FREE SEMIGROUP IN H_S

A.M. Aminpour

ABSTRACT: *This paper presents an important new technique for study of a particular compact right topological semigroup H_S, an important semigroup which is constracted as the semigroup in the Stone-Cech compactification of a particular partial semigroup, S called an oid (see [8]). principal result is that H_S contains a copy of the free semigroups on 2^c generators. Also we conclude by establishing some properties (both algebraic and topological) about the structure of H_S.*

Keywords: *Commutative oid, Compact right topological semigroup; Free semigroup; Cardinal function; Support.*

2000 Mathematics subject classification: *22A15; 22A20.*

1. INTRODUCTION

We shall present our theory in a fairly concrete setting so that our methods and results will be more readily accessible. Throughout this paper we will let S be a commutative oid. A more detailed analysis of *commutative* oids can be found in [8]. Also we assume that S is discrete, so that the points of βS the Stone-Cech compactifcation of S can be considered as the principal ultrafilters on S. The known theory for an oid S shows that how to find a subset H_S of βS which is a compact right topological semigroup (that is, if whenever $\eta \in H_S$ and $\xi_\alpha \to \xi$ in H_S, then $\xi_\alpha\eta \to \xi\eta$ in H_S) [8]. The main purpose of this paper is to show that H_S contains copies of the free semigroups (Theorem 3.9). To prove that for each n, H_n (see Section 2 for precise definition) generates a free semigroup in H_S is an immediate consequence of the stronger result that its contains a cancellative semigroup (Theorems 3.4, 3.8). Also we find (Theorem 3.13) that H_S contains a copy of the free semigroups on 2^c generators.

We conclude by establishing some properties of H_S, for example $(H_S)^2 \neq H_S$ (Theorem 3.22), $(l^\beta)^{-1}(\infty) \cap H_S$ and H_∞ are not nowhere dense in H_S (Theorems 3.24, 3.25).

2. DEFINITION AND PRELIMINARIES

Let $\left(\mathrm{A}_1\right)_{i=1}^{\infty}$ be a sequence of sets such that for each $i \in N$, A_i contains distinguished element 1 and will be supposed to have at least two elements.

Write $1x = x1 = x$ for all $x \in A_i$. Take the sequence $x = \left(x_1\right)_{i=1}^{\infty}$ with $x_i \in A_i$ for each $i \in N$. We define

$$\text{supp}(x_i)_{i\in N} = \{i \in N : x_i \neq 1\}.$$

Write

$$S = \{(x_i)_{i\in N} : \text{supp}(x_i)_{i\in N} \text{ is finite and non-empty}\}.$$

A standard *commutative* oid is the set S together with the product xy defined in S if and only if (supp x)∩(supp y) = Ø to be $(x_i y_i)$ (see [8], for definition). Thus the product $x_i y_i$ is required to be defined only is either $x_i = 1$ or $y_i = 1$. Of course the product in S is associative where defined and supp(xy) = (supp x)∪(supp y) whenever xy is define in S. supp $x(\alpha) \to \infty$ for some net $(x(\alpha))$ in S will mean that the minimum element in the support of $(x(\alpha))$ tends to infinity. Then for a fixed t in S, eventually supp $t <$ supp $x(\alpha)$ and so eventually, $tx(\alpha)$ is defined in S (see [7], for definition). The compact space S is the Stone-Cech compactification of the discrete space S. For each $k \in N$, write $S_k = \{(x_i)_{i\in N} \in S : x_i = 1 \text{ for } i < k\}$. Since S is a discrete space then both S_k and its complement is open in S, so that $cl_{\beta S}S_k$ is both open and closed in βS ([5], 6.9). Then βS produces a compact right topological semigroup H_S defined by

$$H_S = \bigcap_{k=1}^{\infty} cl_{\beta S} S_k,$$

with the multiplication $\mu v = \lim_\alpha \lim_\beta x(\alpha)y(\beta)$ if $\mu = \lim_\alpha x(\alpha)$ and $v = \lim_\beta y(\beta)$ for some nets $(x(\alpha))$, $(y(\beta))$ in S where supp $x(\alpha) \to \infty$, supp $y(\beta) \to \infty$ [1]. Given a function $f : S \to T$ where T is a compact Hausdorff space, the unique continuous extension of f to βS is denoted by f^β. The cardinal function is the map $c : S \to N$ given by $c(x)$ = card (supp x) (That is, the number of elements of the support of x). Then if (supp x)∩(supp y) = Ø so that xy defined in S, $c(xy) = c(x) + c(y)$. It follows easily that c extends to a homomorphism c^β from H_S into the one-point compactification $N \cup \{\infty\}$. Now write $H_n = \{\mu \in H_S : c^\beta(\mu) = n\}$, $n \in N$ and $H_\infty = \{\mu \in H_S : c^\beta(\mu) = \infty\}$. Then $H_S = H_1 \cup H_2 \cup \cdots \cup H\infty$. Thus H_n is clopen and each $\mu \in H_n$ is a limit of a net $(x(\alpha))$ with $c(x(\alpha)) = n$ for each α. Moreover, $H_n H_m \subseteq H_{m\times n} \subseteq H_{m+n}$ for all $n, m \in N$ so that $H_1 \cup H_2 \cup \cdots \cup H_m \cdots$ is a subsemigroup of H_S.

3. FREE SEMIGROUP

In this section we shall be concerned with proving the existence in H_S of free semigroups generated by H_n for each $n \in N$. The proof that H_S contains copies of the free semigroups is much easier than the proof for βN the Stone-Cech compactifcation of the positive integers [6].

Definition 3.1: For $n \in N$ we define a map $d_n : S \to S$ as follows. Let $x \in S$ with supp $x = \{i_1, i_2, \cdots, i_k\}$ so that $c(x) = k$. If $k \le n$ we put $d_n(x) = x$. If $n < k$ we put $d_n(x) = y$ where $y_{i_1} = x_{i_1}, \cdots, y_{i_{k-n}} = x_{i_{k-n}}$ and $y_i = 1$ for all other values of n. Then d_n extends to a unique continuous mapping $d_n^\beta : \beta S \to \beta S$.

Theorem 3.2: For $\xi \in \beta S$, $\eta \in H_S$ with $c^\beta(\eta) = n$, $n \in N$ then $d_n^\beta(\xi\eta) = \xi$.

Proof: There are nets $(x(\alpha))$, $(y(\beta))$ in S such that $x(\alpha) \to \xi$, $y(\beta) \to \eta$ with supp $y(\beta) \to \infty$. Then eventually, $c(y(\beta)) = n$. Now for a fixed α, eventually supp $x(\alpha)$ < supp $y(\beta)$ so that $\lim_\beta d_n(x(\alpha)y(\beta)) = x(\alpha)$. Since $d_n^\beta(\xi\eta) = \lim_\alpha \lim_\beta d_n(x(\alpha)y(\beta))$ it follows that $d_n^\beta(\xi\eta) = \xi$ as claimed.

Lemma 3.3: For $\xi_1, \xi_2 \in H_S$, $\eta \in H_n$, $n \in N$ then $\xi_1\eta = \xi_2\eta$ implies that $\xi_1 = \xi_2$.

Proof: By Theorem 3.2, $\xi_1 = d_n^\beta(\xi_1\eta) = d_n^\beta(\xi_2\eta) = \xi_2$ and the result follows. The next result is an immediate consequence of Lemma 3.3.

Theorem 3.4: The semigroup $H_1 \cup H_2 \cup \cdots \cup H_m \cup \cdots$ is right cancellative.

Definition 3.5: Let $x \in S$ with supp $x = \{i_1, i_2, \cdots, i_k\}$ so that $c(x) = k$ and let $n \in N$. We define a map $e_n : S \to S$ by $e_n(x) = x$ if $k \le n$, otherwise $e_n(x) = y$ where $y_{i_{n+1}} = x_{i_{n+1}}, \cdots$, $y_{i_k} = x_{i_k}$ and $y_i = 1$ for all other values of n. Then e_n has unique continuous extension e_n^β from βS into itself.

Theorem 3.6: For $\xi \in \beta S$ with $c^\beta(\xi) = n$, $n \in N$, $\eta \in H_S$ then $e_n^\beta(\xi\eta) = \eta$.

Proof: Analogous to that of Theorem 3.2.

Lemma 3.7: For $\eta_1, \eta_2 \in H_S$, $\xi \in H_n$, $n \in N$ then $\xi\eta_1 = \xi\eta_2$ implies that $\eta_1 = \eta_2$.

Proof: Analogous to that of Lemma 3.3. As a consequence of Lemma 3.7, we have the following result.

Theorem 3.8: The *semigroup* $H_1 \cup H_2 \cup \cdots \cup H_m \cdots$ is left cancellative.

Theorem 3.9: The *semigroup generated* by H_n, $n \in N$ is a free semigroup in H_S.

Proof: If $\xi_1, \xi_2 \cdots \xi_p = \eta_1 \eta_2 \cdots \eta_q$ with ξ, s and η, s in H_n then $c^\beta(\xi_1, \xi_2 \cdots \xi_p) = np$ and $c^\beta(\eta_1\eta_2 \cdots \eta_q) = nq$, so that $p = q$. We now apply e_n^β (similarly for d_n^β) to both sides, to get $\xi_2\xi_3 \cdots \xi_p = e_n^\beta(\xi_1\xi_2\cdots\xi_p) = e_n^\beta(\eta_1\eta_2\cdots\eta_q)$. An application of theorem 3.4 completes the proof.

Definition 3.10: For $n \in N$, we define a map $r_n : S \to S$ as follows. Let $x \in S$ with supp $x = \{i_1, i_2, \cdots, i_k\}$ so that $c(x) = k$ If $k \le n$ we put $r_n(x) = x$. If $n < k$, we put $r_n(x) = y$ where $y_{i_1} = x_{i_1}, \cdots, y_{i_n} = x_{i_n}$ and $y_i = 1$ for all other values of n. Then r_n extends to a unique continuous function r_n^β of βS into itself, so that $r_n^\beta(\xi\eta) = r_n^\beta(\xi) = \xi$ whenever

$\xi \in \beta S$ with $c^\beta(\xi) = n$, $\eta \in H_S$. Thus obtain an alternative proof of Theorem 3.9 by using r_n^β, $n \in N$.

We now distinguish a second element in each A_i (arbitrary) and denote it by "a" for each $i \in N$. Write

$$\delta_{ij} = \begin{cases} 1 & i = j \\ a & i \neq j \end{cases}$$

Put $\Delta = \left\{ \left(\delta_{in}\right)_{i=1}^{\infty}, n \in N \right\}$. Then Δ is a countable subset of S. Further if $\xi \in \mathrm{H_S}$ is a limit point of a subnet of Δ then $c^\beta(\xi) = 1$.

Lemma 3.11: Let $\xi_1, \xi_2, \cdots, \xi_k \in H_S$ be limit of subnets of Δ. Then

(i) $c^\beta(\xi_1\xi_2 \cdots \xi_k) = k$;

(ii) for $n \leq k$, $r_n^\beta(\xi_1\xi_2 \cdots \xi_k) = \xi_1\xi_2 \cdots \xi_n$;

(iii) for $n \leq k$ and any $\eta \in H_S$, $r_n^\beta(\xi_1\xi_2 \cdots \xi_k\eta) = \xi_1\xi_2 \cdots \xi_n$.

The proof is straightforward.

Remark 3.12: Let G be the set of limit points of Δ. Then $G = (cl_{\beta S}\Delta) \setminus \Delta$. Since Δ is countable and discrete, it follows that $cl_{\beta S}\Delta$ is homeomorphic to βN and so $(cl_{\beta S}\Delta) \setminus \Delta$ is homeomorphic to $N^*(= \beta N \setminus N)$. Thus card$(G) = 2^c$ where c is the cardinality of the continuum.

Theorem 3.13: The set G generates a free semigroup in H_S on 2^c generators.

Proof: Analogous to that Theorem 3.9.

To end of this section we give a number of result, (both algebraic and topological) for the compact right topological semigroup H_S.

Theorem 3.14: For each $n \in N$, $\{\xi H_S : \xi \in H_n\}$ is a family of disjoint right ideals in H_S.

Proof: If $\xi_1 \neq \xi_2$ in H_n then $r_n^\beta(\xi_1 H_S) = \xi_1$, $r_n^\beta(\xi_2 H_S) = \xi_2$ by the definition 3.10. Thus $(\xi_1 H_S) \cap (\xi_2 H_S) = \emptyset$, as claimed.

Theorem 3.15: H_S contains 2^c disjoint right ideals of the from ξH_S where $\xi \in G$.

Proof: Indeed by Remark 3.12, card$(G) = 2^c$. This proves our assertion.

Theorem 3.16: For each $n \in N$, $\{H_S v : v \in H_n\}$ is a family of disjoint left ideals in H_S.

Proof: It is a straightforward argument to prove that $e_n^\beta(\xi v) = e_n^\beta(v) = v,$ whenever $\xi \in \beta S$, $v \in H_n$, $n \in N$. Now the proof is similar to that proof of Theorem 3.14.

As a consequence of Remark 3.12 and Theorem 3.16, we have the following result.

Theorem 3.17: H_S contains 2^c disjoint left ideals of the from $H_S v$, whenever $v \in G$.

Definition 3.18: For $x \in S$ with supp $x = \{i_1, i_2, \cdots, i_k\}$, we define by $l(x)$ (the length of the support of x) the integer $i_k - i_1 + 1$. Then l extends to a unique continuous function l^β from βS into the one-point compactification $\mathrm{N} \cup \{\infty\}$. So for fixed $t \in S$, eventually supp $t <$ supp $y(\beta)$, whenever $(y(\beta))$ is a net in S with supp $y(\beta) \to \infty$ and so, eventually $l(ty(\beta)) = \infty$. Since $l^\beta(\xi\eta) = \lim_\alpha \lim_\beta l(x(\alpha)y(\beta))$ where $\xi \in \beta S$, $\eta \in H_S$, $x(\alpha) \to \xi$ and $y(\beta) \to \eta$ with supp $y(\beta) \to \infty$, it follows that $l^\beta(\xi\eta) = \infty$. Now we have the following result.

Proposition 3.19: Let $\xi \in H_n$ for some $n \in N$ with $l^\beta(\xi) < \infty$. Then ξ is not a product.

Proposition 3.20: H_S has no left identity, no right identity (and hence has no identity).

Proof: If e is a left identity for H_S, $\xi = e\xi$ for all $\xi \in H_S$, which is impossible by proposition 3.19. The other part is similar.

Remark 3.21: (i) Clearly, $l^\beta(e) = \infty$ where e is an idempotent in H_S. We denote the set of all idempotents in H_S by $E(H_S)$. Thus we obtain that $E(H_S) \subseteq \{\eta \in H_S : l^\beta(\eta) = \infty\}$. (ii) H_S has no left zero. If ξ is a left zero, it follows that ξ is in every left ideal in H_S, which is impossible by Theorem 3.17. Now using Theorem 3.15. By a similar argument, H_S has no right zero. Hence H_S is not a left [right] zero semigroup.

Theorem 3.22: $(H_S)^2$ is not dense in H_S.

Proof: Let $\xi, \eta \in H_S$. Then $l^\beta(\xi\eta) = \infty$ so that $(H_S)^2 \cap (l^\beta)^{-1}(1) = \emptyset$. But $(l^\beta)^{-1}(1)$ is a non-empty open set in βS which contains elements of H_S.

Theorem 3.23: The set $\{\xi \in H_S : l^\beta(\xi) < \infty\}$ is not dense in H_S.

Proof: Take a sequence $(s_n)_{n=1}^\infty$ in S with $s_n = (\delta_{in})_{i=1}^\infty (\delta_{in^2})_{i=1}^\infty$. Set $X = \{s_n : n \in N\}$. Let 1_X be the indicator function of X (that is the function on S whose value is 1 on X and 0 on $S \setminus X$) and let η be a cluster point of $(s_n)_{n=1}^\infty$ in βS. Then $\eta \in H_S$, so that $1_X^\beta(\eta) = 1$. Now take $\xi \in H_S$ such that $l^\beta(\xi) = k$ for some $k \in N$. Let $x(\alpha) \to \xi$ with supp $x(\alpha) \to \infty$. Then eventually $l(x(\alpha)) = k$, so that eventually $x(\alpha) \notin X$. Hence $1_X^\beta(\xi) = 0$. It follows that η is not the limit of a net of elements of $\{\xi \in H_S : l^\beta(\xi) < \infty\}$ and the result follows.

Theorem 3.24: $(l^\beta)^{-1}(\infty) \cap H_S$ is not nowhere dense in H_S.

Proof: Let $(s_n)_{n=1}^{\infty}$, X and η be as in Theorem 3.23. Then $\left(1_X^{\beta}\right)^{-1} \cap H_S$ is a non-empty open set in H_S. Now X is countable and discrete,so that $cl_{\beta S}X$ is homeomorphic to N and $(cl_{\beta S}X \backslash X)$ is homeomorphic to N^*. Thus $\xi \in (cl_{\beta S}X)\backslash X$ if and only if $\xi = \lim_i s_{n_i}$, for some subnet (s_{n_i}) of $(s_n)_{n=1}^{\infty}$ with $n_i \to \infty$ and supp $s_{n_i} \to \infty$. Further, $(cl_{\beta S}X)\backslash X = \left(1_X^{\beta}\right)^{-1} \cap H_S$. Now $l(s_n) = n^2 - n + 1$ so that $l(s_n) \to \infty$, as $n \to \infty$. Thus from $\xi \in (cl_{\beta S}X)\backslash X$, it follows that $l^{\beta}(\xi) = \infty$ and so $\left(1_X^{\beta}\right)^{-1} \cap H_S \subseteq \left(l^{\beta}\right)^{-1}(\infty)$, as claimed.

Theorem 3.25: H_{∞} is not nowhere dense in H_S.

Proof: Clearly, $H_{\infty} = (c^{\beta})^{-1}(\infty) \setminus H_S$. Take a sequence $(t_n)_{n=1}^{\infty}$ in S with $t_n = (\delta_{in})_{i=1}^{\infty}$ $(\delta_{in+1})_{i=1}^{\infty} \cdots (\delta_{in^2})_{i=1}^{\infty}$. Let η be the cluster point of $(t_n)_{n=1}^{\infty}$ in βS. Then $\eta \in H_S$ and $c^{\beta}(\eta) = \infty$. Put $X = \{t_n : n \in N\}$. Now by a similar argument as the proof of Theorem 3.24, the result follows.

REFERENCES

[1] A. M. Aminpour, Space of Functions Determined by Iterated Limits “At Infinity” On an Oid, Proc. Cambridge Phill. Soc. **111**, (1992), 127–142.

[2] A. M. Aminpour, A Subsemigroup of Some Stone-Cech Compactification, Math. Nachr. **158**, (1992), 207–218.

[3] A. M. Aminpour, Free Semigroups and Idempotents in T^{∞}, *J. Sci. I. R. Iran*, **9(1)**, Winter 1998.

[4] J. F. Berglund, H. D. Junghenn, and P. Milnes, Analysis on semigroups (wiley, 1989).

[5] L. Gillman, and M. Jerison, Rings of Continuous Functions, Van Nostrand (1960).

[6] N. Hindman, and J. S. Pym, Free Groups and Semigroups in βN, Semigroup Forum, **30**, (1984), 421–428.

[7] T. Papazyan, Oids, Finite Sums and the Structure of the Stone-Cech Compactification of a Discrete Semigroup, Semigroup Forum, **42**, (1991), 265–277.

[8] J. S. Pym, Semigroup Structure in Stone-Cech Compactifications, *J. London Math.*, Soc. (2) **36**, (1987) 421–428.

[9] D. Strauss, Semigroup Structure on βN, Semigroup Forum, **44**, (1992), 238–244.

A. M. Aminpour
Department of Mathematics
College of Mathematics and Computer Sciences
Shahid Chamran University, Ahwaz, Iran
E-mail: Aminpour@scu.ac.ir