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THE DOUBLE BARRIER PROBLEM WITH DOUBLE EXPONENTIAL JUMP DIFFUSION

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Abstract. We consider a perturbed double exponential jump diffusion process which starts at a fixed level $u > 0$. We derive an integral equation, an integro-differential equation, and a general form for the probability that the double exponential jump diffusion process reaches a fixed level $b > u$ before it ruins. The analytic solution we obtain can be implemented with mathematical software.

1. Introduction

Let $(\Omega, \mathbb{F}, (\mathbb{F})_{t\in T}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and containing all defined objects. The following processes which are adapted to the aforementioned filtration are critical to this article: let $\{W_t : t \geq 0\}$ be a standard Brownian motion with $W_0 = 0$ and let $\{\overline{N}_t : t \geq 0\}$ and $\{\widehat{N}_t : t \geq 0\}$ be two independent Poisson processes with parameters $\lambda, \lambda_1 \geq 0$ such that $\overline{N}_0 = 0$, $\widehat{N}_0 = 0$ respectively. Also, let $\{C_i : i \geq 1\}$ and $\{Y_i : i \geq 1\}$ be two sequences of independently identically distributed random variables with with exponential densities $f_1(c) = \alpha \exp(-\alpha c)$, $f_2(y) = \beta \exp(-\beta y)$, $\alpha, \beta \geq 0$ respectively. All defined stochastic quantities are independent of each other. We can define a jump diffusion process of the form

$$R_t = u + \sum_{i=1}^{\overline{N}_t} C_i - \sum_{i=1}^{\widehat{N}_t} Y_i + \sigma W_t; \quad R_0 = 0, t \geq 0, \tag{1.1}$$

where u is the initial capital of an insurance company, $\{\overline{N}_t : t \geq 0\}$ the number of insurance policies bought during $[0, t]$, $\{C_i : i \geq 1\}$ the sizes of premiums paid for corresponding policies, $\{\widehat{N}_t : t \geq 0\}$ the number of claims in the interval $[0, t]$ and $\{Y_i : i \geq 1\}$ the corresponding claim sizes. The Brownian motion $\{W_t : t \geq 0\}$ captures random changes or fluctuations in the insurance company and $\sigma > 0$ is a constant.

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From properties of Poisson processes, the sum of two compound Poisson processes is a compound Poisson process. Thus,

$$\sum_{i=1}^{\overline{N}_t} C_i - \sum_{i=1}^{\widehat{N}_t} Y_i = \sum_{i=1}^{N_t} X_i, \tag{1.2}$$

where $\{N_t : t \geq 0\}$, $N_0 = 0$ is a Poisson process with parameter $\lambda_2 = \lambda_1 + \lambda$ and $\{X_i : i \geq 1\}$ is a sequence of independently identically distributed random variables independent of $\{N_t : t \geq 0\}$ and drawn from the distribution

$$f(x) = p\alpha \exp(-\alpha x) I_{\{x \geq 0\}} + q\beta \exp(\beta x) I_{\{x<0\}} \tag{1.3}$$

with $p = \frac{\lambda_1}{\lambda_1+\lambda}$, $q = \frac{\lambda}{\lambda_1+\lambda}$, $f(x) = pf_1(x) + qf_2(x)$ and I_A is the indicator function of a set A. Note that $f_2(y) = \beta \exp(-\beta y) I_{\{y>0\}} = \beta \exp(\beta y) I_{\{y<0\}}$.

Proof. Let $V = \sum_{i=1}^{\overline{N}_t} C_i$ and $K = \sum_{i=1}^{\widehat{N}_t} Y_i$ where $\overline{N}_t$ and $\widehat{N}_t$ are Poisson with parameters λ and λ_1 respectively. Let the moment generating functions of C_i be $M_C(t)$ and that of Y_i be $M_Y(-t)$. Then the moment generating function of V and K are $M_V(t) = \exp\{\lambda(M_Y(-t) - 1)\}$, $M_K(t) = \exp\{\lambda_1(M_C(t) - 1)\}$ respectively. The moment generating function of $V + K$ is

$$\begin{aligned} M_{V+K}(t) &= M_Y(-t)M_K(t) \\ &= \exp\{\lambda(M_Y(-t) - 1)\} \exp\{\lambda_1(M_C(t) - 1)\} \\ &= \exp\left(\lambda_2\left(\frac{\lambda_1 M_C(t)}{\lambda_2} + \frac{\lambda M_Y(-t)}{\lambda_2} - 1\right)\right), \quad \lambda_2 = \lambda + \lambda_1, \end{aligned}$$

which is Poisson with parameter λ_2 and moment generating function

$$\frac{\lambda_1 M_C(t)}{\lambda_2} + \frac{\lambda M_Y(-t)}{\lambda_2} = pM_C(t) + qM_Y(-t).$$

□

The sequence $\{X_i : i \geq 1\}$ drawn from $f(x)$ in (1.3) captures jumps and the Laplace transform of the jump size distribution of X_1, calculated from

$$\xi(s) = \int_{-\infty}^{\infty} \exp(-st) f(t) dt, \quad s > 0 \tag{1.4}$$

is

$$\xi(s) = \begin{cases} \frac{p\alpha}{\alpha+s} + \frac{q\beta}{\beta-s}, & -\alpha < s < \beta \\ \infty, & \text{otherwise.} \end{cases}$$

Considering (1.2), we can rewrite (1.1) as

$$R_t = u + \sum_{i=1}^{N_t} X_i + \sigma W_t; \quad R_0 = 0, t \geq 0. \tag{1.5}$$

Let the jumps of the Poisson process $\{N_t : t \geq 0\}$ occur at random times $T_1, T_2, T_3, \cdots$ and let that $\{N_t : t \geq 0\}$, $\{W_t : t \geq 0\}$ and $\{X_i : i \geq 1\}$ in (1.5) be independent. Ruin occurs when $R_t \leq 0$. Assuming $u = 0$, the expected value of R_t over time period $(0, t]$ is given by

$$\mathbb{E}[R_t] = \left(\frac{p}{\alpha} - \frac{q}{\beta}\right)t.$$

Define $\overline{u} = \left(\frac{p}{\alpha} - \frac{q}{\beta}\right)$. We are concerned with finding the probability that the risk process (1.5) reaches a fixed level $b > u$ before it ruins.

Lokk and Pärna [3] studied a risk process of the form

$$R_t = u + ct - \sum_{i=1}^{\widehat{N}_t} Y_i; \quad t \geq 0, \tag{1.6}$$

where $u \geq 0$ is the initial capital, c is the gross premium rate, $\widehat{N}_t$ is a Poisson process with intensity $\lambda_1 > 0$ independent from the claims represented by Y_i such that $f_2(y) = \beta e^{-\beta y}$, $\beta > 0$. They obtained an integro-differential equation for the probability that the risk process which begins at $u > 0$ reaches an upper level $b > u$ before the ruin occurs. Furthermore, they obtained an analytic solution for the case with exponential claim size distribution.

One of our goals in this work is to modify their risk process (1.6) to the process (1.5). This is done by replacing the gross premium rate c with stochastic premiums of the form $\sum_{i=1}^{\overline{N}_t} C_i$, where $\{\overline{N}_t : t \geq 0\}$ is Poisson with parameter $\lambda > 0$, independent of premiums $\{C_i : i \geq 1\}$ drawn from a distribution $f_1(c) = \alpha \exp(-\alpha c), \alpha > 0$ and also independent of $\sum_{i=1}^{\widehat{N}_t} Y_i$. Further, fluctuations representing random changes to the insurance company are captured by a Brownian motion component $\{W_t : t \geq 0\}$, essentially giving us the model (1.5).

Nadiia and Zinchenko [4] studied the process (1.5) without the Brownian motion perturbation term. By martingale theorems, they proved the Lundberg inequality for ruin probability when both claims and premiums were exponential. Their model was a generalization of Boykov's results [1], who derived exact formulas for ruin and non-ruin probabilities in the special case where both claims and premiums were exponentially distributed.

In the context of risk processes with two-sided jumps, various ruin-related quantities of the double exponential jump diffusion process have been studied [10]. Zhang et al [11] derived the Laplace transforms and defective renewal equations of the discounted penalty function. They also derived the asymptotic estimate for the ruin probability in the case where downward jumps were heavy-tailed and upward jumps had a rational Laplace transform. Kou and Wang [2] obtained explicit solutions for the Laplace transforms of the distribution of the first passage times of the upper barrier and the process and its overshoot. Yin et al [9] extended the results of Kou and Wang [2] to the case where the downward jumps were mixed exponential, and applied these to look-back and barrier options. These results are interesting, but none of them tackles the probability that a double exponential jump diffusion process which starts from level $u > 0$ reaches a fixed level $b > u$ before it ruins. The closest to this is discussed by Sepp [7] in the context of option pricing. Using Laplace transforms, Sepp studied several path dependent options for which the underlying stochastic process was double exponential jump diffusion.

For one-sided jumps, the problem is easier as the jumps are generally opposite to the barrier and there are no overshoot problems. Wang and Wu [8] obtained explicit solutions for the probability that the risk process (1.6) perturbed by diffusion starting from level $u > 0$ reaches a fixed level $b > u$ before it ruins.

In the context of queuing theory, stochastic processes with two-sided jumps can be interpreted as queuing systems with ordinary workload (customers) arrivals and instantaneous work removals causing upward and downward jumps respectively. In stochastic cash management context, the cash flow is modeled by a superposition of Brownian motion and compound Poisson processes with positive and negative jumps representing big deposits and withdrawals respectively. Perry and Stadje [5] derived explicit formulas for the bankruptcy time (time of ruin in insurance context), the time until reaching a pre-specified level without bankruptcy, the maximum cash amount in the system, and the expected discounted revenue generated by the system. The problem tackled by Perry and Stadje [5] closely resembles ours. The only difference is their process stops when it either ruins or attains the upper level b.

Our second aim in this work is to derive integral and integro-differential equations for the probability that a double exponential jump diffusion process which starts from level $u > 0$ reaches a fixed level $b > u$ before it ruins. This is helpful because such a probability tells us that the insurance company can make profits for a particular number of years before it eventually goes burst. It is also important in the underwriting of barrier options whose underlying stochastic process may be a variation of (1.5) as is the case discussed by Sepp [7] in the context of option pricing.

In section 2, we derive integral and integro-differential equations for the probability that the risk process (1.5) reaches or surpasses a fixed level $b > u$ before it ruins. In section 3, we solve the integro-differential equation for the probability that the risk process (1.5) reaches or surpasses a fixed level $b > u$ before it ruins and obtain a general solution. We conclude in section 4.

2. Derivation of Integral and Integro-differential Equations

In this section, we derive integral and integro differential equations for the risk process given by (1.5). Define the time of ruin $T_u = inf\{t \geq 0 : R_t \leq 0\}$ and $T_u = +\infty$ if $R_t \geq 0$ for all $t \geq 0$. Similarly, the ruin probability $\Psi(u)$ is defined as $\Psi(u) = \mathbb{P}(T_u < \infty)$ and survival probability $\Phi(u) = 1 - \Psi(u)$. We are interested in the supremum value before ruin reaching or surpassing a fixed level $b > 0$ when ruin occurs i.e.

$$G(u, b) = \mathbb{P}\bigg(\sup_{0 \leq t \leq T_u} R_t \geq b, T_u < \infty \bigg). \tag{2.1}$$

2.1. Integral equation for $G(u, b)$. We make use of the following functions as in Wang and Wu [8]. Let $a > 0$, define $\tau_a = inf\{s : |W_s| = a\}$, where W_s is the value of Brownian motion at time s. For $y \in [-a, a]$, let

$$H(a, t, y) = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[\exp\bigg\{ -\frac{(y + 4ka)^2}{2t} \bigg\} - exp\bigg\{ -\frac{(y - 2a - 4ka)^2}{2t} \bigg\} \right],$$

$$h(a,t) = \frac{at^{-\frac{3}{2}}}{2\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \Bigg[(4k+1)exp\Bigg\{-\frac{a^2(1+4k)^2}{2t}\Bigg\} + (4k-3)exp\Bigg\{-\frac{a^2(4k-3)^2}{2t}\Bigg\} - 2(4k-1)exp\Bigg\{-\frac{a^2(4k-1)^2}{2t}\Bigg\}\Bigg]. \quad (2.2)$$

Then it follows from [6](pp. 109-110) that $\mathbb{P}(W_s \in dy, \tau_a > s) = H(a,s,y)dy$, and $\mathbb{P}(\tau_a \in ds) = h(a,s)ds$.

For $t \geq 0$, let θ be the shift operator from Ω to itself defined by $R_s(\theta_t w) = R_{s+t}(w)$. For a finite stopping time T, we define the map θ_T from Ω to itself by $\theta_T(w) = \theta_t(w)$ if $T(w) = t$ (see [6] pp. 36, 44, 102). Clearly, $R_t \circ \theta_T = R_{t+T}$.

Proposition 2.1. *Let $b > u > 0$ and $\mathbb{E}[R_t - u] > 0$. Then $G(u,b)$ in (2.1) satisfies the following integral equation:*

$$\begin{aligned} G(u,b) &= \frac{1}{2}\int_0^{+\infty} \exp\{-\lambda_2 t\}\Big[G(u+\sigma a,b) + G(u-\sigma a,b)\Big]h(a,t)dt \\ &+ q\int_0^{+\infty} \lambda_2 \exp\{-T_1\lambda_2\}dT_1 \\ &\times \left(\int_0^{u+\sigma y}\int_{-a}^{+a} H(a,T_1,y)G(u+\sigma y - x,b)dF^-(x)dy\right) \\ &+ \quad p\int_0^{+\infty} \lambda_2 \exp\{-T_1\lambda_2\}dT_1 \\ &\times \left(\int_0^{u+\sigma y}\int_{-a}^{+a} H(a,T_1,y)G(u+\sigma y + x,b)dF^+(x)dy\right), \end{aligned} \quad (2.3)$$

where $0 < a < \left(\frac{(b-u)\wedge u}{\sigma}\right)$.

Proof. Let T_1 be the time of the first jump, $\tau_a = \inf\{t : |W_t| = a\}$, and set $T = \tau_a \wedge T_1$. For $t \in (0,T)$, we have $0 < R_t < b$, thus $\mathbb{P}(T \leq T_u) = 1$. Therefore, we have $T_u = T + T_u \circ \theta_t$ on $(T_u < \infty)$. By the homogenous strong Markov property of R_t (see [6] pp.44 Definition 4.7, Proposition 4.8-4.10 and pp.102-103, Theorem 3.1 and Proposition 3.3), we get

$$\begin{aligned} G(u,b) &= \mathbb{E}\left[I\left(\sup_{0\leq t\leq T_u} R_t \geq b, T_u < \infty\right)\right] \\ &= \mathbb{E}\left[I\left(\sup_{T\leq t\leq T_u} R_t \geq b, T \leq T_u < \infty\right)\right] \\ &= \mathbb{E}\left[I\left(\sup_{0\leq t\leq T_u\circ\theta_T} R_t\circ\theta_T \geq b, 0 \leq T_u\circ\theta_T < \infty\right)\right] \\ &= \mathbb{E}\left[I\left(\sup_{0\leq t\leq T_u} R_t \geq b, T_u < \infty\right)\circ\theta_T\right] = \mathbb{E}\Big[G(R_T,b)\Big] \end{aligned} \quad (2.4)$$

Therefore,

$$G(u,b) = \mathbb{E}\Big[G(R_{\tau_a},b), \tau_a < T_1\Big] + \mathbb{E}\Big[G(R_{T_1},b), \tau_a \geq T_1\Big]. \quad (2.5)$$

Now,

$$\begin{aligned}\mathbb{E}\Big[G(R_{\tau_a},b)\Big] &= \mathbb{E}\Big[G(u+\sigma W_{\tau_a},b)I_{(\tau_a<T_1)}\Big]\\ &= \mathbb{E}\Big[G(u+\sigma a,b)I_{(W_{\tau_a}=a)}I_{(\tau_a<T_1)}\Big]\\ &\quad+\mathbb{E}\Big[G(u-\sigma a,b)I_{(W_{\tau_a}=-a)}I_{(\tau_a<T_1)}\Big]\\ &= \int_0^{+\infty}\exp\{-\lambda_2 t\}G(u+\sigma a,b)\mathbb{P}\big(W_{\tau_a}=a,\tau_a\in dt\big)\\ &\quad+\int_0^{+\infty}\exp\{-\lambda_2 t\}G(u-\sigma a,b)\mathbb{P}\big(W_{\tau_a}=-a,\tau_a\in dt\big)\\ &= \frac{1}{2}\int_0^{+\infty}\Big[G(u+\sigma a,b)+G(u-\sigma a,b)\Big]e^{-\lambda_2 t}h(a,t)dt \qquad (2.6)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big[G(R_{T_1},b),T_1\le\tau_a\Big]\\ &= \mathbb{E}^-\Big[G(u+\sigma W_{T_1}-x,b)I_{(T_1\le\tau_a)}\Big]\\ &\quad+\mathbb{E}^+\Big[G(u+\sigma W_{T_1}+x,b)I_{(T_1\le\tau_a)}\Big]\\ &= q\int_0^{+\infty}\lambda_2\exp\{-T_1\lambda_2\}dT_1\\ &\qquad\times\left(\int_0^{u+\sigma y}\int_{-a}^{+a}H(a,T_1,y)G(u+\sigma y-x,b)dF^-(x)dy\right)\\ &\quad+p\int_0^{+\infty}\lambda_2\exp\{-T_1\lambda_2\}dT_1\\ &\qquad\times\left(\int_0^{u+\sigma y}\int_{-a}^{+a}H(a,T_1,y)G(u+\sigma y+x,b)dF^+(x)dy\right). \qquad (2.7)\end{aligned}$$

The equation (2.3) follows from (2.5), (2.6), and (2.7). □

Equation (2.3) is challenging to solve. Therefore, we focus on deriving integro-differential equations for the process (1.5).

2.2. Integro-differential equation for $G(u,b)$.

Proposition 2.2. *Let $b>u>0$ and $\mathbb{E}[R_t-u]>0$. Assume that $G(u,b)$ in (2.1) is twice continuously differentiable $\forall u\in(0,+\infty)$. Then $G(u,b)$ satisfies the following integral equation:*

$$\frac{\sigma^2}{2}G''(u,b)=\lambda_2\Big[G(u,b)-q\int_0^u G(u-x,b)dF^-(x)-p\int_0^u G(u+x,b)dF^+(x).\Big] \qquad (2.8)$$

Proof. Let ϵ, $t > 0$ such that $\epsilon < u < b - \epsilon$ and define

$$T_t^\epsilon = inf\{s : u + W_s \notin (\epsilon, b - \epsilon)\} \wedge t.$$

Set $T = T_t^\epsilon \wedge T_1$. As in the proof of proposition 2.1,

$$G(u,b) = \mathbb{E}\Big[G(R_T, b)\Big] = \mathbb{E}\Big[G(R_{T_t^\epsilon \wedge T_1}, b)\Big].$$

Therefore,

$$G(u,b) = \mathbb{E}\Big[G(R_{T_t^\epsilon}, b), T_t^\epsilon < T_1\Big] + \mathbb{E}\Big[G(R_{T_1}, b), T_t^\epsilon \geq T_1\Big]. \tag{2.9}$$

We will proceed by conditioning on the first jump and the time it occurs. Before time T_1, there is no jump, therefore

$$\begin{aligned}\mathbb{E}\Big[G(R_{T_t^\epsilon}, b), T_t^\epsilon < T_1\Big] &= \exp\{-t\lambda_2\}\mathbb{E}\Big[G(u + \sigma W_{T_t^\epsilon}, b)\Big] \\ &= \exp\{-t\lambda_2\}\bigg(G(u,b) + \mathbb{E}\Big[\int_0^{T_t^\epsilon} \frac{\sigma^2}{2} G^{''}(u + \sigma W_s, b) ds\Big]\bigg)\end{aligned} \tag{2.10}$$

by Itô's formula. Likewise, at T_1, there is one jump that can either be positive or negative. After T_1, the process propagates till we reach time t. Hence,

$$\begin{aligned}&\mathbb{E}\Big[G(R_{T_1}, b), T_t^\epsilon \geq T_1\Big] \\ &= \int_0^t \lambda_2 \exp\{-s\lambda_2\}\bigg\{\mathbb{E}\Big[G(u + \sigma W_{T_s^\epsilon}, b), T_s^\epsilon > T_1\Big]\bigg\} \\ &\quad + q\int_0^{u+\sigma W_s} \mathbb{E}\Big[G(u + \sigma W_s - x, b), T_s^\epsilon = T_1\Big] dF^-(x) \\ &\quad + p\int_0^{u+\sigma W_s} \mathbb{E}\Big[G(u + \sigma W_s + x, b), T_s^\epsilon = T_1\Big] dF^+(x)\bigg\} ds.\end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) in (2.9) and dividing by t, we get

$$\begin{aligned}&\left(\frac{1 - \exp\{-t\lambda_2\}}{t}\right) G(u,b) \\ &= \exp\{-t\lambda_2\}\mathbb{E}\Big[\frac{1}{t}\int_0^{T_t^\epsilon} \frac{\sigma^2}{2} G^{''}(u + \sigma W_s, b) ds\Big] \\ &\quad + \frac{1}{t}\int_0^t \lambda_2 \exp\{-s\lambda_2\}\bigg\{\mathbb{E}\Big[G(u + \sigma W_{T_s^\epsilon}, b), T_s^\epsilon > T_1\Big] \\ &\quad + q\int_0^{u+\sigma W_s} \mathbb{E}\Big[G(u + \sigma W_s - x, b), T_s^\epsilon = T_1\Big] dF^-(x) \\ &\quad + p\int_0^{u+\sigma W_s} \mathbb{E}\Big[G(u + \sigma W_s + x, b), T_s^\epsilon = T_1\Big] dF^+(x)\bigg\} ds.\end{aligned} \tag{2.12}$$

By letting $t \to 0$ in (2.12), we obtain (2.8). □

3. General Solution for $G(u, b)$

The Laplace exponent of the Lévy process $\sigma W_1 + X_1$ is

$$\varphi(s) = \log\left(\mathbb{E}\left[\exp\{-s(\sigma W_1 + X_1)\}\right]\right)$$
$$= \frac{(s\sigma)^2}{2} + \frac{\alpha\lambda_1}{s+\alpha} + \frac{\beta\lambda}{\beta - s} - 1, \tag{3.1}$$

where $-\alpha < s < \beta$. For every $r \geq 0$, the equation $\varphi(s) = r$ has exactly four roots i.e.

$$r = \frac{(s\sigma)^2}{2} + \frac{\alpha\lambda_1}{s+\alpha} + \frac{\beta\lambda}{\beta - s} - 1. \tag{3.2}$$

Simplifying equation 3.2, we have

$$0 = -s^4\sigma^2 + s^3\sigma^2(\beta - \alpha) + s^2(\sigma^2\alpha\beta + 2(r+1))$$
$$+ 2s(\alpha(r - \lambda_1 + 1) + \beta(\lambda - r - 1)) + 2\alpha\beta(\lambda_1 + \lambda - r - 1) \tag{3.3}$$

clearly a fourth order polynomial of the form $as^4 + bs^3 + cs^2 + ds + e = 0$ where none of a, b, c, d, e is equal to zero. We will solve for and analyze the roots of this fourth order polynomial. Let

$$P_1 = 2c^3 - 9bcd + 27ad^2 + 27be^2 - 72ace,$$
$$P_2 = P_1 + \sqrt{-4(c^2 - 3bd + 12ae)^3 + P_1^2},$$
$$P_3 = \frac{c^2 - 3bd + 12ae}{3a\sqrt[3]{P_2/2}} + \frac{\sqrt[3]{P_2/2}}{3a},$$
$$P_4 = \sqrt{\frac{b^2}{4a} - \frac{2c}{3a} + P_3},$$
$$P_5 = \frac{b^2}{2a^2} - \frac{4c}{3a} - P_3,$$
$$P_6 = \frac{-(b/a)^3 + 4bc/a^2 - 8d/a}{4P_4},$$

then

$$\overline{\theta_{1,2}} = \frac{-b}{4a} - \frac{P_4}{2} \mp \frac{\sqrt{P_5 - P_6}}{2}, \tag{3.4}$$

$$\overline{\theta_{3,4}} = \frac{-b}{4a} + \frac{P_4}{2} \mp \frac{\sqrt{P_5 - P_6}}{2}. \tag{3.5}$$

We can use the transformations

$$\nu_i = \mp\frac{\sqrt{P_5 - P_6}}{2} = \overline{\theta_i} + \frac{b}{4a} + \frac{P_4}{2}, i = 1, 2$$
$$\eta_i = \mp\frac{\sqrt{P_5 - P_6}}{2} = \overline{\theta_i} + \frac{b}{4a} - \frac{P_4}{2}, i = 3, 4$$

to see that $Re(\nu_1) = -Re(\nu_2)$ and $Re(\eta_1) = -Re(\eta_2)$, where Re stands for real part. This will mean there are two positive roots and two negative roots. In the Laplace space, we focus on the positive real part of the two positive roots designated $\overline{\theta_1}$ and $\overline{\theta_3}$.

$$As \quad r \to 0, \quad \overline{\theta_1} \to \left\{ \begin{array}{ll} 0, & \text{if } \overline{u} \geq 0 \\ \overline{\theta_1^*}, & \text{if } \overline{u} < 0 \end{array} \right. , \overline{\theta_3} \to \overline{\theta_3^*} \quad likewise. \}$$

We have defined $\overline{u} = \left(\frac{p}{\alpha} - \frac{q}{\beta}\right)$, $\overline{\theta_1^*}$ and $\overline{\theta_3^*}$ are unique positive real roots i.e.

$$\varphi(\overline{\theta_1^*}) = 0 = \varphi(\overline{\theta_3^*}), \quad 0 < \overline{\theta_1^*} < \beta < \overline{\theta_3^*} < \infty. \tag{3.6}$$

Thus, we can suggest that $G(u,b)$ is of the form

$$G(u,b) = \left\{ \begin{array}{ll} A\exp\{\overline{\theta_1^*}(b-u)\} + B\exp\{-\overline{\theta_3^*}(b-u)\}, & \text{u < b} \\ 1, & u \geq b \end{array} \right\} \tag{3.7}$$

with constants A and B. Our primary aim now will be to determine the nature of the constants A and B.

From definition,

$$G(u,u) = \Psi(u).$$

Therefore,

$$A + B = \Psi(u). \tag{3.8}$$

Lets calculate $\Psi(u)$, the ruin probability. We assume that $\overline{u} > 0$. We use [5] the martingale defined by

$$m(t) = (\varphi(s) - r)\int_0^t \exp\{-sR_t - ts\}dt + \exp\{-su\} - \exp\{-sR_t - rt\},$$

$r, t \geq 0, \quad -\alpha < Re(s) < \beta$. Using the fact that $\mathbb{E}[m(T_u)] = \mathbb{E}[m(0)]$, we obtain

$$(\varphi(s) - r)\int_0^{T_u} \exp\{-sR_t - ts\}dt = -\exp\{-su\} + \mathbb{E}\big[\exp\{-sR_{T_u} - rT_u\}\big]. \tag{3.9}$$

There are just two ways R_t can go below zero i.e. continuously or by a jump. Therefore,

$$\mathbb{E}\big[\exp\{-sR_{T_u} - rT_u\}\big] = \mathbb{E}\big[\exp\{-rT_u\}I_{(R_{T_u}=0)}\big] + \frac{\beta}{\beta - s}\mathbb{E}\big[\exp\{-rT_u\}I_{(R_{T_u}<0)}\big]. \tag{3.10}$$

Let $\rho_1 = \mathbb{E}\big[\exp\{-rT_u\}I_{(R_{T_u}=0)}\big]$ and $\rho_2 = \mathbb{E}\big[\exp\{-rT_u\}I_{(R_{T_u}<0)}\big]$. Using unique roots $\overline{\theta_1^*}$ and $\overline{\theta_3^*}$ in (3.9), we get

$$\rho_1 = \exp\{-\overline{\theta_1^*}u\} + \frac{\beta}{\overline{\theta_1^*} - \beta}\rho_2,$$

$$\rho_2 = \frac{(\beta - \overline{\theta_1^*})(\overline{\theta_3^*} - \beta)}{\beta(\overline{\theta_3^*} - \overline{\theta_1^*})}\bigg(\exp\{-\overline{\theta_1^*}u\} - \exp\{-\overline{\theta_3^*}u\}\bigg).$$

As well,

$$\Psi(u) = \rho_1 + \rho_2 \tag{3.11}$$

because the inverse Laplace transform of the constant 1 is the Dirac delta function and $T_u < \infty$ for all $u < \infty$. Let us return and consider the various parts of 2.8 i.e. the various parts of

$$\frac{\sigma^2}{2}G^{''}(u,b) = \lambda_2\Big[G(u,b) - q\int_0^u G(u-x,b)dF^-(x) - p\int_0^u G(u+x,b)dF^+(x)\Big].$$

$$G(u,b) = A\underbrace{\exp\{\overline{\theta_1^*}(b-u)\}}_{N_0} + B\underbrace{\exp\{-\overline{\theta_3^*}(b-u)\}}_{N_1}. \tag{3.12}$$

$$G^{''}(u,b) = A\underbrace{\frac{\exp\{(b-u)\overline{\theta_1^*}\}}{\overline{\theta_1^*}^2}}_{K_0} + B\underbrace{\frac{\exp\{-(b-u)\overline{\theta_3^*}\}}{\overline{\theta_3^*}^2}}_{L_0}. \tag{3.13}$$

$$\begin{aligned}\int_0^u G(u-x,b)dF^-(x) &= -\int_0^u \Big[A\exp\{(b-(u-x))\overline{\theta_1^*}\} \\ &\quad + Be^{-(b-(u-x))\overline{\theta_3^*}}\Big]\beta\exp\{\beta x\}dx \qquad (3.14) \\ &= A\underbrace{\frac{\beta\exp\{(b-u)\overline{\theta_1^*}\}}{\beta+\overline{\theta_1^*}}\Big(\exp\{(\overline{\theta_1^*}+\beta)u\}-1\Big)}_{K_1} \\ &\quad + B\underbrace{\frac{\beta\exp\{-(b-u)\overline{\theta_3^*}\}}{\beta-\overline{\theta_3^*}}\Big(\exp\{(\overline{\theta_3^*}+\beta)u\}-1\Big)}_{L_1}. \qquad (3.15)\end{aligned}$$

$$\begin{aligned}\int_0^u G(u+x,b)dF^+(x) &= \int_0^u \Big[A\exp\{(b-(u+x))\overline{\theta_1^*}\} \\ &\quad + B\exp\{-(b-(u+x))\overline{\theta_3^*}\}\Big]\alpha\exp\{\alpha x\}dx \qquad (3.16) \\ &= A\underbrace{\frac{\alpha\exp\{(b-u)\overline{\theta_1^*}\}}{\alpha+\overline{\theta_1^*}}\Big(1-\exp\{-(\overline{\theta_1^*}+\alpha)u\}\Big)}_{K_2} \\ &\quad + B\underbrace{\frac{\alpha\exp\{-(b-u)\overline{\theta_3^*}\}}{\overline{\theta_3^*-\alpha}}\Big(1-\exp\{(\overline{\theta_3^*}-\alpha)u\}\Big)}_{L_2}. \qquad (3.17)\end{aligned}$$

Substituting (3.12), (3.13), (3.14), and (3.17) in (2.8) and simplifying,

$$B = A\underbrace{\frac{\frac{\sigma^2 K_0}{2} - \lambda_2(N_0 - qK_1 - pK_2)}{\lambda_2(N_1 - qL_1 - pL_2) - \frac{\sigma^2 L_0}{2}}}_{J} \tag{3.18}$$

Using (3.8), (3.11) and (3.18), the constants A and B are obtained as

$$A = \frac{\rho_1 + \rho_2}{J+1}, \quad B = JA.$$

Therefore we have proved the following proposition:

Proposition 3.1. *Let $\overline{u} < 0$, $b > u \geq 0$ and let $\overline{\theta_1^*}, \overline{\theta_3^*} > 0$. Then,*

$$G(u,b) = \begin{cases} A\exp\{\overline{\theta_1^*}(b-u)\} + B\exp\{-\overline{\theta_3^*}(b-u)\}, & u < b \\ 1, & u \geq b \end{cases}$$

Following the lead of Wang and Wu [8], we can similarly define the probability that the supreme value before ruin reaches or surpasses the level b i.e.

$$\begin{aligned} \Gamma(u,b) &= \mathbb{P}\left(\sup_{0\leq t\leq T_u} R_t \geq b\right) \\ &= \mathbb{P}\left(\sup_{0\leq t< T_u} R_t \geq b, T_u < \infty\right) + \mathbb{P}\left(\sup_{0\leq t< T_u} R_t \geq b, T_u = \infty\right) \\ &= G(u,b) + \Phi(u). \end{aligned}$$

4. Conclusion

In this paper, we have derived an integral equation, an integro-differential equation and a general formula for the probability that a double exponential jump diffusion process which starts from level $u > 0$ reaches a fixed level $b > u$ before it ruins. The general equation obtained can be implemented with mathematical software.

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