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Adaptive Controller Design for the Generalized Projective Synchronization of Circulant Chaotic Systems with Unknown Parameters

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ABSTRACT

This paper builds adaptive controllers for thegeneralized projective synchronization (GPS) of circulant chaotic systems with unknown parameters. Generalized projective synchronization (GPS) of chaotic systems is a new type of chaos synchronization, which generalizes common types of synchronization such as complete synchronization, anti-synchronization, hybrid synchronization and projective synchronization. In this paper, we derive new results for the GPS of identical circulant chaotic systems with unknown system parameters, *viz*. Halvorsen's circulant systems and Thomas circulant systems. Lyapunov stability theory and adaptive control theory have been applied for deriving the new GPS results for the circulant chaotic systems with unknown system parameters. MATLAB simulations have been shown to demonstrate the validity and effectiveness of the adaptive GPS results derived for the circulant chaotic systems.

Keywords: Chaos, chaotic systems, circulant systems, synchronization, generalized projective synchronization.

1. INTRODUCTION

Chaos theory has been developed and extensively studied over the past four decades. A chaotic system is a nonlinear dissipative dynamical system that is highly sensitive to even small perturbations in its initial conditions. This sensitivity of chaotic systems is popularly referred to as the *butterfly effect* [1]. The Lyapunov exponent of a dynamical system is a quantitative measure that characterizes the rate of separation of infinitesimally close trajectories of the system. Thus, a chaotic system is also defined as a dynamical system having at least one positive Lyapunov exponent.

In the chaos literature, there is a great interest in the modelling of novel chaotic systems. Since the discovery of a chaotic system by Lorenz [2], many classical chaotic systems have been found such as Rössler system [3], ACT system [4], Sprott systems [5], Chen system [6], Lü system [7], Liu system [8], Tigan system [9], Cai system [10], etc. Some of the novel chaotic systems found in the recent years can be listed as Li system [11], Sundarapandian-Pehlivan system [12], Sundarapandian system [13], Vaidyanathan systems [14-18], Vaidyanathan-Madhavan system [19], Pehlivan-Moroz-Vaidyanathan system [20], etc.

The synchronization of chaotic system was first studied by Fujisaka and Yemada [21] in 1983. This problem did not receive great attention until Pecora and Carroll [22-23] published their results on chaos synchronization in early 1990s. From then on, chaos synchronization has been rigorously studied in the last four decades.

Chaos theory has been applied to a variety of fields such as lasers [24-25], oscillators [26-27],chemical reactions[28-30], biology [31-32], neural networks [33-35], ecology [36-37], robotics [38-40], etc.

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Chaos synchronization problem can be stated as follows. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically. Chaos synchronization has many applications in secure communications [41-43], cryptosystems [44-45], encryption [46-47] etc.

The chaos synchronization problem has been paid great attention in the literature and a variety of impressive approaches have been proposed. A few important methods for the chaos synchronization problem can be listed as active control method [48-53], adaptive control method [54-60], sampled-data feedback control method [61-62], time-delay feedback approach [63], backstepping method [64-68], sliding mode control method [69-73], etc.

In this paper, we derive new results for the generalized projective synchronization (GPS) for 3-D circulant chaotic systems [74] with unknown parameters. Explicitly, we shall derive new adaptive results for the GPS of Halvorsen's circulant systems and Thomas circulant chaotic systems.

Generalized projective synchronization [75-84] is a new type of synchronization of chaotic systems, which generalizes common types of synchronization such as complete synchronization [48-53], anti-synchronization [85-90], hybrid synchronization [91-101], projective synchronization [102], etc.

The rest of this paper is organized as follows. Section 2 contains a description of the circulant chaotic systems considered in this paper, *viz*. Halvorsen's circulant system and Thomas circulant system. Section 3 describes the adaptive controller design for the GPS of identical Halvorsen's circulant systems. Section 4 describes the adaptive controller design for the GPS of identical Thomas circulant systems. Numerical simulations using MATLAB are shown to illustrate the main results derived in this paper. Section 5summarizes the main results derived in this paper.

2. CIRCULANT CHAOTIC SYSTEMS

In this section, we describe circulant chaotic systems and give details of the circulant chaotic systems proposed by Halvorsen and Thomas [74].

Circulant chaotic systems are chaotic dynamical systems in which the variables are cyclically symmetric. The general form of 3-D circulant chaotic system is given by

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_3, x_1) \\ \dot{x}_3 &= f(x_3, x_2, x_1) \end{aligned} \tag{1}$$

where all the functions are the same except that the variables are rotated.

Classical circulant chaotic systems are the 3-D circulant systems proposed by Halvorsen and Thomas.

Halvorsen's circulant chaotic system is obtained by taking

$$f(x_1, x_2, x_3) = -ax_1 - bx_2 - bx_3 - x_2^2 \tag{2}$$

where $a = 1.27$ and $b = 4$.

Thus, Halvorsen's circulant chaotic system is given by

$$\begin{aligned} \dot{x}_1 &= -ax_1 - bx_2 - bx_3 - x_2^2 \\ \dot{x}_2 &= -ax_2 - bx_3 - bx_1 - x_3^2 \\ \dot{x}_3 &= -ax_3 - bx_1 - bx_2 - x_1^2 \end{aligned} \tag{3}$$

where x_1, x_2, x_3 are the state variables. The Halvorsen's system is *chaotic* when

$$a = 1.27, \quad b = 4 \tag{4}$$

For simulations, we take the initial conditions of the Halvorsen's system (3) as

$$x_1(0) = 1.2, \quad x_2(0) = 0.5, \quad x_3(0) = 2.3 \tag{5}$$

The strange chaotic attractor of the Halvorsen's circulant system (3) is given in Figure 1.

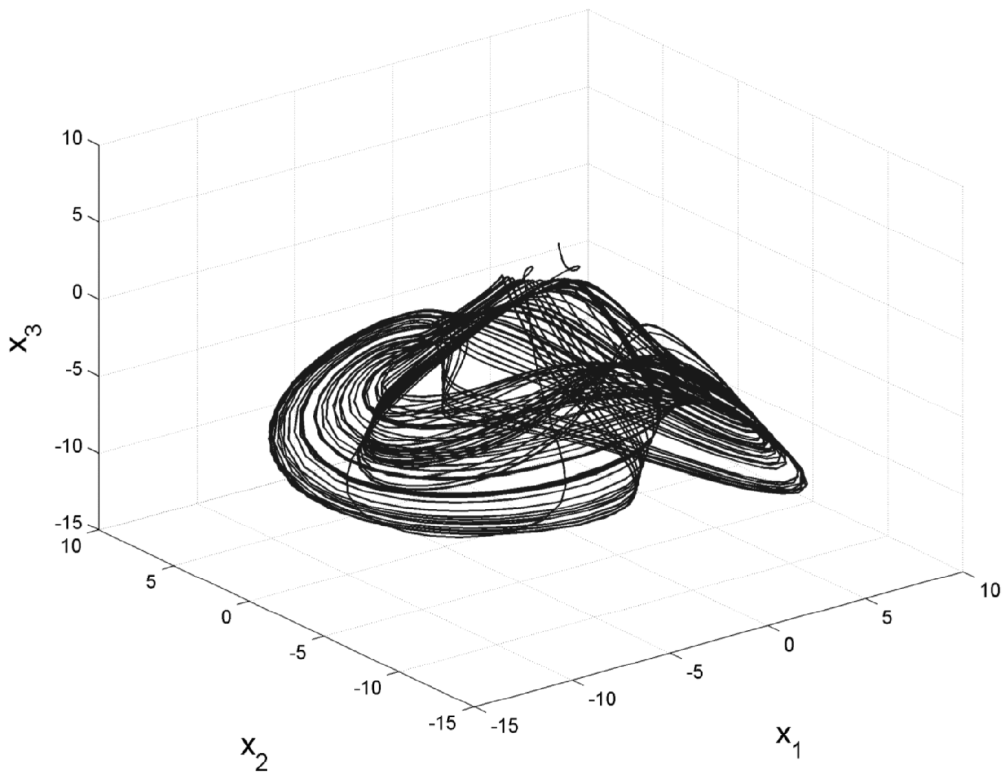

Figure 1: Strange Attractor of the Halvorsen's Circulant Chaotic System

The Lyapunov exponents of the Halvorsen's chaotic system (3) are calculated as

$$L_1 = 0.7979, \quad L_2 \approx 0, \quad L_3 = -4.6087 \tag{4}$$

Thus, the maximal Lyapunov exponent (MLE) of the Halvorsen's chaotic system (3) is given by $L_1 = 0.7979.$ Since the sum of the Lyapunov exponents of the Halvorsen's chaotic system (3) is negative, it follows that Halvorsen's system (3) is a dissipative system.

Also, the Lyapunov dimension of the Halvorsen's chaotic system (3) is calculated as

$$D_L = j + \frac{\sum_{i=1}^{j} L_i}{|L_{j+1}|} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1731, \tag{5}$$

which is fractional.

The dynamics of the Lyapunov exponents of the Halvorsen's chaotic system (3) is depicted in Figure 2.

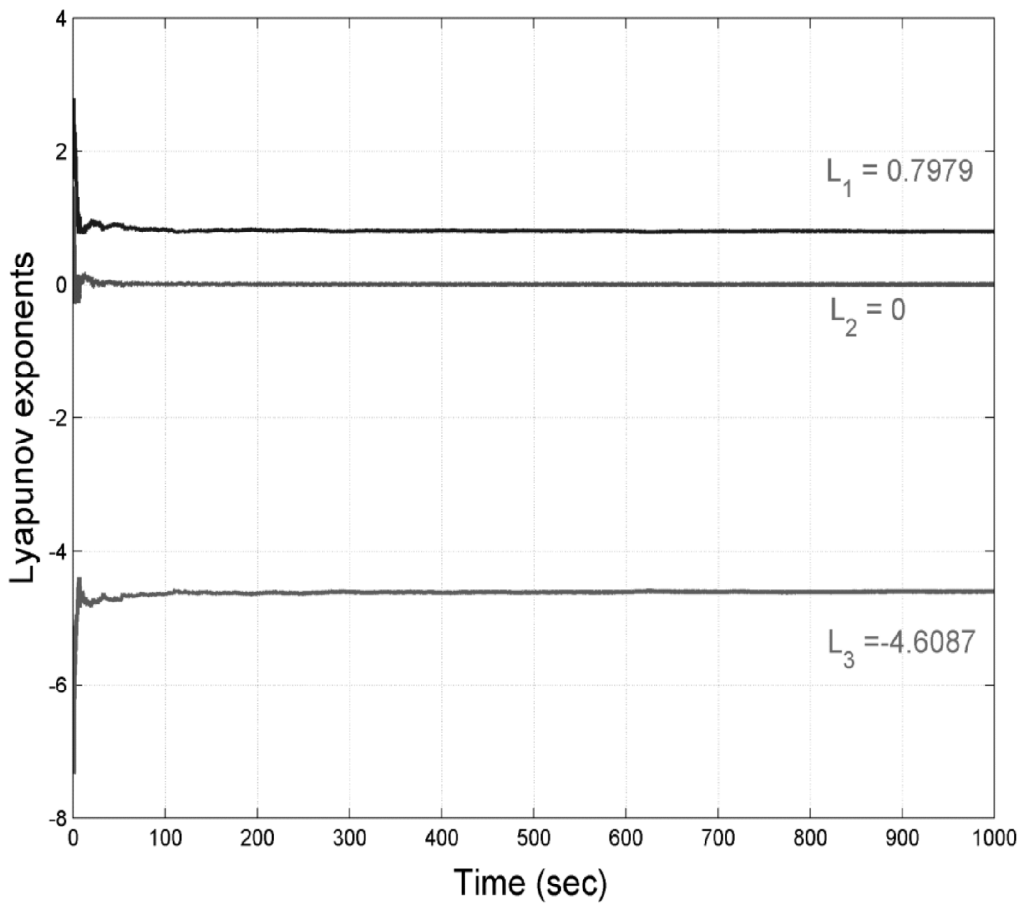

Figure 2: Dynamics of the Lyapunov Exponents of the Halvorsen's Chaotic System

Next, Thomas circulant chaotic system is obtained by taking

$$f(x_1, x_2, x_3) = -ax_1 + bx_2 - x_2^3 \tag{6}$$

where $a = 1$ and $b = 4$.

Thus, Thomascirculant chaotic system is given by

$$\begin{aligned} \dot{x}_1 &= -ax_1 + bx_2 - x_2^3 \\ \dot{x}_2 &= -ax_2 + bx_3 - x_3^3 \\ \dot{x}_3 &= -ax_3 + bx_1 - x_1^3 \end{aligned} \tag{7}$$

where x_1, x_2, x_3 are the state variables.

The Thomas circulant system is *chaotic* when

$$a = 1, \quad b = 4 \tag{8}$$

For simulations, we take the initial conditions of the Thomas system (7) as

$$x_1(0) = 1.2, \;\; x_2(0) = 0.5, \;\; x_3(0) = 2.3 \tag{9}$$

The strange chaotic attractor of the Thomas circulant system (7) is given in Figure 3.

The Lyapunov exponents of the Thomas chaotic system (7) are calculated as

$$L_1 = 0.1714, \;\; L_2 \approx 0, \;\; L_3 = -3.1674 \tag{10}$$

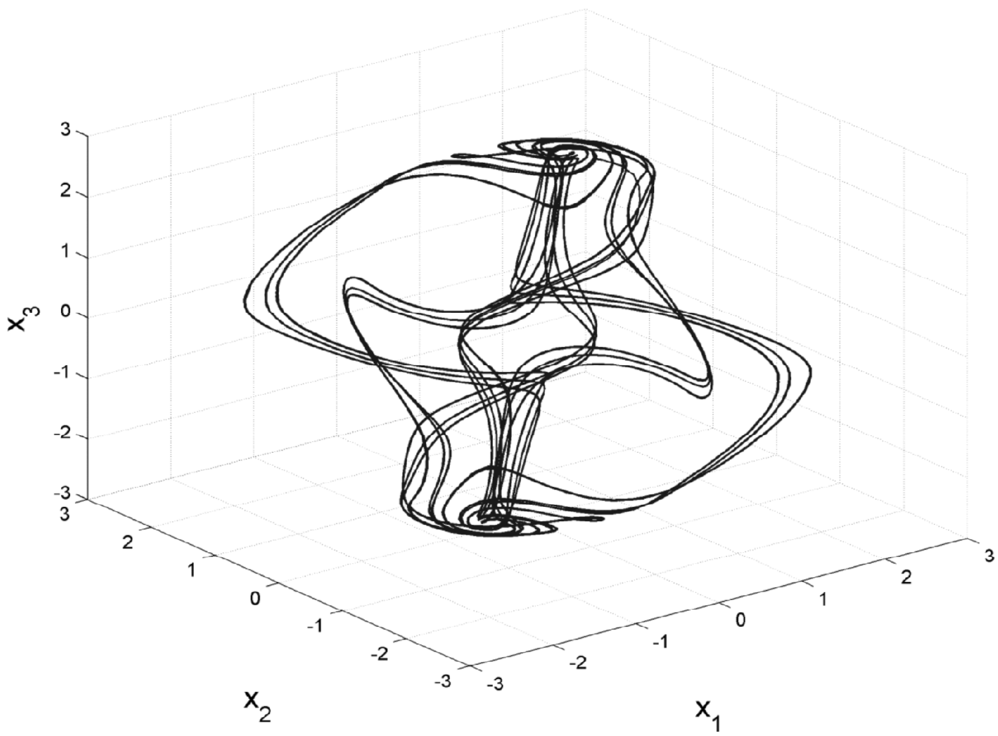

Figure 3: Strange Attractor of the Thomas Circulant Chaotic System

Thus, the maximal Lyapunov exponent (MLE) of the Thomas chaotic system (7) is given by $L_1 = 0.1714$. Since the sum of the Lyapunov exponents of the Thomas chaotic system (7) is negative, it follows that Thomas system (7) is a dissipative system.

Also, the Lyapunov dimension of the Thomas chaotic system (7) is calculated as

$$D_L = j + \frac{\sum_{i=1}^{j} L_i}{|L_{j+1}|} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0541, \tag{11}$$

which is fractional.

The dynamics of the Lyapunov exponents of the Thomas chaotic system (7) is depicted in Figure 4.

3. GPS OF IDENTICAL HALVORSEN'S CIRCULANT CHAOTIC SYSTEMS

In this section, we devise an adaptive controller to achieve generalized projective synchronization (GPS) of identical Halvorsen's circulant chaotic systems.

As the master system, we consider the Halvorsen's chaotic system given by

$$\begin{aligned} \dot{x}_1 &= -ax_1 - bx_2 - bx_3 - x_2^2 \\ \dot{x}_2 &= -ax_2 - bx_3 - bx_1 - x_3^2 \\ \dot{x}_3 &= -ax_3 - bx_1 - bx_2 - x_1^2 \end{aligned} \tag{12}$$

where x_1, x_2, x_3 are state variables and a, b are constant, unknown, parameters of the system.

As the slave system, we consider the controlled Halvorsen's chaotic system given by

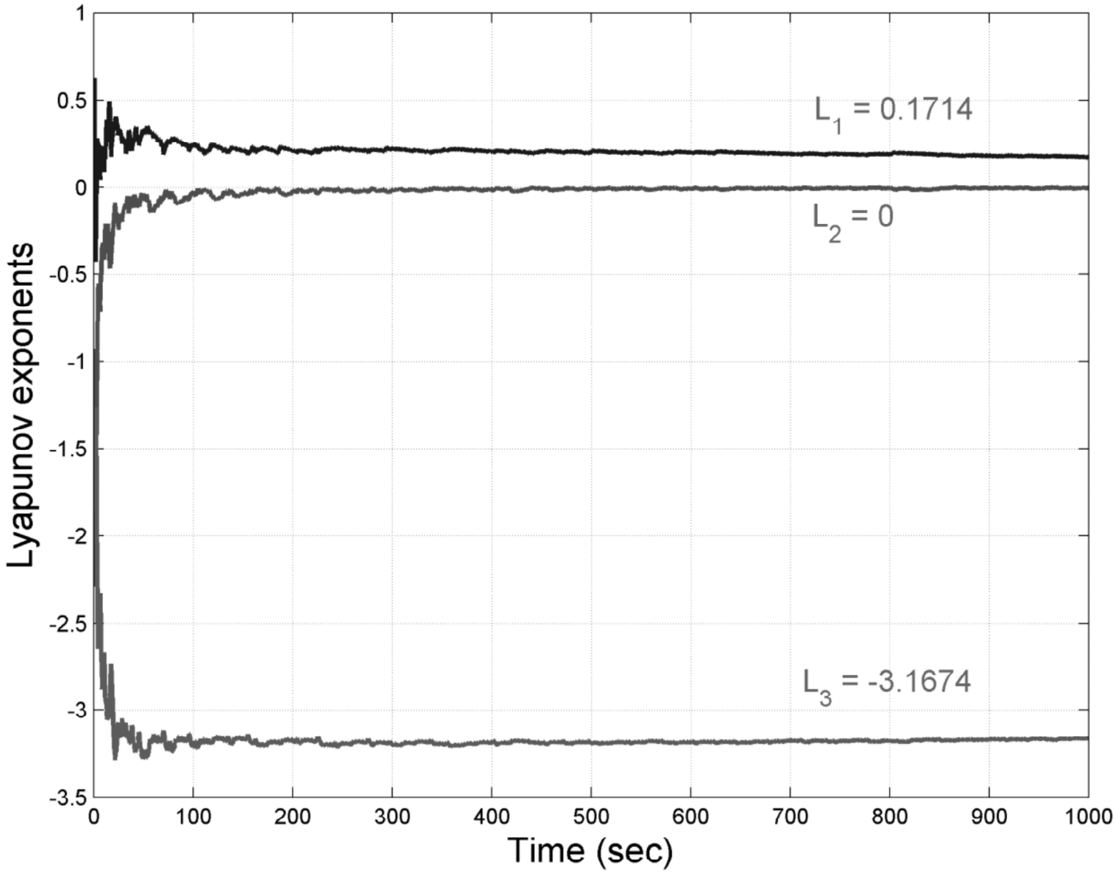

Figure 4: Dynamics of the Lyapunov Exponents of the Thomas Chaotic System

$$\begin{aligned}\dot{y}_1 &= -ay_1 - by_2 - by_3 - y_2^2 + u_1 \\ \dot{y}_2 &= -ay_2 - by_3 - by_1 - y_3^2 + u_2 \\ \dot{y}_3 &= -ay_3 - by_1 - by_2 - y_1^2 + u_3\end{aligned} \tag{13}$$

where y_1, y_2, y_3 are state variables and u_1, u_2, u_3 are adaptive controls to be designed using estimates $A(t), B(t)$ for the unknown parameters a, b, respectively.

The generalized projective synchronization (GPS) error between the Halvorsen's systems (12) and (13) is defined by

$$\begin{aligned}e_1(t) &= y_1(t) - \eta_1 x_1(t) \\ e_2(t) &= y_2(t) - \eta_2 x_1(t) \\ e_3(t) &= y_3(t) - \eta_3 x_1(t)\end{aligned} \tag{14}$$

where the GPS scales η_1, η_2, η_3 are real constants.

The GPS error dynamics is obtained as

$$\begin{aligned}\dot{e}_1 &= -ae_1 - b\left[y_2 + y_3 - \eta_1(x_2 + x_3)\right] - y_2^2 + \eta_1 x_2^2 + u_1 \\ \dot{e}_2 &= -ae_2 - b\left[y_3 + y_1 - \eta_2(x_3 + x_1)\right] - y_3^2 + \eta_2 x_3^2 + u_2 \\ \dot{e}_3 &= -ae_3 - b\left[y_1 + y_2 - \eta_3(x_1 + x_2)\right] - y_1^2 + \eta_3 x_1^2 + u_3\end{aligned} \tag{15}$$

We consider an adaptive controller defined by

$$\begin{aligned} u_1 &= A(t)e_1 + B(t)\left[y_2 + y_3 - \eta_1(x_2 + x_3)\right] + y_2^2 - \eta_1 x_2^2 - k_1 e_1 \\ u_2 &= A(t)e_2 + B(t)\left[y_3 + y_1 - \eta_2(x_3 + x_1)\right] + y_3^2 - \eta_2 x_3^2 - k_2 e_2 \\ u_3 &= A(t)e_3 + B(t)\left[y_1 + y_2 - \eta_3(x_1 + x_2)\right] + y_1^2 - \eta_3 x_1^2 - k_3 e_3 \end{aligned} \tag{16}$$

where the gains k_1, k_2, k_3 are positive constants.

Substituting the control law (16) into (15), we obtain the closed-loop error dynamics

$$\begin{aligned} \dot{e}_1 &= -(a - A(t))e_1 - (b - B(t))\left[y_2 + y_3 - \eta_1(x_2 + x_3)\right] - k_1 e_1 \\ \dot{e}_2 &= -(a - A(t))e_2 - (b - B(t))\left[y_3 + y_1 - \eta_2(x_3 + x_1)\right] - k_2 e_2 \\ \dot{e}_3 &= -(a - A(t))e_3 - (b - B(t))\left[y_1 + y_2 - \eta_3(x_1 + x_2)\right] - k_3 e_3 \end{aligned} \tag{17}$$

The parameter estimation error is defined by

$$\begin{aligned} e_a(t) &= a - A(t) \\ e_b(t) &= b - B(t) \end{aligned} \tag{18}$$

Using (18), the error dynamics (17) can be simplified as

$$\begin{aligned} \dot{e}_1 &= -e_a e_1 - e_b\left[y_2 + y_3 - \eta_1(x_2 + x_3)\right] - k_1 e_1 \\ \dot{e}_2 &= -e_a e_2 - e_b\left[y_3 + y_1 - \eta_2(x_3 + x_1)\right] - k_2 e_2 \\ \dot{e}_3 &= -e_a e_3 - e_b\left[y_1 + y_2 - \eta_3(x_1 + x_2)\right] - k_3 e_3 \end{aligned} \tag{19}$$

Differentiating (18) with respect to t, we get

$$\begin{aligned} \dot{e}_a &= -\dot{A} \\ \dot{e}_b &= -\dot{B} \end{aligned} \tag{20}$$

We consider the Lyapunov function defined by

$$V(e_1, e_2, e_3, e_a, e_b) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2\right), \tag{21}$$

which is a quadratic and positive definite function on R^5.

Taking the time-derivative of V along the trajectories of (19) and (20), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a\left[-e_1^2 - e_2^2 - e_3^2 - \dot{A}\right] \\ &+ e_b \begin{bmatrix} -e_1\left(y_2 + y_3 - \eta_1(x_2 + x_3)\right) - e_2\left(y_3 + y_1 - \eta_2(x_3 + x_1)\right) \\ -e_3\left(y_1 + y_2 - \eta_3(x_1 + x_2)\right) - \dot{B} \end{bmatrix} \end{aligned} \tag{22}$$

In view of Eq. (22), the parameter estimates update law is defined as

$$\begin{aligned} \dot{A} &= -e_1^2 - e_2^2 - e_3^2 \\ \dot{B} &= -e_1\left(y_2 + y_3 - \eta_1(x_2 + x_3)\right) - e_2\left(y_3 + y_1 - \eta_2(x_3 + x_1)\right) \\ &\quad - e_3\left(y_1 + y_2 - \eta_3(x_1 + x_2)\right) \end{aligned} \tag{23}$$

Next, we state and prove the main result of this section.

Theorem 1. The identical Halvorsen'scirculant chaotic systems given by (12) and (13) with unknown parameters a, b are globally and exponentially generalized projective synchronized (GPS) by the adaptive controller (16) and the parameter estimates update law (23), where the gains k_i, (i = 1, 2, 3) are positive constants.

Proof: We use Lyapunov stability theory [65] to prove this result.

Consider the quadratic Lyapunov function V defined by Eq. (21).

By substituting the parameter estimates update law (23) into the dynamics (22), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2, \tag{24}$$

which is a quadratic and negative semi-definite function on R^5.

Thus, it can be concluded that the synchronization vector $e(t)$ and the parameter estimation error are globally bounded, i.e.

$$\left[e_1(t) \quad e_2(t) \quad e_3(t) \quad e_a(t) \quad e_b(t) \right]^T \in L_\infty. \tag{25}$$

We define

$$k = \min\left\{ k_1, k_2 \right\}. \tag{26}$$

Then it follows from (26) that

$$\dot{V} \le -k \left\| e \right\|^2 \text{or}\, k \left\| e \right\|^2 \le -\dot{V}. \tag{27}$$

Integrating the inequality (27) from 0 to t, we get

$$k \int_o^t \left\| e(\tau) \right\|^2 \, d\tau \;\le\; -\int_0^t \dot{V}(\tau) d\tau = V(0) - V(t) \tag{28}$$

Therefore, we can conclude that $e(t) \in L_2$.

Using (19), we can conclude that $\dot{e}(t) \in L_\infty$.

Hence, using Barbalat's lemma, we can conclude that $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in R^3$.

This completes the proof.

Numerical Results

For numerical simulations, the classical fourth-order Runge-Kutta method is used to solve the identical Halvorsen'ssystems (12) and (13) with the adaptive control (16) and the parameter estimates update law (23).

For the Halvorsen's systems (12) and (13), the parameter values are taken as those which result in chaotic behaviour of the systems, *viz.*

$$a = 1.27, \quad b = 4 \tag{29}$$

We take the feedback gains as $k_i = 8$ for $i = 1, 2, 3$.

We take the GPS scales as

$$\eta_1 = 0.6, \quad \eta_2 = -0.8, \quad \eta_3 = 1.2 \tag{30}$$

The initial values of the master system (12) are taken as

$$x_1(0) = 1.2,\ x_2(0) = -0.1,\ x_3(0) = 1.5 \tag{31}$$

The initial values of the slave system (13) are taken as

$$y_1(0) = 1.8,\ y_2(0) = 0.7,\ y_3(0) = -0.2 \tag{32}$$

The initial values of the parameter estimates are taken as

$$A(0) = 5,\ B(0) = 16 \tag{33}$$

Figures 5-7 depict the GPS of the Halvorsen's chaotic systems (12) and (13).

Figure 8 depicts the time-history of the GPS synchronization errors e_1, e_2, e_3.

3. GPS OF IDENTICAL THOMAS CIRCULANT CHAOTIC SYSTEMS

In this section, we devise an adaptive controller to achieve generalized projective synchronization (GPS) of identical Thomas circulant chaotic systems.

As the master system, we consider the Thomas chaotic system given by

$$\begin{aligned}\dot{x}_1 &= -ax_1 + bx_2 - x_2^3 \\ \dot{x}_2 &= -ax_2 + bx_3 - x_3^3 \\ \dot{x}_3 &= -ax_3 + bx_1 - x_1^3\end{aligned} \tag{34}$$

where x_1, x_2, x_3 are state variables and *a, b* are constant, unknown, parameters of the system.

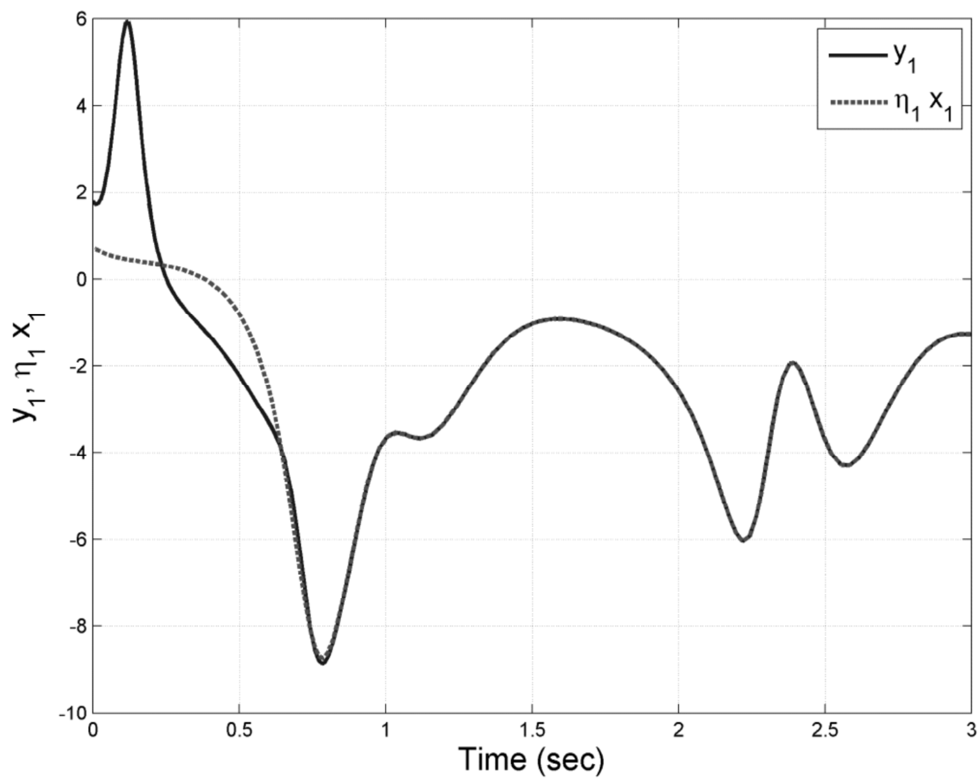

Figure 5: Generalized Projective Synchronization of the States x_1 and y_1

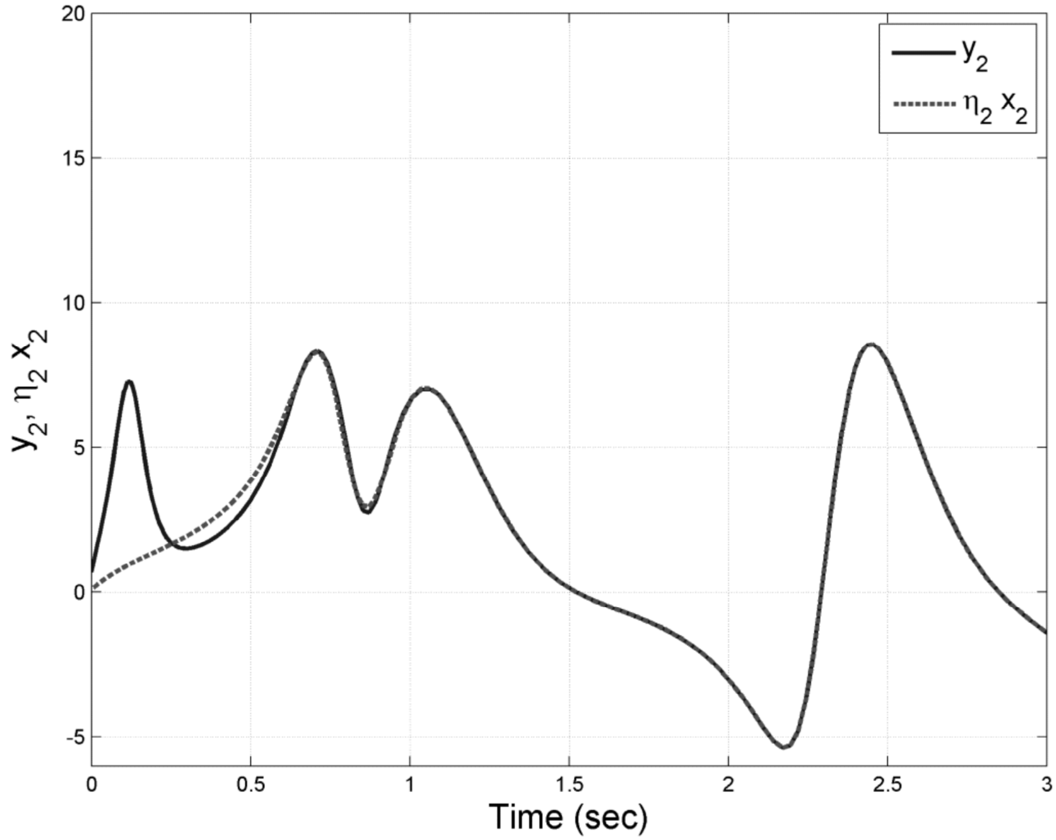

Figure 6: Generalized Projective Synchronization of the States x_2 and y_2

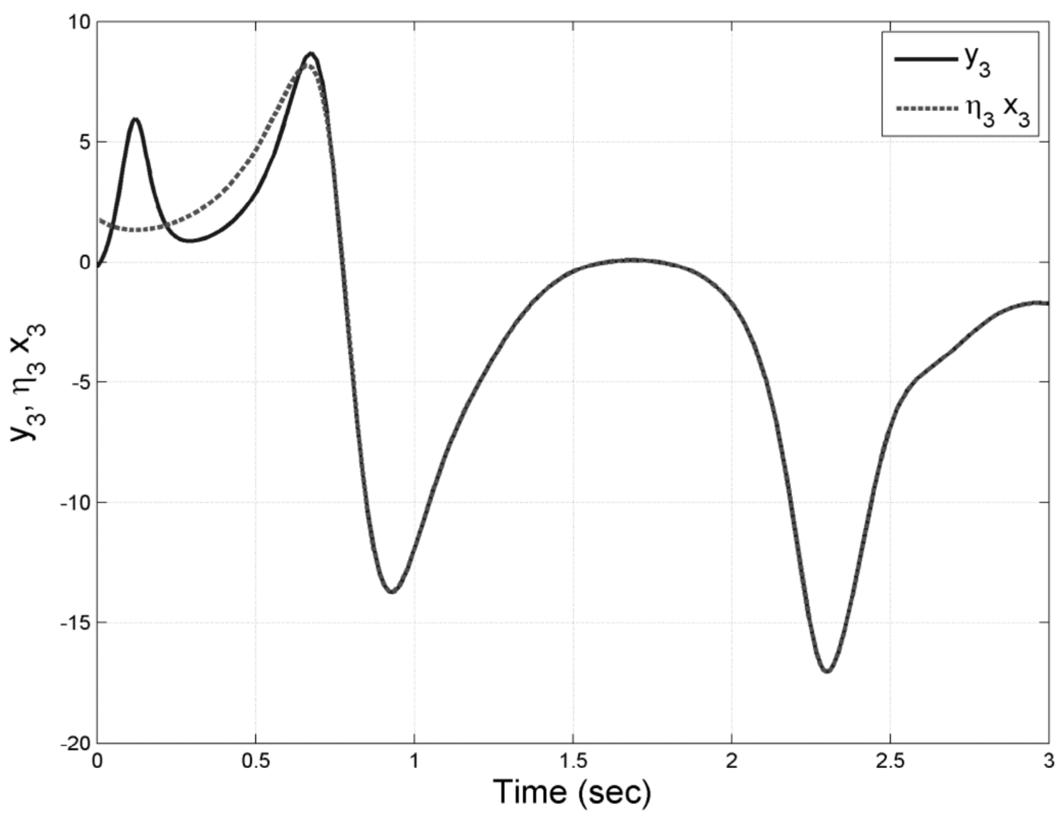

Figure 7: Generalized Projective Synchronization of the States x_3 and y_3

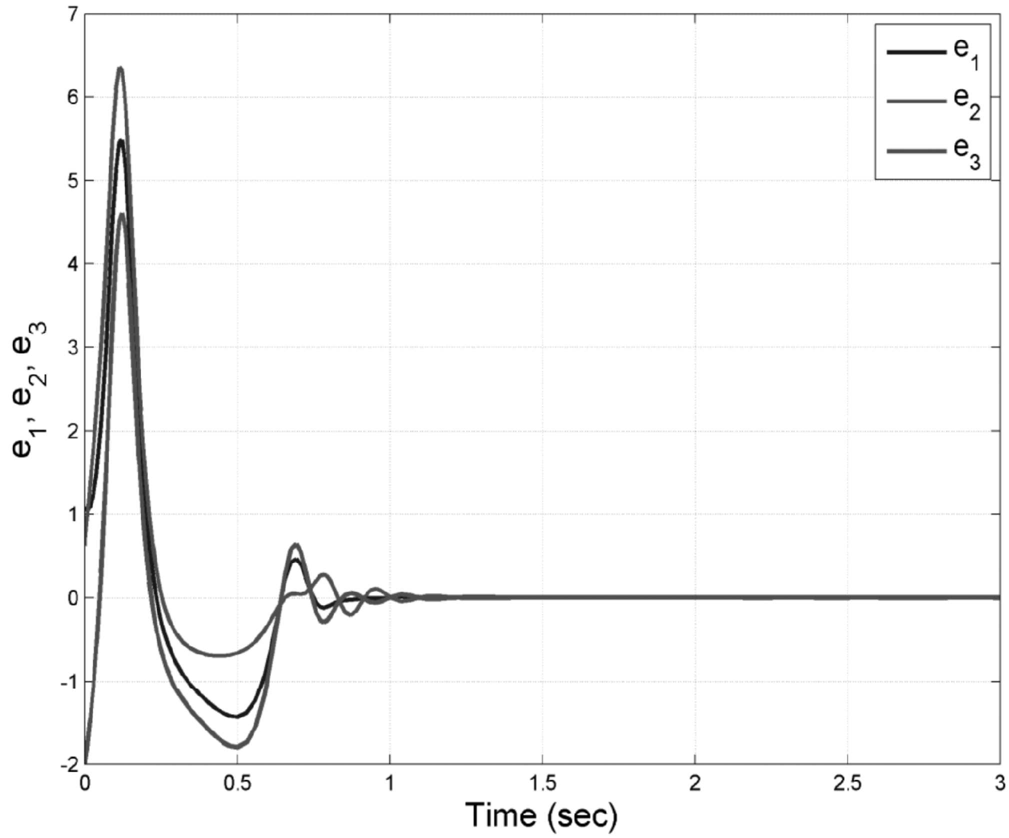

Figure 8: Time-History of the GPS Synchronization Errors e_1, e_2, e_3

As the slave system, we consider the controlled Halvorsen's chaotic system given by

$$\begin{aligned} \dot{y}_1 &= -ay_1 + by_2 - y_2^3 + u_1 \\ \dot{y}_2 &= -ay_2 + by_3 - y_3^3 + u_2 \\ \dot{y}_3 &= -ay_3 + by_1 - y_1^3 + u_3 \end{aligned} \tag{35}$$

where y_1, y_2, y_3 are state variables and u_1, u_2, u_3 are adaptive controls to be designed using estimates $A(t), B(t)$ for the unknown parameters *a*, *b*, respectively.

The generalized projective synchronization (GPS) error is defined by

$$\begin{aligned} e_1(t) &= y_1(t) - \eta_1 x_1(t) \\ e_2(t) &= y_2(t) - \eta_2 x_1(t) \\ e_3(t) &= y_3(t) - \eta_3 x_1(t) \end{aligned} \tag{36}$$

where the GPS scales η_1, η_2, η_3 are real constants.

The GPS error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= -ae_1 + b\left(y_2 - \eta_1 x_2\right) - y_2^3 + \eta_1 x_2^3 + u_1 \\ \dot{e}_2 &= -ae_2 + b\left(y_3 - \eta_2 x_3\right) - y_3^3 + \eta_2 x_3^3 + u_2 \\ \dot{e}_3 &= -ae_3 + b\left(y_1 - \eta_3 x_1\right) - y_1^3 + \eta_3 x_1^3 + u_3 \end{aligned} \tag{37}$$

We consider an adaptive controller defined by

$$\begin{aligned} u_1 &= A(t)e_1 - B(t)\left(y_2 - \eta_1 x_2\right) + y_2^3 - \eta_1 x_2^3 - k_1 e_1 \\ u_2 &= A(t)e_2 - B(t)\left(y_3 - \eta_2 x_3\right) + y_3^3 - \eta_2 x_3^3 - k_2 e_2 \\ u_3 &= A(t)e_3 - B(t)\left(y_1 - \eta_3 x_1\right) + y_1^3 - \eta_3 x_1^3 - k_3 e_3 \end{aligned} \tag{38}$$

where the gains k_1, k_2, k_3 are positive constants.

Substituting the control law (38) into (37), we obtain the closed-loop error dynamics

$$\begin{aligned} \dot{e}_1 &= -(a - A(t))e_1 + (b - B(t))\left(y_2 - \eta_1 x_2\right) - k_1 e_1 \\ \dot{e}_2 &= -(a - A(t))e_2 + (b - B(t))\left(y_3 - \eta_2 x_3\right) - k_2 e_2 \\ \dot{e}_3 &= -(a - A(t))e_3 + (b - B(t))\left(y_1 - \eta_3 x_1\right) - k_3 e_3 \end{aligned} \tag{39}$$

The parameter estimation error is defined by

$$\begin{aligned} e_a(t) &= a - A(t) \\ e_b(t) &= b - B(t) \end{aligned} \tag{40}$$

Using (40), the error dynamics (39) can be simplified as

$$\begin{aligned} \dot{e}_1 &= -e_a e_1 + e_b\left(y_2 - \eta_1 x_2\right) - k_1 e_1 \\ \dot{e}_2 &= -e_a e_2 + e_b\left(y_3 - \eta_2 x_3\right) - k_2 e_2 \\ \dot{e}_3 &= -e_a e_3 + e_b\left(y_1 - \eta_3 x_1\right) - k_3 e_3 \end{aligned} \tag{41}$$

Differentiating (40) with respect to t, we get

$$\begin{aligned} \dot{e}_a &= -\dot{A} \\ \dot{e}_b &= -\dot{B} \end{aligned} \tag{42}$$

We consider the Lyapunov function defined by

$$V(e_1, e_2, e_3, e_a, e_b) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2\right), \tag{43}$$

which is a quadratic and positive definite function on R^5.

Taking the time-derivative of V along the trajectories of (41) and (42), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a\left[-e_1^2 - e_2^2 - e_3^2 - \dot{A}\right] \\ &+ e_b\left[e_1(y_2 - \eta_1 x_2) + e_2(y_3 - \eta_2 x_3) + e_3(y_1 - \eta_3 x_1) - \dot{B}\right] \end{aligned} \tag{44}$$

In view of Eq. (44), the parameter estimates update law is defined as

$$\begin{aligned} \dot{A} &= -e_1^2 - e_2^2 - e_3^2 \\ \dot{B} &= e_1\left(y_2 - \eta_1 x_2\right) + e_2\left(y_3 - \eta_2 x_3\right) + e_3\left(y_1 - \eta_3 x_1\right) \end{aligned} \tag{45}$$

Next, we state and prove the main result of this section.

Theorem 2. The identical Thomas circulant chaotic systems given by (34) and (35) with unknown parameters a, b are globally and exponentially generalized projective synchronized (GPS) by the adaptive controller (38) and the parameter estimates update law (45), where the gains $k_i, (i = 1, 2, 3)$ are positive constants.

Proof: We use Lyapunov stability theory [65] to prove this result.

Consider the quadratic Lyapunov function V defined by Eq. (43).

By substituting the parameter estimates update law (45) into the dynamics (44), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2, \tag{46}$$

which is a quadratic and negative semi-definite function on R^5.

Thus, it can be concluded that the synchronization vector $e(t)$ and the parameter estimation error are globally bounded, i.e.

$$\begin{bmatrix} e_1(t) & e_2(t) & e_3(t) & e_a(t) & e_b(t) \end{bmatrix}^T \in L_\infty. \tag{47}$$

We define

$$k = \min\{k_1, k_2\}. \tag{48}$$

Then it follows from (26) that

$$\dot{V} \le -k\|e\|^2 \text{ or } k\|e\|^2 \le -\dot{V}. \tag{49}$$

Integrating the inequality (49) from 0 to t, we get

$$k\int_o^t \|e(\tau)\|^2 \, d\tau \le -\int_0^t \dot{V}(\tau)d\tau = V(0) - V(t) \tag{50}$$

Therefore, we can conclude that $e(t) \in L_2$.

Using (41), we can conclude that $\dot{e}(t) \in L_\infty$.

Hence, using Barbalat's lemma, we can conclude that $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in R^3$.

This completes the proof.

Numerical Results

For numerical simulations, the classical fourth-order Runge-Kutta method is used to solve the identical Thomas circulant systems (34) and (35) with the adaptive control (38) and the parameter estimates update law (45).

For the Thomas systems (34) and (35), the parameter values are taken as those which result in chaotic behaviour of the systems, *viz.*

$$a = 1, \; b = 4 \tag{51}$$

We take the feedback gains as $k_i = 9$ for $i = 1, 2, 3$.

We take the GPS scales as

$$\eta_1 = 2.1, \quad \eta_2 = 0.7, \quad \eta_3 = -1.4 \tag{52}$$

The initial values of the master system (12) are taken as

$$x_1(0) = -0.3, \; x_2(0) = 1.8, \; x_3(0) = 3.5 \tag{53}$$

The initial values of the slave system (13) are taken as

$$y_1(0) = 1.5, \; y_2(0) = 2.1, \; y_3(0) = -0.8 \tag{54}$$

The initial values of the parameter estimates are taken as

$$A(0) = 25, \ B(0) = 7 \tag{55}$$

Figures 9-11 depict the GPS of the Thomas chaotic systems (34) and (35).

Figure 12 depicts the time-history of the GPS synchronization errors e_1, e_2, e_3.

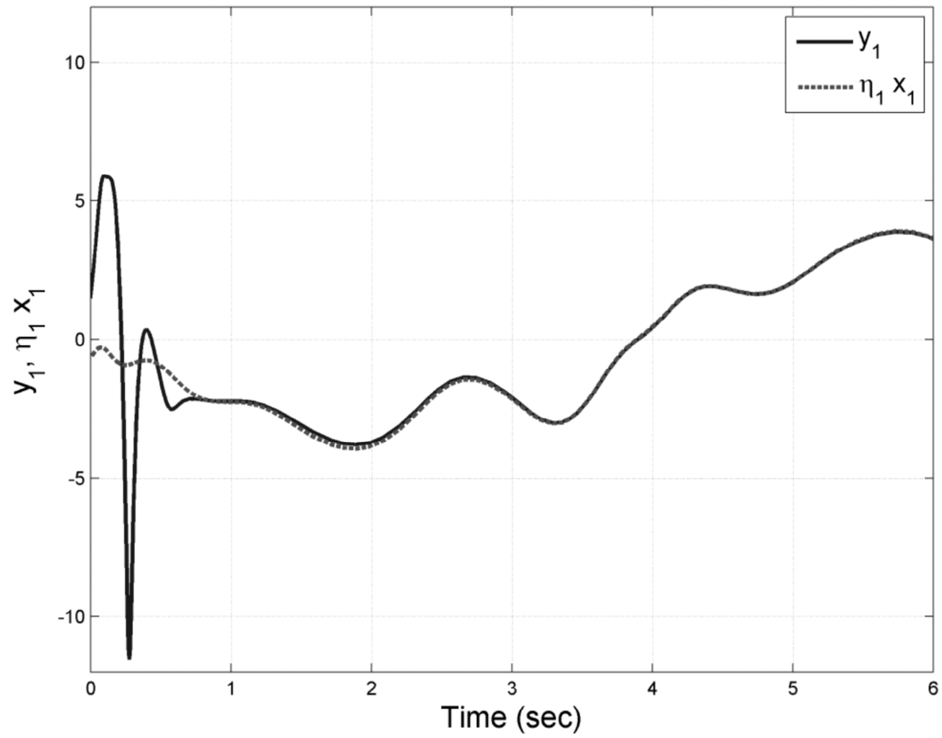

Figure 9: Generalized Projective Synchronization of the States x_1 and y_1

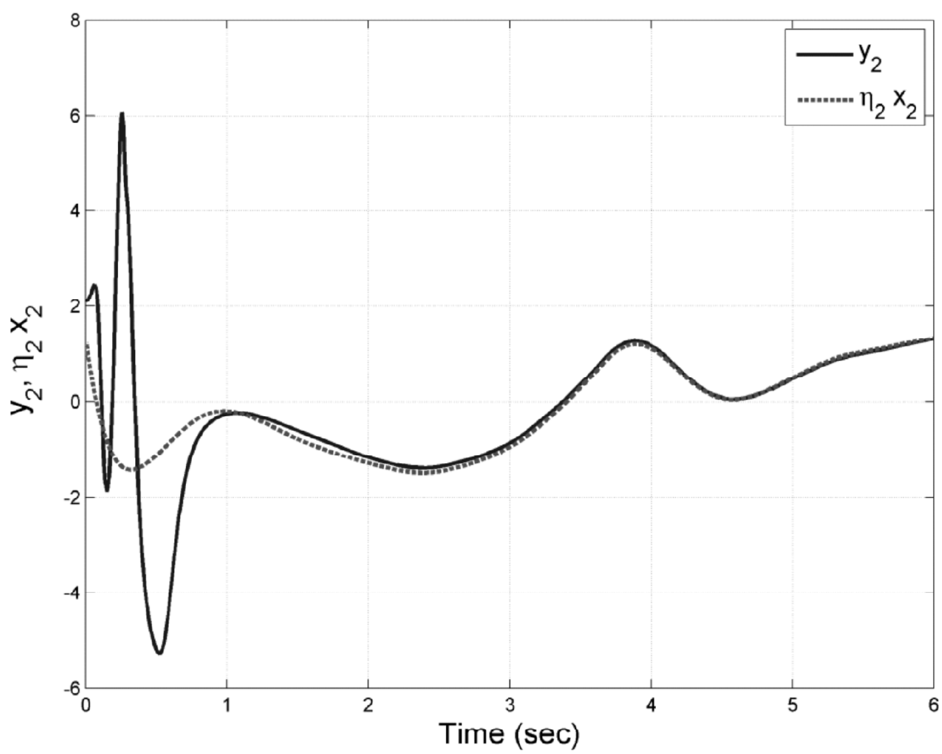

Figure 10: Generalized Projective Synchronization of the States x_2 and y_2

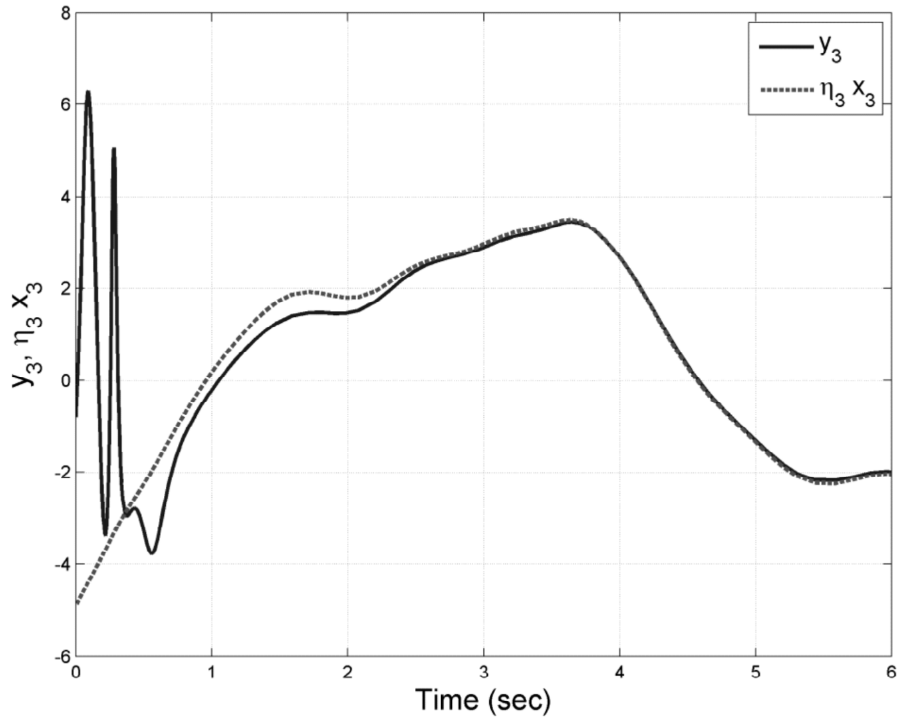

Figure 11: Generalized Projective Synchronization of the States x_3 and y_3

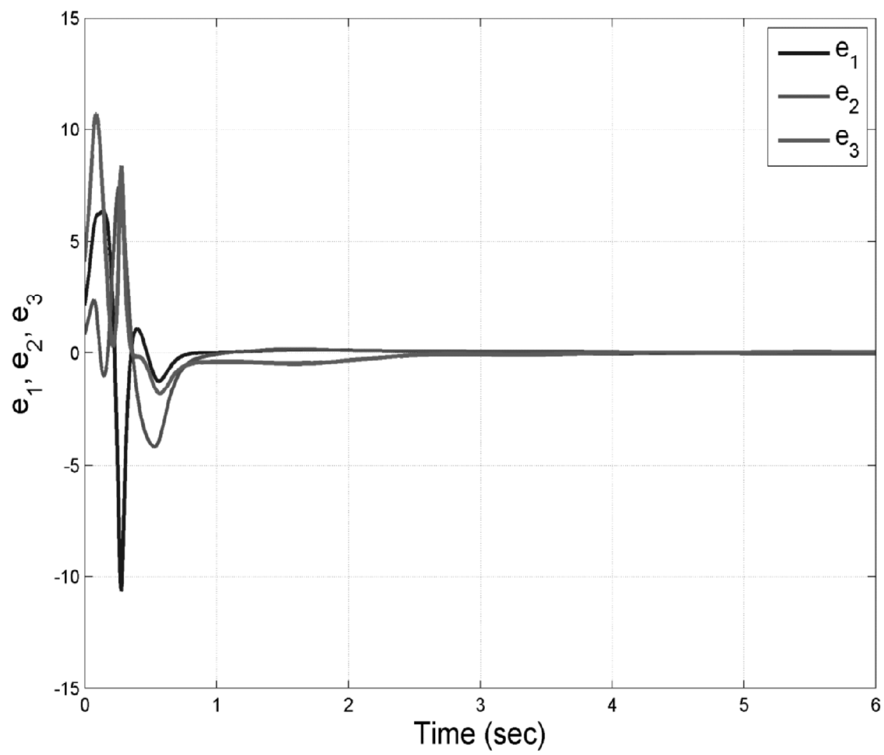

Figure 12: Time-History of the GPS Synchronization errors e_1, e_2, e_3

5. CONCLUSIONS

Generalized projective synchronization is a general type of synchronization, which generalizes common types of synchronization such as complete synchronization (CS), anti-synchronization (AS), hybrid synchronization (HS), projective synchronization (PS), etc. In this paper, we have derived new results for the generalized projective synchronization (GPS) of circulant chaotic systems. Explicitly, we derived adaptive controllers for the generalized projective synchronization of identical Halvorsen's circulant chaotic systems and identical Thomas circulant chaotic systems with unknown parameters. Main results were established using adaptive control theory and Lyapunov stability theory. MATLAB simulations were shown to demonstrate the main results derived in this paper.

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