Discrete Association Schemes

In this section we extend the classical notion of finite association schemes in a natural way. As references for finite association schemes we recommend [4], [3].

Definition 3.1. Let X, D be nonempty, at most countable sets and $(R_i)_{i\in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ for $i \in D$ and the following properties:
(1) There exists $e \in D$ with $R_e = \{(x, x) : x \in X\}$.
(2) There exists an involution $i \mapsto \bar{i}$ on D such that for $i \in D$, $R_{\bar{i}} = \{(y, x) : (x, y) \in R_i\}$.
(3) For all $i, j, k \in D$ and $(x, y) \in R_k$, the number
$$p^k_{i,j} := |\{z \in X : \ (x, z) \in R_i \ \text{ and } \ (z, y) \in R_j\}|$$
is finite and independent of $(x, y) \in R_k$.

Then $\Lambda := (X, D, (R_i)_{i\in D})$ is called *an association scheme* with *intersection numbers* $(p^k_{i,j})_{i,j,k\in D}$ and *identity* e.

An association scheme is called *commutative* if $p^k_{i,j} = p^k_{j,i}$ for all $i, j, k \in D$. It is called *symmetric* (or *hermitian*) if the involution on D is the identity. Moreover, it is called *finite*, if so are X and D.

Finite association schemes above are obviously precisely association schemes in the sense of the monographs [4].

Association schemes have the following interpretation: We regard $X \times X$ as set of all directed paths from points in X to points in X. The set of paths is labeled by D which might be colors, lengths, or difficulties of paths. Then R_e is the set of trivial paths, and $R_{\bar{i}}$ is the set of all reversed paths in R_i. Axiom (3) is some kind of a symmetry condition and the central part of the definition.

Association schemes may be described with the aid of adjacency matrices:

Definition 3.2. The *adjacency matrices* $A_i \in \mathbb{R}^{X\times X}$ $(i \in D)$ of an association scheme $(X, D, (R_i)_{i\in D})$ are given by

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \in D,\ x, y \in X).$$

The adjacency matrices have the following obvious properties:

(1) A_e is the identity matrix I_X.
(2) $\sum_{i\in D} A_i$ is the matrix J_X whose entries are all equal to 1.
(3) $A_i^T = A_{\bar{i}}$ for $i \in D$.
(4) For all $i \in D$ and all rows and columns of A_i, all entries are equal to zero except for finitely many cases (take $k = e$, $j = \bar{i}$ in 3.1(3)!).
(5) For $i, j \in D$, $A_iA_j = \sum_{k\in D} p_{i,j}^k A_k$.
(6) An association scheme Λ is commutative if and only if $A_iA_j = A_jA_i$ for all $i, j \in D$.
(7) An association scheme Λ is symmetric if and only if all A_i are symmetric. In particular, each symmetric association scheme is commutative.

Definition 3.3. Let $(X, D, (R_i)_{i\in D})$ be an association scheme. The *valency* of R_i or $i \in D$ is defined as $\omega_i := p_{i,\bar{i}}^e$. Obviously, the ω_i satisfy

$$\omega_i = |\{z \in X : (x,z) \in R_i\}| \in \mathbb{N} \tag{3.1}$$

for all $x \in X$. In particular, $\omega_e = 1$, and

$$|X| = \textstyle\sum_{i\in K} \omega_i \in \mathbb{N} \cup \{\infty\}. \tag{3.2}$$

Remark 3.4. For $i \in D$, the renormalized matrices $S_i := \frac{1}{\omega_i} A_i \in \mathbb{R}^{X\times X}$ are stochastic, i.e., all rows sum are equal to 1 by (3.1). Notice that also all column sums are finite by 3.2(4). Moreover, by 3.2(5), the stochastic matrices S_i satisfy

$$S_iS_j = \textstyle\sum_{k\in D} \frac{\omega_k}{\omega_i\omega_j} p_{i,j}^k S_k \qquad \text{for} \qquad i, j \in D. \tag{3.3}$$

Since both sides of (3.3) are stochastic, absolute convergence leads to

$$\textstyle\sum_{k\in D} \frac{\omega_k}{\omega_i\omega_j} p_{i,j}^k = 1 \quad \text{for} \quad i, j \in D. \tag{3.4}$$

The formulas (3.3) and (3.4) are the starting point in Section 5 for the extension of association schemes to a continuous setting.

We now collect some relations about the intersection numbers and valencies:

Lemma 3.5. *For all $i, j, k, l, m \in D$:*

(1) $p_{e,i}^j = p_{i,e}^j = \delta_{i,j}$ *and* $p_{i,j}^e = \omega_i \delta_{i,\bar{j}}$.
(2) $p_{i,j}^l = p_{\bar{j},\bar{i}}^{\bar{l}}$.
(3) $\sum_{j\in K} p_{i,j}^l = \omega_i$ *for all* $l \in D$, *and, in particular, for all* $i, l \in D$, $p_{i,j}^l \neq 0$ *holds for finitely many* $j \in K$ *only.*
(4) $\omega_l \cdot p_{i,j}^l = \omega_i \cdot p_{l,\bar{j}}^i$ *and* $\omega_{\bar{j}} \cdot p_{i,l}^j = \omega_{\bar{l}} \cdot p_{i,j}^l$.
(5) $\sum_{l\in K} \omega_l \cdot p_{i,j}^l = \omega_i \cdot \omega_j$ *and* $\sum_{l\in K} \omega_{\bar{l}} \cdot p_{i,j}^l = \omega_{\bar{i}} \cdot \omega_{\bar{j}}$
(6) $\sum_{l\in K} p_{i,j}^l \cdot p_{l,k}^m = \sum_{l\in K} p_{j,k}^l p_{l,k}^m$.
(7) *If* $p_{i,j}^k > 0$, *then* $\dfrac{\omega_k}{\omega_{\bar{k}}} = \dfrac{\omega_i}{\omega_{\bar{i}}} \cdot \dfrac{\omega_j}{\omega_{\bar{j}}}$.

Proof. Part (1) is obvious, and part (2) follows from

$$\sum_l p^{\bar{l}}_{\bar{j},\bar{i}} A_{\bar{l}} = A_j^T A_i^T = (A_i A_j)^T = \sum_l p^l_{i,j} A_l^T = \sum_l p^l_{i,j} A_{\bar{l}}.$$

and a comparison of coefficients. Part (3) is a consequence of

$$\omega_i J_X = A_i J_X = \sum_j A_i A_j = \sum_j \sum_l p^l_{i,j} A_l = \sum_l \Big(\sum_j p^l_{i,j}\Big) A_l.$$

For the proof of the first statement of (4), notice that by part (1), for $x \in X$

$$\omega_l \cdot p^l_{i,j} = \sum_k p^k_{i,j} (A_k A_{\bar{l}})_{x,x} = (A_i A_j A_{\bar{l}})_{x,x} = \sum_k p^k_{j,\bar{l}} (A_i A_k)_{x,x} = \omega_i \cdot p^{\bar{i}}_{j,\bar{l}} = \omega_i \cdot p^i_{l,\bar{j}};$$

the second statement of part (4) follows in a similar way. Moreover, the statements in (5) are consequences of

$$\omega_i \cdot \omega_j J_X = A_i A_j J_X = \sum_l p^l_{i,j} A_l J_X = \sum_l p^l_{i,j} \cdot \omega_l J_X$$

and

$$\omega_{\bar{i}} \cdot \omega_{\bar{j}} J_X = J_X A_i A_j = \sum_l p^l_{i,j} J_X A_l = \sum_l p^l_{i,j} \cdot \omega_{\bar{l}} J_X.$$

Part (6) follows from $(A_i A_j) A_k = A_i (A_j A_k)$ and comparison of coefficients in the expansions. Finally, parts (4) and (2) imply

$$\frac{\omega_k}{\omega_i \cdot \omega_j} p^k_{i,j} = \frac{p^i_{k,\bar{j}}}{\omega_j} = \frac{p^{\bar{j}}_{\bar{k},i}}{\omega_{\bar{i}}} = \frac{p^j_{\bar{i},k}}{\omega_{\bar{i}}} = \frac{\omega_{\bar{k}}}{\omega_{\bar{i}} \omega_{\bar{j}}} p^k_{i,j},$$

which proves (7). □

Typical examples of finite or infinite association schemes are connected with homogeneous spaces G/H in one of the following two ways:

Example 3.6. Let G be a second countable, locally compact group with an compact open subgroup H and with neutral element e. Then the quotient $X := G/H$ as well as the double coset space $D := G//H$ are at most countable, discrete spaces w.r.t. the quotient topology. Consider the partition $(R_i)_{i\in D}$ of $X \times X$ with

$$R_{HgH} := \{(xH, yH) \in X \times X :\ HgH = Hx^{-1}yH\}.$$

Then $(X, D, (R_j)_{j\in D})$ forms an association scheme with neutral element $HeH = H$ and involution $\overline{HgH} := Hg^{-1}H$. In fact, the axioms (1), (2) in 3.1 are obvious. For axiom (3) consider $g, h, x, y, \tilde{x}, \tilde{y} \in G$ with (xH, yH) and $(\tilde{x}H, \tilde{y}H)$ in the same partition $R_{Hx^{-1}yH}$, i.e., with $H\tilde{x}^{-1}\tilde{y}H = Hx^{-1}yH$. Thus, there exist $h_1, h_2 \in H$ and $w \in G$ with $\tilde{x}^{-1} = h_1 x^{-1} w^{-1}$ and $\tilde{y} = wyh_2$. Therefore, for any $zH \in X$, $H\tilde{x}^{-1}zH = Hx^{-1}w^{-1}zH$ and $Hz^{-1}wyH = H(w^{-1}z)^{-1}H$, which means that $zH \mapsto w^{-1}zH$ establishes a bijective mapping between $\{zH :\ Hx^{-1}zH = HgH,\ Hz^{-1}yH = HhH\}$ and $\{zH :\ H\tilde{x}^{-1}zH = HgH,\ Hz^{-1}\tilde{y}H = HhH\}$. This shows that the intersection number $p^{Hx^{-1}yH}_{HgH,HhH}$ is independent of the choice of x, y. Moreover, due to compactness, each double coset HgH decomposes into finitely many cosets xH which shows that $\{zH :\ Hx^{-1}zH = HgH\}$ and thus the intersection number $p^{Hx^{-1}yH}_{HgH,HhH}$ is finite as claimed.

It follows from the definition of R_{HgH} and Eq. (3.1) that for $g \in G$ the valency ω_{HgH} of the double coset $HgH \in G//H$ is given by the finite number of different cosets $xH \in G/H$ contained in HgH.

The preceding example also works for Hecke pairs:

Example 3.7. Let (G, H) be a Hecke pair, i.e., each double coset HgH decomposes into finitely many cosets xH. Then the same partition as in Example 3.6 leads to an association scheme $(G/H, G//H, (R_j)_{j\in D})$. We skip the proof. We remark that again the valency ω_{HgH} of a double coset $HgH \in G//H$ is $ind(HgH)$, i.e., the number of cosets $xH \in G/H$ contained in HgH.

Association schemes lead to discrete hypergroups. The associated convolution algebras are just the Bose-Mesner algebras for finite association schemes in [4].

Proposition 3.8. *Let $\Lambda := (X, D, (R_i)_{i\in D})$ be an association scheme with intersection numbers $p_{i,j}^k$ and valencies ω_i. Then the product $*$ with*

$$\delta_i * \delta_j := \sum_{k\in D} \frac{\omega_k}{\omega_i\omega_j} \cdot p_{i,j}^k \delta_k$$

can be extended uniquely to an associative, bilinear, $\|\,.\,\|_{TV}$-continuous mapping on $M_b(D)$. $(D, *)$ is a discrete hypergroup with the left and right Haar measure

$$\Omega_l := \sum_{i\in D} \omega_i\delta_i \qquad \textit{and} \qquad \Omega_r := \Omega_l^* := \sum_{i\in D} \omega_{\bar{i}}\delta_i \tag{3.5}$$

respectively. This hypergroup is commutative or symmetric if and only if so is Λ.

Proof. By Lemma 3.5(5), $\delta_i * \delta_j$ is a probability measure with finite support for $i, j \in D$. Thus, $*$ can be extended uniquely to a bilinear continuous mapping on $M_b(D)$. The associativity of $*$ follows from Lemma 3.5(6) for point measures, and, in the general case, by the unique bilinear continuous extension. The remaining hypergroup axioms now follow from Lemma 3.5. Clearly, $*$ is commutative if and only if so is Λ. The same holds for symmetry, as the involution on a hypergroup and on an association scheme are unique.

Finally, by Example 2.8, a left Haar measure is given by

$$\Omega_l(\{i\}) = \frac{1}{\tilde{p}_{\bar{i},i}^e} = \frac{\omega_{\bar{i}}\omega_i}{\omega_e p_{\bar{i},i}^e} = \omega_i \quad (i \in D).$$

The same argument works for right Haar measures. □

Example 3.9. Let G be a second countable, locally compact group with a compact open subgroup H. Consider the associated quotient association scheme $(X = G/H, D = G//H, (R_j)_{j\in D})$ as in Example 3.6. Then the associated hypergroup $(D = G//H, *)$ according to Proposition 3.8 is the double coset hypergroup in the sense of Section 2.

For the proof notice that both hypergroups live on the discrete space D. We must show that for all $x, y, g \in G$ the products $(\delta_{HxH} * \delta_{HyH})(\{HgH\})$ are equal.

Let us compute this first for the double coset hypergroup $G//H$: By Example 3.6, HxH decomposes into ω_{HxH} many disjoint cosets $x_1H, \dots, x_{\omega_{HxH}}H$. This shows that HxH also decomposes into $\omega_{Hx^{-1}H}$ many disjoint cosets of the form $H\tilde{x}_1^{-1}, \dots, H\tilde{x}_{\omega_{Hx^{-1}H}}^{-1}$. Moreover, HyH decomposes into ω_{HyH} many disjoint cosets $y_1H,\dots,y_{\omega_{HyH}}H$. Using the normalized Haar measure ω_H of H we obtain

$$((\omega_H*\delta_x*\omega_H)*(\omega_H*\delta_y*\omega_H))(HgH)= \tag{3.6}$$
$$=\frac{1}{\omega_{Hx^{-1}H}\cdot\omega_{HyH}}\sum_{k=1}^{\omega_{Hx^{-1}H}}\sum_{l=1}^{\omega_{HyH}}(\omega_H*\delta_{\tilde{x}_k^{-1}}*\delta_{y_l}*\omega_H)(HgH).$$

Therefore, by the definition of the double coset convolution in Section 2,

$$(\delta_{HxH}*\delta_{HyH})(\{HgH\})=\frac{|\{(k,l):H\tilde{x}_k^{-1}y_lH=HgH\}|}{\omega_{Hx^{-1}H}\cdot\omega_{HyH}}. \tag{3.7}$$

We next check that the hypergroup associated with the association scheme in Example 3.6 leads to the same result. For this we employ the beginning of the proof of 3.5(4) and observe that

$$\omega_{HgH}\cdot p^{HgH}_{HxH,HyH}=(A_{HxH}A_{HyH}A_{Hg^{-1}H})_{eH,eH}$$

for the trivial coset $eH\in X=G/H$. This shows with the notations above that

$$\omega_{HgH}\cdot p^{HgH}_{HxH,HyH}=|\{(k,l):H\tilde{x}_k^{-1}y_lH=HgH\}|.$$

The convolution in Proposition 3.8 now again leads to (3.7) as claimed.

The same result can be obtained for Hecke pairs. We omit the obvious proof.

Example 3.10. Let (G,H) be a Hecke pair. Consider the associated quotient association scheme $(X=G/H, D=G//H,(R_j)_{j\in D}$ as in Example 3.7. Then the associated hypergroup $(D=G//H,*)$ according to Proposition 3.8 is the double coset hypergroup in the sense of Section 2.

We next discuss a property of association schemes which is valid in the finite case, but not necessarily in infinite cases.

Definition 3.11. An association scheme Λ with valencies ω_i is called *unimodular* if $\omega_i=\omega_{\bar{i}}$ for all $i\in D$.

Lemma 3.12.
(1) *If an association scheme is commutative or finite, then it is unimodular.*
(2) *An association scheme $(X,D,(R_i)_{i\in D})$ is unimodular if and only if the associated discrete hypergroup $(D,*)$ is unimodular.*
(3) *If $(X,D,(R_i)_{i\in D})$ is unimodular, then $S_i^T=S_{\bar{i}}\quad(i\in D)$.*

Proof. The commutative case in (1) is trivial. Now let the set X of an association scheme finite. Then, for $h\in D$, the adjacency matrices A_h and $A_{\bar{h}}$ have $|X|\cdot\omega_h$ and $|X|\cdot\omega_{\bar{h}}$ times the entry 1 respectively. $A_{\bar{h}}=A_h^T$ now implies $\omega_h=\omega_{\bar{h}}$.

Part (2) is clear by Eq. (3.5) in Proposition 3.8; (3) is also clear. □

Remark 3.13. (1) There exist non-unimodular association schemes in the group context 3.6. In fact, in [21], examples of totally disconnected groups G with compact open subgroups H are given where the associated double coset hypergroups $(G//H,*)$ are not unimodular. Therefore, by Lemma 3.12(2), the corresponding association schemes are not unimodular. For examples in the context of Hecke pairs see [25].

(2) Lemma 3.5(7) has the interpretation that the modular function $\Delta(i) := \frac{\omega_i}{\omega_{\bar{i}}}$ $(i \in D)$ is multiplicative in a strong form.

Remark 3.14. Let Λ be an unimodular association scheme with the associated hypergroup $(D, *)$ as in Proposition 3.8. Then, by (3.3) and 3.12(3), the (finite) linear span of the matrices S_i $(i \in D)$ forms an $*$-algebra which is via $S_i \longleftrightarrow \delta_i$ $(i \in D)$ isomorphic with the $*$-algebra $(M_f(D), *)$ of all signed measures on D with finite support and the convolution from Proposition 3.8.

We briefly study concepts of automorphisms of association schemes.

Definition 3.15. (1) Let $\Lambda = (X, D = G, (R_j)_{j\in D})$ and $\tilde{\Lambda} = (\tilde{X}, \tilde{D}, (\tilde{R}_j)_{j\in\tilde{D}})$ be association schemes. A pair of (φ, ψ) of bijective mappings $\varphi : X \to \tilde{X}$, $\psi : D \to \tilde{D}$ is called an *isomorphism* from Λ onto $\tilde{\Lambda}$ if for all $x, y \in X$ and $i \in D$ with $(x, y) \in R_i$, $(\varphi(x), \varphi(y)) \in \tilde{R}_{\psi(i)}$.

(2) If $\Lambda = \tilde{\Lambda}$, then an isomorphism is called an *automorphism*.

(3) If (φ, ψ) is an automorphism with ψ as identity, then φ is called a *strong automorphism*. The set $Saut(\Lambda)$ of all strong automorphisms of Λ is obviously a group.

Remark 3.16. Let $\Lambda = (X, D, (R_j)_{j\in D})$ be an association scheme. If we regard $Saut(\Lambda)$ as subspace of the space X^X of all maps from X to X with the product topology, then $Saut(\Lambda)$ obviously is a topological group. For $x_0 \in X$ we consider the stabilizer subgroup $Stab_{x_0} := \{\varphi \in Saut(\Lambda) :\ \varphi(x_0) = x_0\}$ which is obviously open in $Saut(\Lambda)$. $Stab_{x_0}$ is also compact. In fact, for each $g \in Stab_{x_0}$ and $y \in X$ with $(x, y) \in R_i$ with $i \in D$, we have $(x, g(y)) = (g(x), g(y)) \in R_i$. The axioms of an association scheme show that $\{g(y) :\ g \in Stab_{x_0}\}$ is finite $y \in X$. This shows that $Stab_{x_0}$ is compact. In particular, $Stab_{x_0}$ is a compact open neighborhood of the identity of $Saut(\Lambda)$ which shows that $Saut(\Lambda)$ is locally compact.

This argument is completely analog to Example 2.4 for graphs.

Example 3.17. Let G be a locally compact group with compact open subgroup H, or let (G, H) be a Hecke pair. Consider the associated association scheme $\Lambda = (X = G/H, D = G//H, (R_j)_{j\in D})$ as in 3.6 or 3.7. Then for each $g \in G$, the mapping $T_g : G/H \to G/H$, $xH \mapsto gxH$ defines a strong automorphism of Λ. This obviously leads to a group homomorphism $T : G \to Saut(\Lambda)$.

Remark 3.18. Let (φ, ψ) be an automorphism of an association scheme $\Lambda = (X, D, (R_j)_{j\in D})$. Then $\psi : D \to D$ is an automorphism of the associated hypergroup on D according to Proposition 3.8. This follows immediately from the definition of the convolution in 3.8 and the definitions of p^k_{ij} and ω_i.

The constructions of association schemes are parallel above for groups G with compact open subgroups H and for Hecke pairs (G, H). This is not an accident:

If $\tilde{G}$ is a locally compact group with compact open subgroup $\tilde{H}$, then w.r.t. the discrete topology, $(\tilde{G}, \tilde{H})$ is also a Hecke pair, and the association scheme $(\tilde{X} = \tilde{G}/\tilde{H}, \tilde{D} = \tilde{G}//\tilde{H}, (\tilde{R}_j)_{j\in\tilde{D}})$ does not depend on the topologies. In this way, the approach via Hecke pairs seems to be the more general one. On the other hand, the following theorem shows that both approaches are equivalent:

Theorem 3.19. *Let (G, H) be a Hecke pair. Then there is a totally disconnected locally compact group $\bar{G}$ with an compact open subgroup $\bar{H}$ such that the associated association schemes $(G/H, D = G//H, (R_j)_{j\in D})$ and $(\bar{G}/\bar{H}, \bar{D} = \bar{G}//\bar{H}, (\bar{R}_j)_{j\in\bar{D}})$ are isomorphic. The associated double coset hypergroups are also isomorphic.*

Proof. Let (G, H) be a Hecke pair and $\Lambda := (G/H, D = G//H, (R_j)_{j\in D})$ the associated association scheme. Consider the homomorphism $T : G \to Saut(\Lambda)$ according to 3.17, where G acts transitively on $X = G/H$. Hence, the totally disconnected locally compact group $\tilde{G} := Saut(\Lambda)$ of 3.16 acts transitively on $X = G/H$. We define $\tilde{H} := Stab_{eH}$ as a compact open subgroup of $\tilde{G}$.

We next consider the normal subgroup $N := \bigcap_{x\in G} xHx^{-1} \le H$ of G. Then obviously $(G/N)/(H/N) \simeq G/H$, $(G/N)//(H/N) \simeq G//H$, and $(G/N, H/N)$ forms a Hecke pair for which the associated associated association scheme is isomorphic with Λ. Using this division by N, we may assume from now on that $N = \{e\}$.

If this is the case, we obtain that the homomorphism $G \to \tilde{G}, g \mapsto T_g$ from 3.17 with $T_g(xH) = gxH$ is injective. In fact, if for some $g \in G$ and all $x \in G$ we have $T_g(xH) = gxH = xH$, it follows that $g \in N = \{e\}$ as claimed. We thus may assume that G is a subgroup of $\tilde{G}$. We then readily obtain $H = G \cap \tilde{H}$.

We now consider the closures $\bar{G}, \bar{H}$ of G, H in $\tilde{G}$. Then $\bar{G}$ is a locally compact, totally disconnected topological group with $\bar{H} \subset \tilde{H}$ as compact subgroup. Moreover $H = G \cap \tilde{H}$ yields $\bar{H} = \bar{G} \cap \tilde{H}$ which implies that $\bar{H}$ is in fact a compact open subgroup of $\bar{G}$. Moreover, as $\bar{G}$ acts transitively on G/H with $\bar{H}$ as stabilizer subgroup of $eH \in G/H$, we may identify $\bar{G}/\bar{H}$ with G/H. To make this more explicit we claim that

$$\text{for all} \quad \bar{g} \in \bar{G} \quad \text{there exists} \quad g \in G \quad \text{with} \quad \bar{g}\bar{H} = g\bar{H}. \tag{3.8}$$

In fact, each $\bar{g} \in \bar{G}$ is the limit of some $g_\alpha \in G$. As $\bar{g}\bar{H}$ is an open neighborhood of $\bar{g} \in \bar{G}$, we obtain $g_\alpha \in \bar{g}\bar{H}$ for large indices which proves (3.8). Moreover, as for $g_1, g_2 \in G$ the relation $g_1\bar{H} = g_2\bar{H}$ implies $g_1^{-1}g_2 \in \bar{H} \cap G = H$, we obtain with (3.8) that the mapping

$$\varphi : G/K \to \bar{G}/\bar{H}, \quad gH \mapsto g\bar{H}$$

is a well defined bijective mapping. Moreover, it is clear that the orbits of the action of H and $\bar{H}$ on G/H are equal. This shows that also the mapping

$$\psi : G//K \to \bar{G}//\bar{H}, \quad HgH \mapsto \bar{H}g\bar{H}$$

is well defined and bijective. If we compare the pair (φ, ψ) with the construction of the associations schemes in Examples 3.6 and 3.7, we see that the schemes Λ and $(\bar{X} = \bar{G}/\bar{H}, \bar{D} = \bar{G}//\bar{H}, (\bar{R}_j)_{j\in\bar{D}})$ are isomorphic via (φ, ψ) as claimed.

Finally, the statement about the associated double coset hypergroups is clear by Example 3.9. □

We finally present the following trivial commutativity criterion:

Lemma 3.20. *If an association scheme $\Lambda = (X, D, (R_i)_{i\in D})$ admits an automorphism (φ, ψ) with $\psi(i) = \bar{i}$ for $i \in D$, then Λ is commutative.*

Proof. For $i, j, k \in D$, $p^k_{i,j} = p^{\bar{k}}_{\bar{j},\bar{i}} = p^{\psi(k)}_{\psi(j),\psi(i)} = p^k_{j,i}$ where the last equality follows easily from the definition of an automorphism and the definition of $p^k_{j,i}$. □

Remark 3.21. Lemma 3.20 is a natural extension of the following well-known criterion for Gelfand pairs:
Let H be a compact subgroup of a locally compact group G such that there is a continuous automorphism T of G with $x^{-1} \in HT(x)H$ for $x \in G$. Then (G, H) is a Gelfand pair, i.e., $G//H$ is a commutative hypergroup.
If H is compact and open in G, then we obtain this criterion from Lemma 3.20 applied to association scheme $\Lambda = (X = G/H, D = G//H, (R_i)_{i\in D})$ with the automorphism (φ, ψ) with $\varphi(gH) := T(g)H$ and $\psi(HgH) := HT(g)H$.

4. Commutative Association Schemes and Dual Product Formulas

In this section let $\Lambda = (X, D, (R_i)_{i\in D})$ be a commutative association scheme and $(D, *)$ the associated commutative discrete hypergroup as in Proposition 3.8.

By Remark 3.14, the linear span $A(X)$ of the matrices S_i $(i \in D)$ is a commutative $*$-algebra which is isomorphic with the $*$-algebra $(M_f(D), *)$ via

$$\mu \in M_f(D) \longleftrightarrow S_\mu := \sum_{i\in D} \mu(\{i\})S_i \in A(X).$$

Moreover, as $(f * g)\Omega = f\Omega * g\Omega$ and $f^*\Omega = (f\Omega)^*$ for all $f, g \in C_c(D)$ and the Haar measure Ω of $(D, *)$ by [20], the $*$-algebra $A(X)$ is also isomorphic with the commutative $*$-algebra $(C_c(D), *)$ via

$$f \in C_c(D) \longleftrightarrow S_f := \sum_{i\in D} f(i)\omega_i S_i = \sum_{i\in D} f(i)A_i \in A(X).$$

By this construction, $S_f = S_{f\Omega}$ for $f \in C_c(D)$. We notice that even for $f \in C_b(D)$, the matrix $S_f := \sum_{i\in D} f(i)\omega_i S_i$ is well-defined, and that products of S_f with matrices with only finitely many nonzero entries in all rows and columns are well-defined. In particular, we may multiply S_f on both sides with any S_μ, $\mu \in M_f(D)$.

For each function $f \in C_b(D)$ we define the function $F_f : X \times X \to \mathbb{C}$ with $F_f(x, y) = f(i)$ for the unique $i \in D$ with $(x, y) \in R_i$. Then F_f is constant on all partitions R_i, and by our construction,

$$(F_f(x, y))_{x,y\in X} = S_f.$$

In particular, $f(e)$ is equal to the diagonal entries of S_f (where these are equal).

The following definition is standard for kernels.

Definition 4.1. A function $F : X \times X \to \mathbb{C}$ is called *positive definite* on $X \times X$ if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{k,l=1}^{n} c_k \bar{c}_l \cdot F(x_k, x_l) \geq 0,$$

i.e., the matrices $(F(x_k, x_l))_{k,l}$ are positive semidefinite. This in particular implies that these matrices are hermitian, i.e.,

$$F(x, y) = \overline{F(y, x)} \quad \text{for all} \quad x, y \in X.$$

As the pointwise products of positive semidefinite matrices are again positive semidefinite (see e.g. Lemma 3.2 of [5]), we have:

Lemma 4.2. *If $F, G : X \times X \to \mathbb{C}$ are positive definite, then the pointwise product $F \cdot G : X \times X \to \mathbb{C}$ is also positive definite.*

Moreover, for $f \in C_b(D)$, the positive definiteness of f on D is related to that of F_f on $X \times X$. We begin with the following observation:

Lemma 4.3. *Let $f \in C_b(D)$. If F_f is positive definite, then f is positive definite on the hypergroup $(D, *)$.*

Proof. For f define the reflected function $f^-(x) := f(\bar{x})$ for $x \in D$. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ and $c_1, \ldots, c_n \in \mathbb{C}$. Let $\mu := \sum_{k=1}^n c_k \delta_{x_k} \in M_f(D)$ and observe that by the definition of the convolution of a function with a measure and by commutativity,

$$P := \sum_{k,l=1}^{n} c_k \bar{c}_l \cdot f^-(x_k * \bar{x}_l) = \int_D f^- \, d(\mu * \mu^*) = (\mu * \mu^* * f)(e) = (\mu * f * \mu^*)(e).$$

where by the considerations above, this is equal to the diagonal entries of

$$S_{\mu * f * \mu^*} = S_\mu \cdot S_f \cdot \overline{S_\mu}^T = S_\mu \cdot F_f(x,y)_{x,y \in X} \cdot \overline{S_\mu}^T.$$

This matrix product exists and is positive semidefinite, as so is $F_f(y,x)_{x,y \in X}$ by our assumption. As the diagonal entries of a positive semidefinite matrix are nonnegative, we obtain $P \geq 0$. Hence, f^- and thus f is positive definite. $\square$

Here is a partially reverse statement.

Lemma 4.4. *Let $f \in C_c(D)$. Then, by 2.10(2), $f * f^*$ is positive definite on D, and F_{f*f^*} is positive definite.*

Proof. By our considerations above, $(F_{f*f^*}(x,y))_{x,y \in X} = S_{f*f^*} = S_f \overline{S_f}^T$. This proves that F_{f*f^*} is positive definite. $\square$

We now combine Lemma 4.4 with Fact 2.10(4) and the trivial observation that pointwise limits of positive definite functions are positive definite. This shows:

Corollary 4.5. *Let $\alpha \in S \subset \hat{D}$ be a character in the support of the Plancherel measure. Then $F_\alpha : X \times X \to \mathbb{C}$ is positive definite.*

Lemmas 4.2 and 4.3 now lead to the following central result of this paper:

Theorem 4.6. *Let $(D, *)$ be a commutative discrete hypergroup which is associated with some association scheme $\Lambda = (X, D, (R_i)_{i \in D})$. Let $\alpha, \beta \in S \subset \hat{D}$ be characters in the support of the Plancherel measure. Then $\alpha \cdot \beta$ is positive definite on D, and there exists a unique probability measure $\delta_\alpha \hat{*} \delta_\beta \in M^1(\hat{D})$ with $(\delta_\alpha \hat{*} \delta_\beta)^\vee = \alpha \cdot \beta$. The support of this measure is contained in S.*

Furthermore, for all $\alpha \in S$, the unique positive character α_0 in S according to 2.10(5) is contained in the support of $\delta_\alpha \hat{*} \delta_{\bar{\alpha}}$.

Proof. 4.5, 4.2, and 4.3 show that $\alpha \cdot \beta$ is positive definite. Bochner's theorem 2.10(1) now leads to the probability measure $\delta_\alpha \hat{*} \delta_\beta \in M^1(\hat{D})$. The assertions about the supports of $\delta_\alpha \hat{*} \delta_\beta$ and $\delta_\alpha \hat{*} \delta_{\bar{\alpha}}$ follow from Theorem 2.1(4) of [36]. $\square$

For finite association schemes we have a stronger result. It is shown in Section 2.10 of [4] in another, but finally equivalent way:

Theorem 4.7. *Let $(D, *)$ be a finite commutative hypergroup which is associated with some association scheme $\Lambda = (X, D, (R_i)_{i\in D})$. Then $(\hat{D}, \hat{*})$ is a hypergroup.*

Proof. For finite hypergroups we have $S = \hat{D}$, and the unique positive character in S is the identity $\mathbf{1}$. Therefore, if we take $\mathbf{1}$ as identity and complex conjugation as involution, we see that almost all hypergroup properties of $(\hat{D}, \hat{*})$ follow from Theorem 4.6. We only have to check that for $\alpha \neq \beta \in \hat{D}$, $\mathbf{1}$ is not contained in the support of $\delta_\alpha \hat{*} \delta_{\bar{\beta}}$. For this we recapitulate that for all $\gamma, \rho \in \hat{D}$, $\hat{\gamma}(\rho) = \int \gamma\bar{\rho}\, d\Omega = \|\gamma\|_2^2 \delta_{\gamma,\rho}$ with the Kronecker-δ. Therefore, with 12.16 of [20],

$$\begin{aligned}(\delta_\alpha \hat{*} \delta_{\bar{\beta}})(\{\mathbf{1}\}) &= \int_{\hat{D}} \mathbf{1}_{\{\mathbf{1}\}}\, d(\delta_\alpha \hat{*} \delta_{\bar{\beta}}) = \int_{\hat{D}} \hat{\mathbf{1}}\, d(\delta_\alpha \hat{*} \delta_{\bar{\beta}}) = \\ &= \int_D \mathbf{1}\, (\delta_\alpha \hat{*} \delta_{\bar{\beta}})^\vee\, d\Omega = \int_D \alpha\bar{\beta}\, d\Omega = \|\gamma\|_2^2 \delta_{\alpha,\beta} = 0.\end{aligned}$$

This completes the proof. □

We present some examples related to Gelfand pairs which show that the infinite case is more involved than in Theorem 4.7; for details see [37].

Example 4.8. Let $a, b \geq 2$ be integers. Let C_b the complete undirected graph graph with b vertices, i.e., all vertices of C_b are connected. We now consider the infinite graph $\Gamma := \Gamma(a, b)$ where precisely a copies of the graph C_b are tacked together at each vertex in a tree-like way, i.e., there are no other cycles in Γ than those in a copy of C_b. For $b = 2$, Γ is the homogeneous tree of valency a. We denote the distance function on Γ by d.

It is clear that the group $G := Aut(\Gamma)$ of all graph automorphisms acts on Γ in a distance-transitive way, i.e., for all $v_1, v_2, v_3, v_4 \in \Gamma$ with $d(v_1, v_3) = d(v_3, v_4)$ there exists $g \in \Gamma$ with $g(v_1) = v_3$ and $g(v_2) = v_4$. $Aut(\Gamma)$ is a totally disconnected, locally compact group w.r.t. the topology of pointwise convergence, and the stabilizer subgroup $H \subset G$ of any fixed vertex $e \in \Gamma$ is compact and open. We identify G/H with Γ, and $G//H$ with $\mathbb{N}_0$ by distance transitivity. We now study the association scheme $\Lambda = (\Gamma \simeq G/H, \mathbb{N}_0 = G//H, (R_i)_{i\in\mathbb{N}_0})$ and the double coset hypergroup $(\mathbb{N}_0 \simeq G//H, *)$. As in the case of finite distance-transitive graphs in [4], Λ and $(\mathbb{N}_0, *)$ are commutative and associated with a sequence of orthogonal polynomials in the Askey scheme [2] .

More precisely, by [34], the hypergroup convolution is given by

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n} g_{m,n,k}\delta_k \in M^1(\mathbb{N}_0) \qquad (m, n \in \mathbb{N}_0) \tag{4.1}$$

with

$$g_{m,n,m+n} = \frac{a-1}{a} > 0, \quad g_{m,n,|m-n|} = \frac{1}{a(a-1)^{m\wedge n-1}(b-1)^{m\wedge n}} > 0,$$

$$g_{m,n,|m-n|+2k+1} = \frac{b-2}{a(a-1)^{m\wedge n-k-1}(b-1)^{m\wedge n-k}} \geq 0 \quad (k = 0, \ldots, m \wedge n - 1),$$

$$g_{m,n,|m-n|+2k+2} = \frac{a-2}{(a-1)^{m\wedge n-k-1}(b-1)^{m\wedge n-k-1}} \geq 0 \quad (k=0,\ldots,m\wedge n-2).$$

The Haar weights are given by $\omega_0 := 1$, $\omega_n = a(a-1)^{n-1}(b-1)^n \quad (n \geq 1)$. Using

$$g_{n,1,n+1} = \frac{a-1}{a}, \quad g_{n,1,n} = \frac{b-2}{a(b-1)}, \quad g_{n,1,n-1} = \frac{1}{a(b-1)},$$

we define a sequence of orthogonal polynomials $(P_n^{(a,b)})_{n\geq 0}$ by

$$P_0^{(a,b)} := 1, \qquad P_1^{(a,b)}(x) := \frac{2}{a}\cdot\sqrt{\frac{a-1}{b-1}}\cdot x + \frac{b-2}{a(b-1)},$$

and the three-term-recurrence relation

$$P_1^{(a,b)}P_n^{(a,b)} = \frac{1}{a(b-1)}P_{n-1}^{(a,b)} + \frac{b-2}{a(b-1)}P_n^{(a,b)} + \frac{a-1}{a}P_{n+1}^{(a,b)} \qquad (n\geq 1). \tag{4.2}$$

Then,

$$P_m^{(a,b)}P_n^{(a,b)} = \sum\nolimits_{k=|m-n|}^{m+n} g_{m,n,k}P_k^{(a,b)} \quad (m,n\geq 0). \tag{4.3}$$

We also notice that the formulas above are correct for all $a,b\in[2,\infty[$, and that Eq. (4.1) then still defines commutative hypergroups $(\mathbb{N}_0,*)$.

We discuss some properties of the $P_n^{(a,b)}$ from [34], [37]. Eq. (4.2) yields

$$P_n^{(a,b)}\Big(\frac{z+z^{-1}}{2}\Big) = \frac{c(z)z^n + c(z^{-1})z^{-n}}{((a-1)(b-1))^{n/2}} \qquad \text{for } z\in\mathbb{C}\setminus\{0,\pm1\} \tag{4.4}$$

with

$$c(z) := \tfrac{(a-1)z-z^{-1}+(b-2)(a-1)^{1/2}(b-1)^{-1/2}}{a(z-z^{-1})}. \tag{4.5}$$

We define

$$s_0 := s_0^{(a,b)} := \frac{2-a-b}{2\sqrt{(a-1)(b-1)}}, \qquad s_1 := s_1^{(a,b)} := \frac{ab-a-b+2}{2\sqrt{(a-1)(b-1)}}. \tag{4.6}$$

Then

$$P_n^{(a,b)}(s_1) = 1, \qquad P_n^{(a,b)}(s_0) = (1-b)^{-n} \qquad (n\geq 0). \tag{4.7}$$

It is shown in [37] that the $P_n^{(a,b)}$ fit into the Askey-Wilson scheme (pp. 26–28 of [2]). By the orthogonality relations in [2], the normalized orthogonality measure $\rho=\rho^{(a,b)}\in M^1(\mathbb{R})$ is

$$d\rho^{(a,b)}(x) = w^{(a,b)}(x)dx\Big|_{[-1,1]} \qquad \text{for} \quad a\geq b\geq 2 \tag{4.8}$$

and

$$d\rho^{(a,b)}(x) = w^{(a,b)}(x)dx\Big|_{[-1,1]} + \frac{b-a}{b}d\delta_{s_0} \quad \text{for} \quad b>a\geq 2 \tag{4.9}$$

with

$$w^{(a,b)}(x) := \frac{a}{2\pi}\cdot\frac{(1-x^2)^{1/2}}{(s_1-x)(x-s_0)}.$$

For $a,b\in\mathbb{R}$ with $a,b\geq 2$, the numbers s_0,s_1 satisfy

$$-s_1\leq s_0\leq -1<1\leq s_1.$$

By Eq. (4.4), we have the dual space

$$\hat{D}\simeq\{x\in\mathbb{R}:\ (P_n^{(a,b)}(x))_{n\geq 0} \quad \text{is bounded}\} = [-s_1,s_1].$$

This interval contains the support $S := supp\, \rho^{(a,b)}$ of the orthogonality measure, which is also equal to the Plancherel measure. We have $S = \hat{D}$ precisely for $a = b = 2$. The following theorem from [37] shows that for these examples several phenomena appear, and that Theorem 4.6 cannot be improved considerably.

Theorem 4.9. *Let $a, b \geq 2$ be integers. Let G, H, Γ, and Λ be given as above. Then the following statements are equivalent for $x \in \mathbb{R}$:*

(1) $x \in [s_0^{(a,b)}, s_1^{(a,b)}]$.
(2) *The mapping $\Gamma \times \Gamma \to \mathbb{R}$, $(v_1, v_2) \longmapsto P_{d(v_1,v_2)}^{(a,b)}(x)$ is positive definite;*
(3) *The mapping $g \longmapsto P_{d(gH,e)}^{(a,b)}(x)$ is positive definite on G.*

Moreover, for all $x, y \in [s_0^{(a,b)}, s_1^{(a,b)}]$ there exists a unique probability measure $\mu_{x,y} \in M^1([-s_1^{(a,b)}, s_1^{(a,b)}])$ with

$$P_n^{(a,b)}(x) \cdot P_n^{(a,b)}(y) = \int_{-s_1^{(a,b)}}^{s_1^{(a,b)}} P_n^{(a,b)}(z)\, d\mu_{x,y}(z) \quad \textit{for all} \quad n \in \mathbb{N}_0. \tag{4.10}$$

Furthermore, for certain integers $b > a$ there exist $x, y \in [-s_1^{(a,b)}, s_0^{(a,b)}[$ for which no probability measure $\mu_{x,y} \in M^1(\mathbb{R})$ exists which satisfies (4.10).

Remark 4.10. The preceding examples show that for Gelfand pairs (G, H) with H a compact open subgroup, characters $\alpha \in (G//H)^\wedge$ of the double coset hypergroup may not correspond to a positive definite function on G and not to a positive definite kernel on G/H. On the other hand, as in the examples in parts (2) and (3) of Theorem 4.9, these positive definiteness conditions are equivalent in general. This well-known fact follows immediately from the definitions of positive definite functions on G and positive definite kernel on G/H.

5. Generalized Discrete Association Schemes

In this section we propose a generalization of the definition of association schemes from Section 3 via stochastic matrices. We focus on the commutative case and have applications in mind to dual product formulas as in Theorem 4.6.

One might expect at a first glance that for each discrete commutative hypergroup $(D, *)$ there is a natural kind of a generalized association scheme with $X = D$ and transition matrices S_i with $(S_i)_{j,k} := (\delta_i * \delta_j)(\{k\})$ for $i, j, k \in D$. We then would obtain $F_f(x, y) = f(x * y)$ for $x, y \in D$ and $f \in C(D)$ with the notations in Section 4. These matrices S_i and the functions F_f have much in common with the construction above. However, one central point is different in the view of dual product formulas: In Section 4 we used the fact that the R_i form partitions which led to the central identity $F_{fg} = F_f F_g$ for functions f, g on D. This is obviously usually not correct for $F_f(x, y) = f(x * y)$. In particular, such a simple approach would lead to a contradiction with Example 9.1C of Jewett [20].

Having this problem in mind, we propose a definition which skips the integrality condition and which keeps the partition condition. The integrality conditions will be replaced by the existence of a positive invariant measure on X which replaces the counting measure on X in Sections 3 and 4.

Definition 5.1. Let X, D be nonempty, at most countable sets and $(R_i)_{i\in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ for $i \in D$. Let $\tilde{S}_i \in \mathbb{R}^{X\times X}$ for $i \in D$ be stochastic matrices. Assume that:

(1) For all $i, j, k \in D$ and $(x, y) \in R_k$, the number
$$p_{i,j}^k := |\{z \in X : (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$$
is finite and independent of $(x, y) \in R_k$.

(2) For all $i \in D$ and $x, y \in X$, $\tilde{S}_i(x, y) > 0$ if and only if $(x, y) \in R_i$.

(3) For all $i, j, k \in D$ there exist (necessarily nonnegative) numbers $\tilde{p}_{i,j}^k$ with $\tilde{S}_i\tilde{S}_j = \sum_{k\in D} \tilde{p}_{i,j}^k \tilde{S}_k$.

(4) There exists an identity $e \in D$ with $\tilde{S}_e = I_X$ as identity matrix.

(5) There exists a positive measure $\pi \in M^+(X)$ with *supp* $\pi = X$ and an involution $i \mapsto \bar{i}$ on D such that for all $i \in D$, $x, y \in X$,
$$\pi(\{y\})\tilde{S}_{\bar{i}}(y, x) = \pi(\{x\})\tilde{S}_i(x, y).$$

Then $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ is called a *generalized association scheme.* Λ is called *commutative* if $\tilde{S}_i\tilde{S}_j = \tilde{S}_j\tilde{S}_i$ for all $i, j \in D$. It is called *symmetric* if the involution is the identity. Λ is called *finite*, if so are X and D.

Example 5.2. If $\Lambda = (X, D, (R_i)_{i\in D})$ is an unimodular association scheme as in Section 3, and if we take the stochastic matrices $(S_i)_{i\in D}$ as in Remark 3.4, then $(X, D, (R_i)_{i\in D}, (S_i)_{i\in D})$ is a generalized association scheme. In fact, axioms (1)-(4) are clear, and for axiom (5) we take the involution of Λ and π as the counting measure on X. Lemma 3.12(3) and unimodularity then imply axiom (5).

Lemma 5.3. *Let $(X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ be a generalized association scheme. Then:*

(1) *The triplet $(X, D, (R_i)_{i\in D})$ is an association scheme.*

(2) *The positive measure $\pi \in M^+(X)$ from axiom (5) is invariant, i.e., for all $y \in X$ and $i \in D$, $\sum_{x\in X} \pi(\{x\})\tilde{S}_i(x, y) = \pi(\{y\})$.*

(3) *The deformed intersection numbers $\tilde{p}_{i,j}^k$ satisfy $\sum_{k\in D} \tilde{p}_{i,j}^k = 1$ and*
$$\tilde{p}_{i,j}^k > 0 \quad \iff \quad p_{i,j}^k > 0 \qquad (i, j, k \in D).$$

(4) *For all $p \in [1, \infty[$ and $i \in D$, the transition operator*
$$T_{\tilde{S}_i} f(x) := \sum_{y\in X} \tilde{S}_i(x, y)f(y) \quad (x \in X)$$
is a continuous linear operator on $L^p(X, \pi)$ with $\|T_{\tilde{S}_i}\| \leq 1$. Moreover, for $p = 2$, the $T_{\tilde{S}_i}$ satisfy the adjoint relation $T_{\tilde{S}_{\bar{i}}} = T^*_{\tilde{S}_i}$.

Proof. Part (1) is clear; we only remark that the axiom regarding the involution on D follows from axioms (2) and (5) above.

For part (2) we use axiom (5) which implies for $i \in D$, $y \in X$ that
$$\sum_{x\in X} \pi(\{x\})S_i(x, y) = \pi(\{x\}) \sum_{x\in X} S_i(y, x) = \pi(\{y\}).$$
Part (3) and the adjoint relation in (4) are obvious. Moreover, for $p \geq 1$, the invariance of π w.r.t. S_i and Hölder's inequality easily imply $\|T_{\tilde{S}_i}\| \leq 1$. $\square$

In summary, we may regard generalized association schemes as deformations of given classical association schemes with deformed intersection numbers $\tilde{p}^k_{i,j}$.

Generalized association schemes also lead to discrete hypergroups:

Proposition 5.4. *Let $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ be a generalized association scheme with deformed intersection numbers $\tilde{p}^k_{i,j}$. Then the product $\tilde{*}$ with*

$$\delta_i \tilde{*} \delta_j := \textstyle\sum_{k\in D} \tilde{p}^k_{i,j} \delta_k$$

can be extended uniquely to an associative, bilinear, $\| \,.\, \|_{TV}$-continuous mapping on $M_b(D)$. $(D, \tilde{*})$ is a discrete hypergroup with the left and right Haar measure

$$\Omega_l := \textstyle\sum_{i\in D} \omega_i \delta_i \quad \textit{and} \quad \Omega_r := \sum_{i\in D} \omega_{\bar{i}} \delta_i \quad \textit{with} \quad \omega_i := \frac{1}{\tilde{p}^e_{i,\bar{i}}} > 0$$

respectively. This hypergroup is commutative or symmetric if and only if so is Λ.

Proof. As the product of matrices is associative, the convolution $\tilde{*}$ is associative. Moreover, by Lemma 5.3(3), $\delta_i \tilde{*} \delta_j$ is a probability measure for $i, j \in D$ whose support is the same as for the hypergroup convolution of the association scheme $(X, D, (R_i)_{i\in D})$ in Proposition 3.8. This readily shows that $(D, \tilde{*})$ is a hypergroup.

Clearly, $\tilde{*}$ is commutative or symmetric if and only if so is Λ. Finally, the statement about the Haar measures follows from Example 2.8(1). □

We now extend the approach of Section 4 to dual positive product formulas for discrete commutative hypergroups $(D, \tilde{*})$ which are associated with generalized commutative association schemes $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$.

For this we study the linear span $A(X)$ of the matrices $\tilde{S}_i$ $(i \in D)$. If we identify the $\tilde{S}_i$ with the transition operators $T_{\tilde{S}_i}$, we may regard $A(X)$ as a commutative $*$-subalgebra of the C^*-algebra $\mathcal{B}(L^2(X, \pi))$ of all bounded linear operators on $L^2(X, \pi)$ by Lemma 5.3(4). As in Section 4, $A(X)$ is isomorphic with the $*$-algebra $(M_f(D), \tilde{*})$ of all measures with finite support via

$$\mu \in M_f(D) \longleftrightarrow \tilde{S}_\mu := \textstyle\sum_{i\in D} \mu(\{i\}) \tilde{S}_i \in A(X).$$

Moreover, using the Haar measure Ω of $(D, \tilde{*})$, $A(X)$ is also isomorphic with the commutative $*$-algebra $(C_c(D), \tilde{*})$ via

$$f \in C_c(D) \longleftrightarrow \tilde{S}_f := \textstyle\sum_{i\in D} f(i) \omega_i \tilde{S}_i \in A(X).$$

As in Section 4, we have $\tilde{S}_f = \tilde{S}_{f\Omega}$ for $f \in C_c(D)$.

We even may define the matrices $\tilde{S}_f := \sum_{i\in D} f(i)\omega_i \tilde{S}_i$ for $f \in C_b(D)$, and we may form matrix products of $\tilde{S}_f$ with matrices with only finitely many nonzero entries in all rows and columns. In particular, we may multiply $\tilde{S}_f$ on both sides with any $\tilde{S}_\mu$, $\mu \in M_f(D)$. We need the following notion of positive definiteness:

Definition 5.5. Let $A \in \mathbb{C}^{X\times X}$ be a matrix. Then for all $g_1, g_2 \in C_f(X) := \{g : X \to \mathbb{C} \quad \text{with finite support}\}$, we may form

$$\langle Ag_1, g_2\rangle_\pi := \textstyle\sum_{x\in X} (Ag_1)(x) \cdot \overline{g_2(x)} \cdot \pi(\{x\}) \in \mathbb{C}.$$

We say that A is *π-positive definite*, if $\langle Ag, g\rangle_\pi \geq 0$ for all $g \in C_f(X)$. This is obviously equivalent to the fact that the matrix $(\pi(\{x\}) \cdot A_{x,y})_{x,y\in X}$ is positive semidefinite in the usual way.

For each $f \in C_b(D)$ we define the function $F_f : X \times X \to \mathbb{C}$ with $F_f(x,y) = f(i)$ for the unique $i \in D$ with $(x,y) \in R_i$ as in Section 4. Then different from Section 4, we usually have $(F_f(x,y))_{x,y\in X} \neq \tilde{S}_f$. We thus have to state some results from Section 4 for the matrices $\tilde{S}_f$ instead of $(F_f(x,y))_{x,y\in X}$.

Lemmas 4.3 and 4.4 now read as follows:

Lemma 5.6. *Let $f \in C_b(D)$. If $\tilde{S}_f$ is π-positive definite, then f is positive definite on the hypergroup $(D, \tilde{*})$.*

Proof. For f define $f^-(x) := f(\bar{x})$ for $x \in D$. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ and $c_1, \ldots, c_n \in \mathbb{C}$. Let $\mu := \sum_{k=1}^n c_k \delta_{x_k} \in M_f(D)$ and observe that as in the proof of Lemma 4.3, $P := \sum_{k,l=1}^n c_k \bar{c}_l \cdot f^-(x_k \tilde{*} \bar{x}_l) = (\mu \tilde{*} f \tilde{*} \mu^*)(e)$ where this is equal to the diagonal entries of

$$\tilde{S}_{\mu\tilde{*}f\tilde{*}\mu^*} = \tilde{S}_\mu \cdot \tilde{S}_f \cdot \tilde{S}_\mu^{\ *}.$$

This matrix product exists and is π-positive semidefinite. Therefore, by 5.5, the diagonal entries of this matrix are nonnegative and thus $P \geq 0$. Hence, f^- and thus f is positive definite. □

Lemma 5.7. *Let $f \in C_c(D)$. Then, by 2.10(2), $f \tilde{*} f^*$ is positive definite on $(D, \tilde{*})$, and $S_{f\tilde{*}f^*}$ is π-positive definite.*

Proof. By our considerations above, $S_{f\tilde{*}f^*} = \tilde{S}_f \tilde{S}_f^{\ *}$. This proves the claim. □

We now combine Lemma 5.7 with 2.10(4) and conclude as in Corollary 4.5:

Corollary 5.8. *Let $\alpha \in (D, \tilde{*})^\wedge$ be a character in the support of the Plancherel measure. Then $\tilde{S}_\alpha$ is π-positive definite.*

We now want to combine Corollary 5.8 with Lemmas 4.2 and 5.6 to derive an extension of Theorem 4.6. Here, however we would need $(F_f(x,y))_{x,y\in X} = \tilde{S}_f$. We can overcome this problem with some additional condition which relates the characters of $(D, \tilde{*})$ with the characters of the commutative hypergroup $(D, *)$ which is associated with the association scheme $(X, D, (R_i)_{i\in D})$:

Definition 5.9. Let $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ be a generalized commutative association scheme with the associated hypergroups $(D, \tilde{*})$ and $(D, *)$ as above. We say that $\alpha \in (D, \tilde{*})^\wedge$ has the *positive connection property* if α is positive definite on $(D, *)$, and if the associated kernel $F_\alpha : X \times X \to \mathbb{C}$ is positive semidefinite.

We say that Λ has the *positive connection property*, if all $\alpha \in (D, \tilde{*})^\wedge$ have the positive connection property.

To understand this condition, consider a finite generalized commutative association scheme Λ. If a character $\alpha \in (D, \tilde{*})^\wedge$ can be written as a nonnegative linear combination of the characters of $(D, *)$, then α is positive definite on $(D, *)$, and by Corollary 4.5, the associated kernel F_α is positive semidefinite.

In this way, the positive connection property of Λ roughly means that all $\alpha \in (D, \tilde{*})^\wedge$ admit nonnegative integral representations w.r.t. the characters of $(D, *)$. Such positive integral representations are well known for many families of special functions and often easier to prove than positive product formulas.

We now use the positive connection property and obtain the following extension of Theorem 4.6:

Theorem 5.10. *Let $(D, \tilde{*})$ be a commutative discrete hypergroup which corresponds to the generalized association scheme $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$. Let $\alpha, \beta \in (D, \tilde{*})^\wedge$ be characters such that α is in the support S of the Plancherel measure of $(D, \tilde{*})$ and β has the positive connection property. Then $\alpha \cdot \beta$ is positive definite on $(D, \tilde{*})$, and there is a unique probability measure $\delta_\alpha \hat{\tilde{*}} \delta_\beta \in M^1((D, \tilde{*})^\wedge)$ with $(\delta_\alpha \hat{\tilde{*}} \delta_\beta)^\vee = \alpha \cdot \beta$. The support of this measure is contained in S.*

Proof. By Corollary 5.8, $\tilde{S}_\alpha$ is π-positive definite, i.e., $(\pi(\{x\})(\tilde{S}_\alpha)_{x,y})_{x,y\in X}$ is a positive semidefinite matrix. Furthermore, as β has the positive connection property, $(F_\beta(x,y))_{x,y\in X}$ is also positive semidefinite. We conclude from Lemma 4.2 that the pointwise product $(\pi(\{x\})(\tilde{S}_\alpha)_{x,y}F_\beta(x,y))_{x,y\in X}$ is also positive semidefinite. On the other hand, for all $x, y \in X$ and $i \in D$ with $(x,y) \in R_i$, we have

$$(\tilde{S}_\alpha)_{x,y}F_\beta(x,y) = \beta(i)\alpha(i)\omega_i(\tilde{S}_i)_{x,y} = (\tilde{S}_{\alpha\beta})_{x,y}.$$

Therefore, $\tilde{S}_{\alpha\beta}$ is π-positive definite, and, by Lemma 5.6, $\alpha\beta$ is positive definite on $(D, \tilde{*})$. As in the proof of Theorem 4.6, Bochner's theorem 2.10(1) together with Theorem 2.1(4) of [36] lead to the theorem. □

We remark that Theorem 4.7 can be also established for finite generalized commutative association schemes similar to Theorem 5.10.

We complete the paper with an example.

Example 5.11. Consider the group $(\mathbb{Z}, +)$, on which the group $H := \mathbb{Z}_2 = \{\pm 1\}$ acts multiplicatively as group of automorphisms. Consider the semidirect product $G := \mathbb{Z} \rtimes \mathbb{Z}_2$ with $H := \mathbb{Z}_2$ as finite subgroup. We then consider $X := G/H = \mathbb{Z}$ and $D := G//H = \mathbb{N}_0$ (with the canonical identifications) as well as the corresponding commutative association scheme $(\mathbb{Z}, \mathbb{N}_0, (R_k)_{k\in\mathbb{N}_0})$ with the associated double coset hypergroup $(\mathbb{N}_0, *)$ and the associated transition matrices

$$S_0 = I_{\mathbb{Z}}, \quad S_k(x,y) = \frac{1}{2}\delta_{k,|x-y|} \quad (k \in \mathbb{N},\ x, y \in \mathbb{Z})$$

with the Kronecker-δ.

Now fix some parameter $r > 0$ and put $p := e^r/(e^r + e^{-r}) \in]0,1[$. Define the deformed stochastic matrices

$$\tilde{S}_0 = I_{\mathbb{Z}}, \quad \tilde{S}_k(x,y) = \frac{1}{p^k + (1-p)^k}\Big(p^k\delta_{k,y-x} + (1-p)^k\delta_{k,x-y}\Big)$$

for $k \in \mathbb{N}_0$, $x, y \in \mathbb{Z}$. It can be easily checked that for $k \in \mathbb{N}$,

$$\begin{aligned}\tilde{S}_k\tilde{S}_1 &= \frac{p^{k+1} + (1-p)^{k+1}}{p^k + (1-p)^k}\tilde{S}_{k+1} + \Big(1 - \frac{p^{k+1} + (1-p)^{k+1}}{p^k + (1-p)^k}\Big)\tilde{S}_{k-1} \\ &= \frac{\cosh((k+1)r)}{2\cosh(kr)\cosh(r)}\tilde{S}_{k+1} + \frac{\cosh((k-1)r)}{2\cosh(kr)\cosh(r)}\tilde{S}_{k-1}. \qquad (5.1)\end{aligned}$$

Induction yields that for $k, l \in \mathbb{N}_0$,

$$\tilde{S}_k\tilde{S}_l = \frac{\cosh((k+l)r)}{2\cosh(kr)\cosh(lr)}\tilde{S}_{k+l} + \frac{\cosh((k-l)r)}{2\cosh(kr)\cosh(lr)}\tilde{S}_{|k-l|}. \qquad (5.2)$$

We thus obtain the axioms (1)-(4) of 5.1. Moreover, with the measure $\pi(\{x\}) := \left(\frac{p}{1-p}\right)^x = e^{2rx}$ $(x \in \mathbb{Z})$ and the identity as involution on $\mathbb{N}_0$ we also obtain axiom

5.1(5). We conclude from (5.2) that the associated hypergroup $(\mathbb{N}_0, \tilde{*})$ is the so called discrete cosh-hypergroup; see [40] and 3.4.7 and 3.5.72 of [7]. The characters of $(\mathbb{N}_0, \tilde{*})$ are given by

$$\alpha_\lambda(n) := \frac{\cos(\lambda n)}{\cosh(rn)} \quad (n \in \mathbb{N}_0,\ \ \lambda \in [0,\pi] \cup i\cdot[0,r] \cup \{\pi + iz:\ z \in [0,r]\})$$

where in this parameterization, α_λ is in the support S of the Plancherel measure precisely for $\lambda \in [0,\pi]$. Using

$$\frac{\cos(\lambda n)}{\cosh(rn)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(trn)}{\cosh((t+\lambda/r)\pi/2)}\, dt \quad \text{for} \quad \lambda \in \mathbb{C},\ |\Im\lambda| < r$$

(see (1) in [40] and references there) and degenerated formulas for $\lambda = ir, \pi + ir$, we see readily that each character of $(\mathbb{N}_0, \tilde{*})$ has a positive integral representation w.r.t. characters of $(\mathbb{N}_0, *)$. As for the hypergroup $(\mathbb{N}_0, *)$, the support of the Plancherel measure is equal to $(\mathbb{N}_0, *)^\wedge$, we conclude that the generalized association scheme $(\mathbb{Z}, \mathbb{N}_0, (R_i)_{k\in\mathbb{N}_0}, (\tilde{S}_k)_{k\in\mathbb{N}_0})$ has the positive connection property. We thus may apply Theorem 5.10. The associated dual convolution can be determined explicitely similar to [40].

Notice that the hypergroups $(\mathbb{N}_0, *)$ and $(\mathbb{N}_0, \tilde{*})$ are related by a deformation of the convolution via some positive character as described in [33].

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