A NOTE ON NONSMOOTH PARTIAL VECTOR INVEXITY IN n-SET OPTIMIZATION

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***Abstract*:** Sufficient optimality conditions and duality results for a class of minmax programming problems involving n-set functions using the notion of generalized vector invexity for nonsmooth n-set functions are established.

Key words: Minmax Programming, V-invexity, Duality results.

1. INTRODUCTION

Minmax programming problems arise frequently in many areas particularly in game theory and facility location problems. A concrete theory for minmax programming problems was developed by Damyanov and Malozemov [6]. Later, Bector and Chandra [2] presented duality results for minmax programs involving pseudolinear functions. Notion of V-invexity was first introduced by Jayakumar and Mond [7] and was successfully applied by Bector, Chandra and Kumar [3] to establish sufficient optimality conditions and duality results for minmax programs.

On the other hand, optimization problems involving n-set functions was extensively studied in the past and a rich literature is available on optimality conditions and duality results for such problems. For such problems one can refer to [1,5,8,9,10] for more details. Recently, Bhatia and Kumar [4] introduced the notions of invexity and its generalizations for n-set functions and established duality results for minmax programming problem involving invex functions. In this note, we define the notion of generalized V-invexity for nonsmooth n-set functions. Sufficient optimality conditions and duality results are derived for the following minmax program (P) with generalized V-invexity assumptions on the functions involved:

(P) $$\min_{S} \max_{1 \leqq j \leqq p} F_j(S)$$

subject to

$$H_k(S) \leqq 0,\ k \in K = \{1,2,\ldots,m\}$$

$$S = (S_1, S_2, \ldots, S_n) \in \boldsymbol{A}^n$$

where $\boldsymbol{A}^n$ is the n-fold product of σ-algebra $\boldsymbol{A}$ of subsets of a given set X F_j, $j \in J = \{1,2,\ldots,p\}$ and H_k, $k \in K$ are real-valued n-set nonsmooth functions defined on $\boldsymbol{A}^n$.

Throughout the paper, we assume that $(X,\boldsymbol{A},\mu)$ is a finite atom less measure space with $L_1(X,\boldsymbol{A},\mu)$ separable. $\boldsymbol{A}^n$ is n-fold product of a σ-algebra $\boldsymbol{A}$ of subsets of set X, a pseudometric d on $\boldsymbol{A}^n$ is defined by

$$d(S, T) = \left(\sum_{i=1}^{n} \mu(S_i \Delta T_i)^2\right)^{1/2}$$

where $S = (S_1, S_2, \ldots, S_n) \in \boldsymbol{A^n}$, $T = (T_1, T_2, \ldots, T_n) \in \boldsymbol{A^n}$, and $S_i \Delta T_i$ denote symmetric difference of sets S_i and T_i. For $h \in L_1(X, \mathcal{A}, \mu)$ and $S_i \in \mathcal{A}$, the integral $\int_{S_i} h\, d\mu$ will be denoted by $\langle h, \chi_{S_i} \rangle$, where χ_{S_i} is the characteristic function of S_i.

2. NONSMOOTH V-INVEXITY

In this section, we first recall some preliminary results which are well established in the literature. Some observations are made in the results. These observations motivated us to define the notion of V-invexity and generalized V-invexity for nonsmooth n-set functions. Imposing the conditions of generalized V-invexity on few of the functions, duality results will be developed in the later section.

We first define the subdifferential for a nonsmooth n-set function.

Definition 2.1 For a real-valued n-set function $F : \boldsymbol{A^n} \to \mathbf{R}$, not differentiable at $\overline{S} \in \boldsymbol{A^n}$, the subdifferential of F at $\overline{S}$ is defined as

$$\partial F(\overline{S}) = \{ f \in L_1^n(X, \boldsymbol{A^n}, \mu) : F(S) - F(\overline{S}) \geq \langle f, \chi_S - \chi_{\overline{S}} \rangle \}$$

If the set $\partial F(\overline{S})$ is nonempty then we say F is subdifferentiable at $\overline{S}$.

It is evident that if $\overline{S}$ solves an optimization problem min F(S) then $0 \in \partial F(\overline{S})$. Calculus properties of the subdifferential of a nonsmooth convex n-set function F have been studied in the past. These properties are in turn extensively used to study nonsmooth n-set optimization problems. The aim of this short note is to study minmax optimization problem (P) under more relaxed assumptions of V-invexity. In this context it may be emphasized that the purpose is not merely to generalize the duality results obtained in [4] but rather to establish primal-dual relationship under V-invexity conditions only on very few functions involved in the problem (P). The note may be treated as supplement to the basic ideas of [4].

To achieve the desire results, we associate another optimization problem (EP) with (P) as follows

(EP) min q

subject to $F_j(S) \leq q$, $j \in J$

$H_k(S) \leq 0$, $k \in K$

$S \in \boldsymbol{A^n}$

Equivalence between the problems (P) and (EP) are well known. We recall here the corresponding results for the sake of completeness.

Lemma 2.1 [12] If $\bar{S}$ is an optimal solution of (P) then $(\bar{S}, \bar{q})$ with $\bar{q} = \max_{1 \leqq j \leqq p} F_j(\bar{S})$ is an optimal solution of (EP). Conversely, if $(\bar{S}, \bar{q})$ is an optimal solution of (EP) then $\bar{S}$ is an optimal solution of (P).

In view of the above Lemma, studying the problem (P) and characterizing its optimal solution is equivalent to studying the problem (EP) and characterizing its optimal solution. Hence, from now onwards, we concentrate on the problem (EP).

Following theorem, providing necessary conditions for the existence of an optimal solution of (P) or (EP), follows immediately from the above discussion.

Theorem 2.1 Let $\bar{S}$ be an optimal solution of (P). Then $\exists\ \bar{\lambda} \in \mathbf{R}^p$ and $\bar{\mu} \in \mathbf{R}^m$, $\bar{q} \in \mathbf{R}_+$ with $\bar{q} = \max_{1 \leqq j \leqq p} F_j(\bar{S})$ such that

$$0 \in \sum_{j=1}^{p} \bar{\lambda}_j \partial F_j(\bar{S}) + \sum_{k=1}^{p} \bar{\mu}_k \partial H_k(\bar{S}) \tag{1}$$

$$\bar{\lambda}_j (F_j(\bar{S}) - \bar{q}) = 0, \ j \in J \tag{2}$$

$$\bar{\mu}_k H_k(\bar{S}) = 0, \ k \in K \tag{3}$$

$$\bar{\lambda} \geqq 0, \ \bar{\mu} \geqq 0, (\bar{\lambda}, \bar{\mu}) \neq 0.$$

If an appropriate constraint qualification or regularity condition holds for the problem (EP), then we can take $\sum_{j=1}^{p} \bar{\lambda}_j = 1$.

Also, observe from the complementarity conditions (2) and (3), if $J(\bar{S}) \cup K(\bar{S})$ denotes the set of active constraints of (EP) at $\bar{S}$ then

for $j \notin J(\bar{S}) = \{ j \in J : F_j(\bar{S}) = \max_{1 \leqq j \leqq p} F_j(\bar{S}) \}$, $\bar{\lambda}_j = 0$

and for $k \notin K(\bar{S}) = \{ k \in K : H_k(\bar{S}) = 0 \}$, $\bar{\mu}_k = 0$.

So, (1) can be rewritten as

$$0 \in \sum_{j \in J(\bar{S})} \bar{\lambda}_j \partial F_j(\bar{S}) + \sum_{k \in K(\bar{S})} \bar{\mu}_k \partial H_k(\bar{S}) \tag{4}$$

Moreover, note that for $j \in J(\overline{S})$, we may have $\overline{\lambda}_j = 0$ and for $k \in K(\overline{S})$, we may have $\overline{\mu}_k = 0$. Defining two sets

$$J_1(\overline{S}) = \{ j \in J(\overline{S}) : \overline{\lambda}_j > 0 \}, \quad K_1(\overline{S}) = \{ k \in K(\overline{S}) : \overline{\mu}_k > 0 \},$$

we can write (4) as

$$0 \in \sum_{j \in J_1(\overline{S})} \overline{\lambda}_j \partial F_j(\overline{S}) + \sum_{k \in K_1(\overline{S})} \overline{\mu}_k \partial H_k(\overline{S}).$$

Thus we conclude the following:

Theorem 2.2 Let $\overline{S}$ be an optimal solution of (P) and appropriate regularity condition hold for (EP) then there exist $\overline{\lambda} \in \mathbf{R}^p$ and $\overline{\mu} \in \mathbf{R}^m$ such that

$$0 \in \sum_{j \in J_1(\overline{S})} \overline{\lambda}_j \partial F_j(\overline{S}) + \sum_{k \in K_1(\overline{S})} \overline{\mu}_k \partial H_k(\overline{S})$$

$$\overline{\lambda}_j > 0, \ \overline{\mu}_k > 0$$

where $J_1(\overline{S}) = \{ j \in J(\overline{S}) : \overline{\lambda}_j > 0\}, \ K_1(\overline{S}) = \{ k \in K(\overline{S}) : \overline{\mu}_k > 0\}.$

Regularity condition on (EP) at $\overline{S}$ ensures the non emptiness of the set $J_1(\overline{S})$.

The above necessary conditions for the problem (P) motivate us to impose V-invexity conditions on few of the functions of the problem (P) unlike [4] where the authors discussed duality results by imposing condition of invexity on all the n-set functions involved in (P).

Another fact that we may observe is that while establishing optimality conditions and duality results for the problem (P) under V-invexity assumptions one actually does not require the explicit knowledge of the functions η or θ. For given points S, T $\in \boldsymbol{A}^n$, existence of vectors η and θ (depending on S and T) satisfying the appropriate inequality is sufficient to ensure the optimality conditions and duality results.

In view of the above observations, we first define the notion of V-invexity for nonsmooth n-set functions.

If $G : \boldsymbol{A^n} \rightarrow \mathbf{R^r}$ is a nonsmooth vector function, say, $G = (G_1, G_2, \ldots, G_r)$, then for each $j \in J$, $\partial G_j(\overline{S}) \subset L_1^n(X, \boldsymbol{A^n}, \mu)$ and we define $\partial G(\overline{S}) = \partial G_1(\overline{S}) \times \partial G_2(\overline{S}) \times \ldots \times \partial G_r(\overline{S})$. Thus, each element of $\partial G(\overline{S})$ is a matrix $M = (\xi_{ij})$ of the order $n \times r$ in which the elements of the j^{th} column are the elements of the set $\partial G_j(\overline{S})$.

i.e., for $\xi_j \in \partial G_j(\overline{S}) = \{\xi \in L_1^n(X, \boldsymbol{A^n}, \mu) : G_j(T) - G_j(\overline{S}) \geqq \langle f, \chi_T - \chi_{\overline{S}} \rangle\}$

writing $\xi_j = (\xi_{1j}, \xi_{2j}, \ldots, \xi_{nj})^t \in \partial G_j(\overline{S})$.

Definition 2.2 A vector function $G : \boldsymbol{A^n} \rightarrow \mathbf{R^r}$ is said to be V-pseudoinvex at $\overline{S}$ if for every $T \in \boldsymbol{A^n}$ $\exists$ vectors $\eta \in \mathbf{R^n}$ and $\theta \in \text{int}(\mathbf{R}_+^r)$, depending on T, such that

$$\sum_{j=1}^{r}\Big(\sum_{i=1}^{n} \eta_i \left\langle \xi_{ij}, \chi_{T_i} - \chi_{\overline{S}_i} \right\rangle\Big) \geqq 0 \;\Rightarrow\; \sum_{j=1}^{r}\theta_j\, G_j(T) \geqq \sum_{j=1}^{r}\theta_j\, G_j(\overline{S}),$$

for some $(\xi_{ij}) \in \partial G(\overline{S})$

Definition 2.3 A vector function $G : \boldsymbol{A^n} \rightarrow \mathbf{R^r}$ is said to be V-quasiinvex at $\overline{S}$ if for every $T \in \boldsymbol{A^n}$ there exist vectors $\eta \in \mathbf{R^n}$ and $\theta \in \text{int}(\mathbf{R}_+^r)$, depending on T, such that

$$\sum_{j=1}^{r}\theta_j\, G_j(T) \leqq \sum_{j=1}^{r}\theta_j\, G_j(\overline{S})$$

$$\Rightarrow \sum_{j=1}^{r}\Big(\sum_{i=1}^{n}\big(\eta_i \left\langle \xi_{ij}, \chi_{T_i} - \chi_{\overline{S}_i} \right\rangle\big)\Big) \leqq 0 \;\; \forall\; (\xi_{ij}) \in \partial G(\overline{S}).$$

If for every $T \in \boldsymbol{A^n}$, vectors $\eta = \theta = e \in \mathbf{R^r}$ then the above definitions reduces to that of pseudoinvexity and quasiinvexity, respectively, for the nonsmooth case.

Observe that in the above two definitions we have assumed that for every given $T \in \boldsymbol{A^n}$ we can find $\eta \in \mathbf{R^n}$ and $\theta \in \text{int}(\mathbf{R}_+^r)$ satisfying the appropriate inequalities. With change in T, values of η and θ obviously change. So, both vectors η and θ are functions of T. However, we need not know the explicit expressions formulas for the two functions η and θ in the above definitions.

3. DUALITY

We now associate a Mond-Weir type dual (D) to the problem (P) and establish duality results under generalized V-invexity assumptions on few of the functions.

(D) Max ξ

subject to

$$0 \in \sum_{j=1}^{p} \lambda_j \, \partial F_j(T) + \sum_{k=1}^{m} \mu_k \, \partial H_k(T) \tag{5}$$

$$\lambda_j (F_j(T) - \xi) \geq 0, \; j \in J \tag{6}$$

$$\mu_k H_k(T) \geq 0, \; k \in K \tag{7}$$

$$T \in \mathcal{A}^n, \; \lambda \in \mathbf{R}_+^p, \; \lambda^t e = 1, \; \mu \in \mathbf{R}_+^m, \; \xi \in \mathbf{R}_+ \tag{8}$$

Theorem 3.1(Weak Duality) Let S be feasible for (P) and $q = \max_{1 \leq j \leq p} F_j(S)$. Let (T, ξ, λ, μ) be feasible for (D). Further, let $(\lambda_{J_1} F_{J_1})$ be V-pseudoinvex and $(\mu_{K_1} H_{K_1})$ be V-quasiinvex at T with respect to a common vector η, where $J_1 \equiv J_1(T)$ and $K_1 \equiv K_1(T)$. Then $q \geq \xi$.

Proof. Note that $J_1 \equiv J_1(T) = \{ j : \lambda_j > 0 \}$

$$K_1 \equiv K_1(T) = \{ k : \mu_k > 0 \}.$$

So, from (5) it follows that $\exists \; \xi_j \in \partial F_j(T)$, $j \in J_1$ and $\tilde{\xi}_k \in \partial H_k(T)$, $k \in K_1$ such that

$$0 = \sum_{j \in J_1} \lambda_j \, \xi_j + \sum_{k \in K_1} \mu_k \, \tilde{\xi}_k \tag{9}$$

We now prove the result by contradiction. Let $q < \xi$.
Then from (2), (6) and (8), we get

$$\lambda_j F_j(S) < \lambda_j F_j(T), \; j \in J_1$$

$\Rightarrow \sum_{j \in J_1} \theta_j \lambda_j F_j(S) < \sum_{j \in J_1} \theta_j \lambda_j F_j(T)$, $\theta_j \geq 0$, $j \in J_1$ with at least one $\theta_j > 0$.

From V-pseudoinvexity assumptions it follows that

$$\sum_{j \in J_1} \sum_{i=1}^{n} \eta_i \left\langle \lambda_j \, \xi_{ij}, \; \chi_{S_i} - \chi_{T_i} \right\rangle < 0 \tag{10}$$

$$\xi_j = (\xi_{1j}, \xi_{2j}, \ldots, \xi_{nj})^t \in \partial F_j(T),\ j \in J_1 .$$

Also, from (3) and (7) we have

$$\mu_k H_k(S) \leq \mu_k H_k(T),\ k \in K_1$$

which along with V-quasiinvexity of $(\mu_{K_1} H_{K_1})$ at T implies

$$\sum_{k \in K_1} \left(\sum_{i=1}^{n} \eta_i \left(\left\langle \mu_k \tilde{\xi}_{ik}, \chi_{S_i} - \chi_{T_i} \right\rangle\right)\right) \leq 0 \tag{11}$$

Contradiction follows from (10), (11) and (9).

Hence $q \geq \xi$.

Theorem 3.2 (Strong Duality) Let $\bar{S}$ be an optimal solution of (P) and regularity condition holds for the problem (EP). Then there exist $\bar{\lambda} \in \mathbf{R}^{\mathbf{p}}_{+}$, $\lambda^t e = 1$, $\bar{\mu} \in \mathbf{R}^{\mathbf{m}}_{+}$, $\bar{q} \in \mathbf{R}_{+}$ such that $(\bar{S}, \bar{q}, \bar{\lambda}, \bar{\mu})$ is feasible for (D). Further, if the conditions of Weak Duality theorem hold then $(\bar{S}, \bar{q}, \bar{\lambda}, \bar{\mu})$ is an optimal solution of (D) and $\max\limits_{1 \leqq j \leqq p} F_j(\bar{S}) = \bar{q}$.

Proof. By Theorem 2.1 there exist $\bar{\lambda} \in \mathbf{R}^{\mathbf{p}}$ and $\bar{\mu} \in \mathbf{R}^{\mathbf{m}}$, $\bar{q} \in \mathbf{R}_{+}$ with $\bar{q} = \max\limits_{1 \leqq j \leqq p} F_j(\bar{S})$ such that

$$0 \in \sum_{j=1}^{p} \bar{\lambda}_j \partial F_j(\bar{S}) + \sum_{k=1}^{p} \bar{\mu}_k \partial H_k(\bar{S})$$

$$\bar{\lambda}_j (F_j(\bar{S}) - \bar{q}) = 0, \quad j \in J$$

$$\bar{\mu}_k H_k(\bar{S}) = 0, \qquad k \in K$$

$$\bar{\lambda} \geqq 0,\ \bar{\mu} \geqq 0,\ (\bar{\lambda}, \bar{\mu}) \neq 0.$$

Further, as regularity condition hold for the problem (EP), we have $\sum\limits_{j=1}^{p} \bar{\lambda}_j = 1.$

This shows that $(\bar{S}, \bar{q}, \bar{\lambda}, \bar{\mu})$ is feasible of (D). Optimality of $(\bar{S}, \bar{q}, \bar{\lambda}, \bar{\mu})$ follows from the Weak Duality theorem.

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