IJAMAA
January-June 2014, Volume 9, No. 1, pp. 67-74

RECURRENCE RELATION FOR SINGLE AND PRODUCT MOMENTS OF RECORD VALUE FROM GENERALIZED EXPONENTIAL DISTRIBUTION AND CHARACTERIZATION

SANJAY KUMAR SINGH

ABSTRACT: In the present study, we give some recurrence relations satisfied by single and product moments of record values from the generalized exponential distribution. Using a recurrence relation for single moments we obtain a characterization of generalized exponential distribution.

Keyword and phrases: order statistics, moments; product moments; upper record values; k-th record value, generalized exponential distribution.

1. INTRODUCTION

A random variable X is said to have generalized exponential distribution of the pdf is of the form

$$f(x) = (1-\alpha x)^{\left(\frac{1}{\alpha}\right)-1}, \quad 0 \le x < \frac{1}{\alpha}, \, 0 \le \alpha < 1, \tag{1.1}$$

and cdf is of the form

$$F(x) = 1-(1-\alpha x)^{\frac{1}{\alpha}}, \quad 0 \le x < \frac{1}{\alpha}, \, 0 \le \alpha < 1, \tag{1.2}$$

generalized exponential distribution in (1.1) has been widely studied by Raqab (2004) among others.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d) random variables from generalized exponential distribution given in (1.1) and (1.2) with pdf and cdf respectively. Set $Y_n = \max(\min)\{X_1, \ldots, X_n\}$, $n \ge 1$. We say X_j is an upper (lower) record value of this sequence if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as a lower record value. The indices at which the upper (lower) record values occurrence given by record times $\{U(n), n \ge 1\}$ where

$U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}\}$, $n > 1$, with $U(1) = 1$ Chandler(1952) introduced record values and record value times.Feller (1966) gave some examples of record values with respect to gambling problems. Properties of record values of i.i.d. r.v.'s have been extensively studied in the literature; for example, see Ahsanullah (1988), Nagaraja (1988),Nevzorov (1987), Arnold and Balakrishnan (1989) , Arnold , Balakrishnan and Nagaraja (1992) and Balakrishnan and Ahsanullah (1994) for recent interviews. One can transform from upper record and lower record by replacing the original sequence by $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(x_i > 0) = 1$ for all i) by $\left\{\frac{1}{X_i}, i \geq 1\right\}$; the lower record values of this sequence will correspond to the upper record values from generalized exponential distribution.

If we let the shape parameter $\alpha \to 0$, the pdf in (1.1) and the cdf in (1.2) becomes e^{-x} and $1-e^{-x}$, respectively, for $x \geq 0$. This is the standard exponential distribution discussed in detail by Joshi (1978, 1982).

The family of generalized exponential distribution is an IFR (Increasing Failure Rate) family. Therefore, the generalized exponential distribution discussed in this section a quite useful as a life span model (see Cohen and Whitten, 1998). Its application include use in the analysis of extreme events, as a failure-time distribution in relaiblility studies, in the mode thing of large insurance claims, and in any situation in which the exponential distribution might be used but in which some robustness is required against heavier tailed or lighter tailed alternatives.

First or all, we may note that for the generalized exponential distribution is (1.1) and (1.2) the characterizing differential equation is

$$(1-\alpha x) f(x) = 1 - F(x), \quad 0 \leq x < \frac{1}{\alpha}, 0 \leq \alpha < 1. \tag{1.3}$$

The relation in (1.3) will be used to desire some simple recurrence relations for the single and product moments of upper record values from generalized exponential distribution. Using these relations we give characterization of generalized exponential distribution (G.E.D.)

2. RELATIONS FOR SINGLE AND PRODUCT MOMENTS

Let $X_{U(1)} < X_{U(2)} < \ldots$ be the sequence of upper record values from (1.1). Fro convenience, we shall also take $X_{U(0)} = 0$. Then the pdf of $X_{U(n)}$, $n = 1, 2, \ldots$, is given by

$$f_n(x) = \frac{1}{(n-1)!}[H(x)]^{n-1} f(x), \qquad 0 \le x < \infty \tag{2.1}$$

where $H(x) = -\ln(1-F(x))$.

The joint density function of $X_{U(m)}$ and $X_{U(n)}$, $1 \le m < n$ is as follows

$$f_{m,n}(x,y) = \frac{1}{(m-1)!(n-m-1)!}[\ln(1-F(x)) - \ln(1-F(y))]^{n-m-1}$$
$$.[-\ln(1-F(x))]^{m-1} \frac{f(x)}{1-F(x)} f(y), \qquad x < y \tag{2.2}$$

Theorem 2.1: For $n \ge 1$ and $r = 0,1,2,\dots$, we have from (1.1) and (1.3)

$$E(X_{U(n)}^{r+1}) - \alpha E(X_{U(n)}^{r+1}) = \frac{1}{(n-1)!}\int_0^\infty x^r (1-\alpha x) f_n(x)dx$$
$$= \frac{1}{(n-1)!}\int_0^\infty x^r (1-\alpha x)\{-\ln(1-F(x))\}^{n-1} f(x)dx$$
$$= \frac{1}{(n-1)!}\int_0^\infty x^r \{-\ln(1-F(x))\}^{n-1}(1-F(x))dx \quad \text{(using (1.3))}$$

Integrating by parts, taking x^r as the point to be integrated we get (2.3)

$$= \frac{1}{(n-1)!}\Bigg[\left|\frac{x^{r+1}}{r+1}\{-\ln(1-F(x))\}^{n-1}(1-F(x))\right|_0^\infty$$
$$-\int_0^\infty \frac{x^{r+1}}{r+1}(-f(x))\{-\ln(1-F(x))\}^{n-1}$$
$$+(1-F(x))(n-1)\{-\ln(1-F(x))\}^{n-2}\frac{f(x)}{1-F(x)}\Bigg]dx$$

$$= \frac{1}{(n-1)!}\int_0^\infty \frac{x^{r+1}}{r+1}\{-\ln(1-F(x))\}^{n-1} f(x)dx$$
$$-\frac{1}{(n-1)!}\int_0^\infty \frac{x^{r+1}}{r+1}\{-\ln(1-F(x))\}^{n-2} f(x)dx$$

$$= \frac{1}{(n-1)!}\int_0^\infty \frac{x^{r+1}}{r+1}\{-\ln(1-F(x))\}^{n-1} f(x)dx$$
$$- \frac{1}{(n-2)!}\int_0^\infty \frac{x^{r+1}}{r+1}\{-\ln(1-F(x))\}^{n-2} f(x)dx$$

$$= \frac{1}{r+1}E(X_{U(n)}^{r+1}) - \frac{1}{r+1}E(X_{U(n-1)}^{r+1}).$$

Therefore

$$E(X_{U(n)}^{r}) - \alpha E(X_{U(n)}^{r+1}) = \frac{1}{r+!}E(X_{U(n)}^{r+1}) - \frac{1}{r+1}E(X_{U(n-1)}^{r+1})$$

or

$$\frac{1}{r+1}E(X_{U(n)}^{r-1}) + \alpha E(X_{U(n)}^{r+1}) = E(X_{U(n)}^{r}) - \frac{1}{r+1}E(X_{U(n-1)}^{r+1})$$

or

$$E(X_{U(n)}^{r-1}) = \frac{r+1}{1+\alpha+\alpha r} + E(X_{U(n)}^{r+1}) + \frac{1}{r+\alpha+\alpha r}E(X_{U(n-1)}^{r+1})$$

which, when rewritten, gives the recurrence relation in (2.3). Then by repeatedly applying the recurrence relation in (2.3), we simply derive the recurrence.

Remark 1: The recurrence relation in Theorem 2.1 can be used in simple recursive manner to compute all single moments of record values. By setting $r = 0$ in (2.3) we get the relation.

Once again, using the characterizing differential equation in (1.3), some simple recurrence relations for the product moments of record values, as follows.

Theorem 2.2: For $m \geq 1$ and $r, s = 0, 1, 2, \ldots,$

$$E(X_{U(m)}^{r+1}, X_{U(m+1)}^{s+1}) = E(X_{U(m)}^{r+s+1}) + (s+1)E(X_{U(m)}^{r} X_{U(m+1)}^{s}) \tag{2.4}$$

and for $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \ldots,$

$$E(X_{U(m)}^{r}, X_{U(n)}^{s+1}) = \frac{s+1}{1+\alpha+\alpha s}E[(X_{U(m)}^{r})(X_{U(n)}^{s})] + \frac{1}{1+\alpha+\alpha s}E[(X_{U(m)}^{r})(X_{U(n)}^{s})] \tag{2.5}$$

Proof. From (2.2) for $1 \leq m \leq n-1$ and $r, s = 0, 1, 2, \ldots,$ and on using (1.3) we get

$$E(X_{U(m)}^{r}, X_{U(n)}) - \alpha E[(X_{U(m)}^{r})(X_{U(n)}^{s+1})]$$
$$= \iint_{0\le x<y<\infty} (x^{r} y - \alpha x^{r} y^{s+1}) f_{m,n}(x,y)dxdy$$
$$= \frac{1}{(m-1)!(n-m-1)!}\int x^{r}\{-\ln(1-F(x))\}^{m01}\frac{f(x)}{1-F(x)}I(x)dx \qquad (2.6)$$

Upon using the relation in (2.3). Integrating now by parts treating y^{s} for integration and the rest of the integrated for differentiation, we obtain

$$I(x) = -\int_{x}^{\infty}\frac{y^{s+1}}{s+1}[(-f(y))\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-1}$$
$$+(n-m-1)\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-2}\frac{f(y)}{1-F(y)}(1-F(y))dy$$
$$= \frac{1}{s+1}\int_{x}^{\infty} y^{s+1}\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-2} f(y)dy$$
$$-\frac{(n-m-1)}{s+1}\int_{x}^{\infty} y^{s+1}\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-2} f(y)dy$$

Upon substituting the above expression of $I(x)$ in equation (2.6) and simplifying, we obtain

$$E[(X_{U(m)}^{r})(X_{U(n)}^{s})] - \alpha E[(X_{U(m)}^{r})(X_{U(n)}^{s+1})]$$
$$= \frac{1}{(m-1)!(n-m-1)!}\int x^{r}\{-\ln(1-F(x))\}^{m-1}\frac{f(x)}{1-F(x)}$$
$$.\frac{1}{s+1}\int y^{s+1}\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-1} f(y)dy$$
$$-\frac{n-m-1}{s+1}\int y^{s+1}\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-1} f(y)dy]$$

or

$$E[(X_{U(m)}^{r})(X_{U(n)}^{s+1})]\left\{\alpha + \frac{1}{s+1}\right\} = E[(X_{U(m)}^{r})(X_{U(n)}^{s})] + \frac{1}{s+1}E[(X_{U(m)}^{r})(X_{U(n-1)}^{s+1})]$$

or

$$E[(X_{U(m)}^{r})(X_{U(n)}^{s+1})] = \frac{s+1}{1+\alpha+\alpha s}E[(X_{U(m)}^{r})(X_{U(n)}^{s+1})] + \frac{1}{1+\alpha+\alpha s}E[(X_{U(m)}^{r})(X_{U(n-1)}^{s+1})]$$

3. CHARACTERIZATION

This section contains characterization of the generalized exponential distribution. We start with the following result of Lin (1986).

Proposition. Let n_0 be any fixed non negative integer, $-\infty \le a < b < \infty$ and $g(x) \ge 0$ and absolutely continuous function with $g'(x) \ne 0$ a.e. on (a,b). Then the functional $\{[g(x)]^n e^{-g(x)}, n \ge n_0\}$ is complete in $L(a,b)$ iff $g(x)$ is strictly monotone on (a,b).

Using above proposition we get a stronger version of Theorem 2.1.

Theorem 3.1. For $n \ge 1$ and let r be non negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1.1) is that

$$\begin{aligned} E[(X_{U(m)}^{r+!})] &= \frac{r+1}{1+\alpha+\alpha r} E(X_{U(m)}^{r}) + \frac{r+1}{1+\alpha+\alpha r} (X_{U(n-1)}^{r+1}) \\ &= \frac{r+1}{1+\alpha+\alpha r} [(r+1)E(X_{U(m)}^{r}) + E(X_{U(n-1)}^{r+1}) \quad for \quad n = 1, 2, \ldots \end{aligned} \tag{3.1}$$

Proof. The necessary part follows immediately from (3.1). On the other hand of the recurrence relation in (3.1) is satisfied, then

$$(1+\alpha+\alpha r)E[(X_{U(m)}^{r+!})] = (r+1)E(X_{U(m)}^{r}) + E(X_{U(n-1)}^{r+1})$$

or

$$(r+1)E(X_{U(m)}^{r+1}) = (1+\alpha+\alpha r)E(X_{U(m)}^{r+!}) - E(X_{U(n-1)}^{r+1})$$

or

$$(r+1)\int_0^\infty x^n f_n(x)dx = (1+\alpha+\alpha r)\int_0^\infty x^n f_n(x)dx - \int_0^\infty x^{r+1} f_{n-1}(x)dx$$

Integrating the last integral by parts, we have, the above integral reduces to

$$\int x^n \{-\ln(1-F(x))\}^{n-1}[f(x) - (1-\alpha x)^{\frac{1}{\alpha}-1}]dx = 0$$

follows from above proposition with

$$g(x) - \ln(1-F(x))$$

and hence

$$f(x) = (1-\alpha x)^{\left(\frac{1}{\alpha}\right)-1}$$

Which is the pdf of the generalized exponential distribution.

4. k -TH RECORD VALUE

The single and product moments of generalized exponential distribution is given as follows

Theorem 4.1. Fix a positive integer $k \geq 1$. For $n \geq 1$ and $r = 0,1,2,...$

$$E(X_{U(n)}^{(k)})^{r+1} = \frac{r+1}{1+\alpha+\alpha r} E(X_{U(n)}^{(k)})^{r+1} + \frac{r+1}{K+\alpha+\alpha r} E(X_{U(n-1)}^{(k)})^{r+1} \tag{4.1}$$

Theorem 4.2. For $1 \leq m \leq n-2$ and $r,s = 0,1,2,...$

$$E[(X_{U(m)}^{(k)})^r (X_{U(n)}^{(k)})^{s+1}] = \frac{s+1}{K+\alpha+\alpha r} E[(X_{U(m)}^{(k)})^r (X_{U(n)}^{(k)})^s] + \frac{1}{K+\alpha+\alpha r} E[(X_{U(m)}^{(k)})^r (X_{U(n-1)}^{(k)})^s] \tag{4.2}$$

and for $m \geq 1$ and

$$E[(X_{U(m)}^{(k)})^r (X_{U(n)}^{(k)})^{s+1}] = (s+1)E[(X_{U(m)}^r X_{U(m+1)}^s)] + E[(X_{U(m)}^{r+s+1})] \tag{4.3}$$

Remark 1. When in Theorem 4.1 and Theorem 4.2, we get the recurrence relation for single and product moments of ordinary record values for generalized exponential distribution which verifies the result of Saran and Pushkarna (2000)and also Saran and Singh(2008).

5. CONCLUSION

In this study some recurrence relations for single and product moments of upper record values from the generalized exponential distribution have been established, which generalize the corresponding results for upper 1-record values from the generalized exponential distribution. Further, these recurrence relations have been utilized to obtain a characterization of the generalized exponential distribution by using a result of Lin (1986). Similar results have been obtained by Saran and Pushkarna(2000).

References

Ahsanullah ,M.(1988). Introduction to Record Values, Ginn Press, Needham Heights,MA.

Arnold, B.C., Balakrishnan ,N. and Nagaraja, H.N. (1992). A First Course in Order Statistics, New York: John Wiley & Sons

Balakrishnan, N., & Ahsanullah, M. (1994). Relations for Single and Product Moments of Record Values from Lomax Distribution The Indian Journal of Statistics, Series B (1960-2002)

Chandler, K.M. (1952). The distribution and frequency of record values. J. Roy. Statist. Soc, 14(5): 220–228. Ser. B

Cohen,C. and Whitten(1998) Handbook of Statistics, volume 17, pp 283-314.

Feller, W. (1966). An Introduction to Probability Theory and Its Applications, Vol. 2, New York: John Wiley and Sons.

Joshi, P. C. (1978). Recurrence relations between moments of order statistics from exponential and truncated exponential distributions. Sankhya, Ser. B, 39, 362-3710.

Joshi (1982). A note on the mixed moments of order statistics from exponential and truncated distributions. vol 6, issue 1, pp 13-16.

Lin,G.D., (1986). On a moment problem. Tohoku Math. J.,38:595-598

Nevzorov, V.B. 1987. Records. Theo. Prob. Appl, 32: 201–228.

Raqab,M.Z.,(2004) Sharp bounds on the expectations of second record values from symmetric populations-Taylor & Francis

Saran, J. and N. Pushkarna, (2000). Relationships for moments of record values from linear-exponential distribution. J. Applied Statist. Sci., 10: 69-76.

Saran,J. and Singh,S.K., (2008). Recurrence Relations for Single and Product Moments of k-th Record Values from Linear-Exponential Distribution and a Characterization. Asian Journal of Mathematics & Statistics, 1: 159-164.

Sanjay Kumar Singh
Department of Statistics,
PGDAV College
University of Delhi,
India