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LATTICES OF TOPOLOGIES

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Abstract : In this paper we show that the meets of any infinite number of dual atoms always admits a finer topology that cannot be obtained as the meets of a countable number of dual atoms.

Keywords: Lattice of topologies, atoms, anti atoms, ecsec space, Stone-Cech compactifi- cation, topological properties.

1. INTRODUCTION

The second countability of topological space is an important topological propery, describable in terms of the lattice of topologies. Now the following question is natural: What are the meets of countable number of dual atoms? It is a simple and interesting result of (Larson and Andima, 1975) that such topologies cannot be first countable (Murdeshwar, 1982). This can be roughly stated as: First countable topologies cannot occur in high positions in the lattice of topologies. However, it is quite possible that, of two comparable topologies, the smaller is a countable meet of dual atoms, but the larger is not. We show that the meet of any infinite number of dual atoms always admits a finer topology that cannot be obtained as a meet of a countable number of dual atoms. It is true that no element of D_1 is first countable where D_1 is the dual $\sigma-$ideal generated by dual atoms. A dual $\sigma-$ideal is a subset stable under the formation of countable meets and bigger elements. This follows from the stronger assertion that every member of D_1 is an ecsec space.

2. COUNTABLE MEETS OF DUAL ATOMS

An ecsec space is one in which every convergent sequence is eventually constant. We prove that a countable meet of ecsec topologies is again a ecsec. Thus the ecsec topologies on a set X always form a dual $\sigma-$ideal in $T(X)$, the lattice of topologies on the set X. Let us denote it by D_2. Thus we have asserted is that $D_1 \subset D_2$.

Theorem 2.1. If an ecsec topology is the join of an increasing sequence of topologies, then all but a finite number of elements in this increasing chain, must themselves be ecsec.

Though the result stated above disclose a close connection between topological and lattice theoretic properties of a topology, they only prove the way for two natural questions:

1. Characterize ecsec topologies purely in lattice-theoretical terms.
2. Characterize the elements of $D1$ in purely topological terms.

It is a known result that there are Hausdorff topological spaces that cannot be embedded in any Hausdorff sequential space. Infact every dual atom has this property. We consider only T_1 topologies.

Theorem 2.2. Let τ be a topology on X such that some non-discrete subspace of (X, τ) is embeddable in a T_2 sequential space. Then atleast c (cardinality of the continuum) dual atoms are required to give τ as their meet.

It is worth-noting here that every topology on a countable set X can be obtained as a meet of atmost c dual atoms, even though there are 2^c dual atoms in $T(X)$. Note that the condition on τ is a meager one, satisfied by all topologies that we often meet. Hence the above theorem says that the members of D_1 are pathalogical. This can be used to prove that not every member of D_2 is in D_1. Thus we note the following chain of statements, each implies the next.

Theorem 2.3. The following statements are equivalent.

a. The topology is obtainable as the meet of a collection of dual atoms having cardinality strictly less than c.

b. Every common subspace of this topological space and a sequential space has to be discrete.

c. No finer non-discrete topology can be embedded in a sequential space.

d. No finer non-discrete topology is sequential.

e. No finer non-discrete topology is first countable.

f. This topology is not first countable.

All these results are proved by non-straight-forward methods, making use of the properties of Stone-Cech compactification (Stone, 1948).

3. MANY INTERVALS CONTAIN L(βN N)

While discussing the lattice structure of $T(X)$, the first natural question is how does it look like. Is it like a patrician lattice? Is it look like a boolian algebra etc. It is known that $T(X)$ is not distributive, not modular, etc. Some embedding theorem proved that $T(X)$ is a complicated lattice. The following result exhibits the complexity of this lattice. Let N be a countably infinite discrete space. Let βN be its Stone-Compactification. Let L be the lattice of all open subsets of the reminder $\beta N \setminus N$. Of cource, L is a huge distributive lattice of cardinality 2^c. Still, we assert that an isomorphic copy of this lattice L can be found in each and every interval of $T(X)$ whose least element is some what nice(say, first countable, T_2 or metrizable).

Theorem 3.1. Let τ_1 and τ_2 be any two topologies on a set X such that τ_1 is sequential, Hausdorff and $< \tau_2$. Then there is a τ_3 such that $\tau_1 < \tau_3 \leq \tau_2$ and such that L is completely isomorphic to the interval.

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Corrolory 3.1 : A space is ecsec if and only if the principal dual ideal of $T(X)$ generated by it, does not contain a copy of L as a dual ideal.

The theorem 3.1 implies that every finite distributive lattice is embeddable in every interval with a lower bound. Also it implies that every sequential Hausdorff topology is the sequential coreflection of atleast 2^c distinct topologies. Let us state this in an elementary fashion. Define two topologies to be equivalent if they have the same family of convergent sequence. Then each equivalence class either contains at least 2^c elements or contains no Hausdorff member. Also we note that the final result of (Larson and Andima, 1975) is the consequence of the theorem 3.1.

TOPOLOGICAL PROPERTIES THROUGH LATTICE JOINTS AND MEETS

Certain topological properties can be characterized as being obtained from simpler or better known properties by performing lattice joins or meets. The following table gives a list of results of this sort.

Sl. No.	The set A	Arbitrary meets of members of A	Complete sublattice generated by A
1	Metrizable	T_1 sequential	T_1
2	First countable	Sequential	All
3	Locally compact T_2	T_1 k-space	T_1
4	Locally coutable	$c-$space	All

The following is a theorem of general pattern containing as particular cases, the results of rows (Larson and Andima, 1975) and (Steiner, 1966).

Let E be ~~a~~ family of topological spaces stable under the formation of continuous images. Then a space has week topology from it subspaces belonging to $\underline{E}$ if and only if its topology is the meet of a collection of topologies for which the interiors of the subspaces belonging to E, form a base. All these results are proved in (Birkhoff, 1967). The first row in the above table answers a question of (Birkhoff, 1936), since it now follows that the product of two LM spaces need not be an LM-space. The fourth row ansers a question of Larson and Zimmerman concerning countably accessible topologies.

Now we give some results that describe a fairly large and natural topological property as the lattice joints of simpler topologies some times homomorphic to a simple space.

Theorem 4.1. Let Q be the set of all rational numbers.

a. The countable non-compact T_3 topologies are precisely the lattice joints of topologies homomorphic to Q.

b. Every subspace of a well-ordered space is obtainable as the lattice join of well-ordered compact topologies.

c. A topology on a countable set is the join of compact Hausdorff topologies if and only if it is scattered and T_3.

d. A topology on a countable set admits a corner compact T_2 topology if and only if for each subset A there is a open W such that is $A \cap W$ singleton.

e. A topology on a countable set is metrizible if and only if it is the join of a countable number of topologies homomorphic to Q.

Thus the lattice joins and meets yield elegant characterization of nice properties.

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