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AR(1) SEQUENCE WITH RANDOM COEFFICIENTS: REGENERATIVE PROPERTIES AND ITS APPLICATION

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Abstract. Let $\{X_n\}_{n\geq 0}$ be a sequence of real valued random variables such that $X_n = \rho_n X_{n-1} + \epsilon_n$, $n = 1, 2, \ldots$, where $\{(\rho_n, \epsilon_n)\}_{n\geq 1}$ are i.i.d. and independent of initial value (possibly random) X_0. In this paper it is shown that, under some natural conditions on the distribution of (ρ_1, ϵ_1), the sequence $\{X_n\}_{n\geq 0}$ is regenerative in the sense that it could be broken up into i.i.d. components. Further, when ρ_1 and ϵ_1 are independent, we construct a non-parametric strongly consistent estimator of the characteristic functions of ρ_1 and ϵ_1.

1. Introduction

Let $\{X_n\}_{n\geq 0}$ be a sequence of real valued random variables satisfying the stochastic recurrence equation

$$X_n = \rho_n X_{n-1} + \epsilon_n, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $\{(\rho_n, \epsilon_n)\}_{n\geq 1}$ are i.i.d. $\mathbb{R}^2$-valued random vectors and independent of the initial random variable X_0. If $E(|X_0|) < \infty$ and $E(\epsilon_n) = 0$, for each $n \geq 1$ and then $E(X_n|X_0, \ldots, X_{n-1}) = E(\rho_n)X_{n-1}$. For this reason the sequence $\{X_n\}$ satisfying (1.1) is often referred to in the time series literature as Random Coefficient Auto Regressive sequence of order one (RCAR(1)) (see [1, 6, 7, 9]). [5] studied a parametric model for (ρ_1, ϵ_1) under the assumption that ρ_1 and ϵ_1 are independent and provided a consistent estimator of the model parameters. In the current paper, we find conditions on the distribution function of (ρ_1, ϵ_1) to ensure that $\{X_n\}$ is a Harris recurrent Markov chain and hence regenerative, i.e., it can be broken up into i.i.d. excursions. We exploit the regenerative property of $\{X_n\}$ to construct a non-parametric consistent estimator of the characteristic functions of ρ_1 and ϵ_1 under the independence assumption of ρ_1 and ϵ_1.

A sequence $\{X_n\}_{n\geq 0}$ is said to be delayed *regenerative* if there exists a sequence $\{T_j\}_{j\geq 1}$ of positive integer valued random variables such that $\mathbb{P}(0 < T_{j+1} - T_j < \infty) = 1$ for all $j \geq 1$ and the random cycles $\eta_j \equiv (\{X_i : T_j \leq i < T_{j+1}\}, T_{j+1} - T_j)$ for $j = 1, 2, \ldots$ are i.i.d. and independent of $\eta_0 \equiv (\{X_i : 0 \leq i < T_1\}, T_1)$. If

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$\{\eta_j\}_{j\geq 0}$ are i.i.d. then $\{X_n\}$ is called non-delayed regenerative sequence. If, in addition, $E(T_2 - T_1) < \infty$ then $\{X_n\}$ is called regenerative and positive recurrent.

If $\{X_n\}$ is a Markov chain with a general state space $(S, \mathcal{S})$, that is Harris irreducible and recurrent (see Definition 3.1) then it can be shown that $\{X_n\}$ is regenerative ([4]). Further if $\{X_n\}$ admits a stationary probability measure (necessarily unique because of irreducibility), then $\{X_n\}$ is positive recurrent regenerative as well.

In Sections 2 and 3, under some condition on the distribution of (ρ_1, ϵ_1) we show that the sequence $\{X_n\}$ satisfying (1.1) is positive recurrent and regenerative by establishing that $\{X_n\}$ admits a stationary distribution and is Harris irreducible, respectively. In Section 4, we show that the distribution of (ρ_1, ϵ_1) can be determined by transition probability function of $\{X_n\}$. We subsequently provide a consistent estimator of transition probability function of $\{X_n\}$ by using the regenerative property. Finally, if ρ_1 and ϵ_1 are independent then we provide a non-parametric consistent estimator of characteristic function of ρ_1 and ϵ_1, based on $\{X_n\}_{n\geq 0}$.

2. Limit Distribution of X_n

We begin with existence of the limiting distribution of X_n in (1.1).

Theorem 2.1. *Let* $-\infty \leq \mathbb{E}(\log|\rho_1|) < 0$ *and* $\mathbb{E}(\log|\epsilon_1|)^+ < \infty$. *Then* $\{X_n\}$ *in* (1.1) *converges in distribution to* X_∞ *as* $n \to \infty$ *where*

$$X_\infty \equiv \epsilon_1 + \rho_1\epsilon_2 + \rho_1\rho_2\epsilon_3 + \ldots + \rho_1 \ldots \rho_n\epsilon_{n+1} + \ldots. \tag{2.1}$$

The infinite series on the right hand side of (2.1) *is absolutely convergent with probability 1.*

The above result can be deduced from [6]. A proof of Theorem 2.1 is given in the appendix. Theorem 2.1 does not indicate nature of limiting distribution of X_n. We show that the distribution of X_∞ is non-atomic when the distribution of (ρ_1, ϵ_1) is non-degenerate.

Theorem 2.2. *Let* $-\infty \leq \mathbb{E}(\log|\rho_1|) < 0$, $\mathbb{E}(\log|\epsilon_1|)^+ < \infty$, $\mathbb{P}(\rho_1 = 0) = 0$ *and* (ρ_1, ϵ_1) *has a non-degenerate distribution. Then* X_∞ *has a non atomic distribution, i.e.,* $\mathbb{P}(X_\infty = a) = 0$ *for all* $a \in \mathbb{R}$.

Proof. Since (ρ_1, ϵ_1) has a nondegenerate distribution, the random variable X_∞ as in (2.1) does not have a degenerate distribution and hence $\sup\{\mathbb{P}(X_\infty = a) : a \in \mathbb{R}\} \equiv p < 1$. Let a_0 be such that $\mathbb{P}(X_\infty = a_0) = p$. Then by Doob's martingale convergence theorem (see page 211 of [3]), we have

$$\mathbb{E}(\mathbb{I}(X_\infty = a_0)|\mathcal{F}_n) \to \mathbb{E}(\mathbb{I}(X_\infty = a_0)|\mathcal{F}_\infty) \quad \text{w. p. 1} \tag{2.2}$$

where, $\mathcal{F}_n \equiv \sigma\{(\rho_i, \epsilon_i) : i = 1, 2, \ldots, n, X_0\}$, the σ-algebra generated by (ρ_i, ϵ_i) for $i = 1, \ldots, n$ and X_0, and $\mathcal{F}_\infty \equiv \sigma\{(\rho_i, \epsilon_i) : i \in \mathbb{N}, X_0\}$. Since X_∞ is measurable with respect to $\mathcal{F}_\infty$, $\mathbb{E}(\mathbb{I}(X_\infty = a_0)|\mathcal{F}_\infty) = \mathbb{I}(X_\infty = a_0)$. Next,

$$\begin{aligned}
&\mathbb{E}(\mathbb{I}(X_\infty = a_0)|\mathcal{F}_n) \\
&= \mathbb{P}(\epsilon_1 + \rho_1\epsilon_2 + \cdots + \rho_1 \cdots \rho_{n-1}\epsilon_n + \rho_1 \cdots \rho_n(\epsilon_{n+1} + \rho_{n+1}\epsilon_{n+2} + \cdots) = a_0|\mathcal{F}_n) \\
&= \mathbb{P}\left(Y_n = \frac{a_0 - \epsilon_1 - \rho_1\epsilon_2 - \cdots - \rho_1\rho_2 \cdots \rho_{n-1}\epsilon_n}{\rho_1\rho_2 \cdots \rho_n}|\mathcal{F}_n\right)
\end{aligned}$$

where $Y_n = \epsilon_{n+1} + \rho_{n+1}\epsilon_{n+2} + \rho_{n+1}\rho_{n+2}\epsilon_{n+3} + \cdots$ and last equality holds since $\mathbb{P}(\rho_1 = 0) = 0$, $|\rho_1 \cdots \rho_n| \neq 0 \;\forall\, n \geq 1$. But Y_n and X_∞ have the same distribution, and Y_n is independent of $\mathcal{F}_n$ and $\frac{a_0 - \epsilon_1 - \rho_1\epsilon_2 - \cdots - \rho_1\rho_2\cdots\rho_{n-1}\epsilon_n}{\rho_1\rho_2\cdots\rho_n}$ is $\mathcal{F}_n$ measurable. So

$$\mathbb{E}(\mathbb{I}(X_\infty = a_0)|\mathcal{F}_n) \leq p < 1 \text{ for all } n \geq 1.$$

From (2.2), it follows that $\mathbb{I}(X_\infty = a_0) \leq p < 1$ with probability 1. Since $\mathbb{I}(X_\infty = a_0)$ is a $\{0, 1\}$ valued random variable, $\mathbb{I}(X_\infty = a_0) = 0$ with probability 1 and hence $\mathbb{P}(X_\infty = a_0) = 0$. Hence, X_∞ has a non atomic distribution. □

A natural question is under what additional conditions on the distribution of (ρ_1, ϵ_1), the sequence $\{X_n\}$ is regenerative. When a Markov sequence is Harris recurrent and σ-algebra is countably generated then it can be established that the sequence exhibits regenerative property (see [4]). We now explore the Harris recurrence property of $\{X_n\}$.

3. Harris Recurrence of X_n

Definition 3.1. A Markov chain $\{X_n\}_{n\geq 0}$ is called *Harris or ϕ-recurrent* if there exists a σ-finite measure ϕ on the state space $(S, \mathcal{S})$ such that

$$\phi(A) > 0 \quad \Longrightarrow \quad \mathbb{P}(\tau_A < \infty|X_0 = x) = 1 \;\forall x \in S, \tag{3.1}$$

where $\tau_A = \min\{n : n \geq 1, X_n \in A\}$.

Note that any irreducible and recurrent Markov chain with a countable state space is Harris recurrent as one can take ϕ to be the δ measure at some $i_0 \in S$. A definition related to Definition 1 is given by [4].

Definition 3.2. A Markov chain $\{X_n\}$ is called (A, ϵ, ϕ, n_0) *recurrent* if there exists a set $A \in \mathcal{S}$, a probability measure ϕ on S, a real number $\epsilon > 0$, and an integer $n_0 > 0$ such that

$$\mathbb{P}(\tau_A < \infty|X_0 = x) \equiv \mathbb{P}_x(\tau_A < \infty) = \mathbb{P}_x(X_n \in A \text{ for some } n \geq 1) = 1 \quad \forall\, x \in S \tag{3.2}$$

and

$$\mathbb{P}(X_{n_0} \in E|X_0 = x) \equiv \mathbb{P}_x(X_{n_0} \in E) = \mathbb{P}^{(n_0)}(x, E) \geq \epsilon\phi(E) \quad \forall\, x \in A\;,\; \forall\, E \subset \mathcal{S}. \tag{3.3}$$

It can be shown by using the C-set lemma of Doob (see [8]) that when $\mathcal{S}$ is countably generated, then Definition 3.1 implies Definition 3.2. That Definition 3.2 implies Definition 3.1 is not difficult to prove.

The following theorem provides a sufficient condition for $\{X_n\}$ in (1.1) to be a Harris recurrent Markov chain.

Theorem 3.3. *Let* $-\infty \leq \mathbb{E}(\log|\rho_1|) < 0$, $\mathbb{E}(\log|\epsilon_1|)^+ < \infty$, $\mathbb{P}(\rho_1 = 0) = 0$ *and* $-\infty < c < d < \infty$ *be such that* $\mathbb{P}(c \leq X_\infty \leq d) > 0$. *Then for all* $x \in \mathbb{R}$,

$$\mathbb{P}(X_n \in [c,d] \quad \textit{for some} \quad n \geq 1 | X_0 = x) = 1. \tag{3.4}$$

In addition, let there exists a finite measure ϕ *on* $\mathcal{R}$ *such that* $\phi([c,d]) > 0$ *and* $0 < \alpha < 1$ *such that*

$$\inf_{c \leq x \leq d} \mathbb{P}(\rho_1 x + \epsilon_1 \in \cdot) \geq \alpha\phi(\cdot). \tag{3.5}$$

Then the Markov chain $\{X_n\}_{n\geq 0}$ *as described in (1.1) is Harris recurrent and hence regenerative.*

Note that since X_n converges in distribution to X_∞ which is a proper real valued random variable, $\{X_n\}_{n\geq 0}$ is positive recurrent as well. Thus under the hypothesis of Theorem 3.3, $\{X_n\}_{n\geq 0}$ is regenerative and positive recurrent. The proof of Theorem 3.3 is based on the following results.

Lemma 3.4. *Let* $-\infty \leq \mathbb{E}(\log|\rho_1|) < 0$, $\mathbb{E}(\log|\epsilon_1|)^+ < \infty$, $\mathbb{P}(\rho_1 = 0) = 0$ *and* $-\infty < c < d < \infty$ *be such that* $\mathbb{P}(c \leq X_\infty \leq d) > 0$. *Then there exist* $\theta > 0$ *and for all* $x \in \mathbb{R}$, *an integer* $n_x \geq 1$ *such that*

$$\mathbb{P}(X_n \in [c,d] | X_0 = x) \geq \theta \quad \textit{for all } n \geq n_x. \tag{3.6}$$

Proof. Iterating (1.1) yields,

$$X_n = \rho_n\rho_{n-1}\cdots\rho_1 X_0 + \rho_n\rho_{n-1}\cdots\rho_2\epsilon_1 + \cdots + \rho_n\epsilon_{n-1} + \epsilon_n \equiv Z_n X_0 + Y_n, \text{ say.}$$

So, if $X_0 = x$ w. p. 1, then

$$\begin{aligned}\mathbb{P}_x(X_n \in [c,d]) &= \mathbb{P}(Y_n + Z_n x \in [c,d]) \\ &\geq \mathbb{P}(Y_n \in [c+\eta, d-\eta], |Z_n x| < \eta) \\ &\geq \mathbb{P}(Y_n \in [c+\eta, d-\eta]) - \mathbb{P}(|Z_n x| \geq \eta),\end{aligned}$$

where $\eta > 0$ such that $c + \eta < d - \eta$. Now, define

$$Y_n' \equiv \epsilon_1 + \rho_1\epsilon_2 + \rho_1\rho_2\epsilon_3 + \cdots + \rho_1 \ldots \rho_{n-1}\epsilon_n. \tag{3.7}$$

Note that the distribution of Y_n and Y_n' are same and from Theorem 2.1, $Y_n' \to X_\infty$ with probability 1. Thus, we have

$$\begin{aligned}&\mathbb{P}_x(X_n \in [c,d]) \\ &\quad\geq \mathbb{P}(Y_n' \in [c+\eta, d-\eta]) - \mathbb{P}(|Z_n x| \geq \eta) \\ &\quad\geq \mathbb{P}(Y_n' \in [c+\eta, d-\eta], |Y_n' - X_\infty| \leq \eta') - \mathbb{P}(|Z_n x| \geq \eta) \\ &\quad\geq \mathbb{P}(X_\infty \in [c+\eta+\eta', d-\eta-\eta']) - \mathbb{P}(|Y_n' - X_\infty| \geq \eta') - \mathbb{P}(|Z_n x| \geq \eta),\end{aligned}$$

where $\eta' > 0$ such that $c + \eta + \eta' < d - \eta - \eta'$.

Now choose n_1 large such that $\mathbb{P}(|Y_n' - X_\infty| \geq \eta') \leq \frac{\delta}{2}$ and n_2 large such that $\mathbb{P}(|Z_{n_2} x| \geq \eta) \leq \frac{\delta}{2}$. Note that choice of n_2 depends on x. Let $n_x = \max(n_1, n_2)$. Then for all $n \geq n_x$,

$$\mathbb{P}_x(X_{n_x} \in [c,d]) \geq \mathbb{P}(X_\infty \in [c+\eta+\eta', d-\eta-\eta']) - \delta.$$

Since X_∞ has a continuous distribution by Theorem 2.2 and $\mathbb{P}(c \le X_\infty \le d) > 0$, first choose η and $\eta^{'}$ and then δ small enough such that

$$\theta \equiv \mathbb{P}(X_\infty \in [c+\eta+\eta^{'}, d-\eta-\eta^{'}]) - \delta > 0.$$

Thus (3.6) is established. □

Lemma 3.5. *Let $\{X_n\}$ be a time homogeneous Markov chain with state space $(S, \mathcal{S})$ and transition function $P(\cdot,\cdot)$. Let there exists $A \in \mathcal{S}$ and $0 < \theta \le 1$ such that for all $x \in S$, there exists an integer $n_x \ge 1$ such that*

$$\mathbb{P}(X_{n_x} \in A | X_0 = x) \ge \theta. \tag{3.8}$$

Then for all $x \in S$,

$$\mathbb{P}(\tau_A < \infty | X_0 = x) = 1 \tag{3.9}$$

where $\tau_A = \min\{n : \ n \ge 1, X_n \in A)\}$.

Proof. Fix $x \in S$. Let $B_0 \equiv \{X_{n_x} \notin A\}$ and $\tau_0 = n_x$. Then $B_0 \equiv \{X_{\tau_0} \notin A\}$. Let us define

$$\begin{aligned} B_1 &\equiv \{X_{\tau_0} \notin A,\ X_{\tau_0 + n_{X_{\tau_0}}} \notin A\} \\ \tau_1 &= \tau_0 + n_{X_{\tau_0}} \\ B_2 &\equiv \{X_{\tau_0} \notin A,\ X_{\tau_1} \notin A,\ X_{\tau_1 + n_{X_{\tau_1}}} \notin A\} \\ \tau_2 &= \tau_1 + n_{X_{\tau_1}}, \end{aligned}$$

and so on. Note $B_1 = \{X_{\tau_0} \notin A, X_{\tau_1} \notin A\}$, $B_2 = \{X_{\tau_0} \notin A, X_{\tau_1} \notin A, X_{\tau_2} \notin A\}$ and for any integer $k \ge 3$,

$$B_k \equiv \{X_{\tau_0} \notin A, X_{\tau_1} \notin A, \ldots, X_{\tau_k} \notin A\},$$

with $\tau_k = \tau_{k-1} + n_{X_{\tau_{k-1}}}$. By hypothesis (3.8), $\mathbb{P}((B_0) \le (1-\theta)$. By the strong Markov property of $\{X_n\}$, $\mathbb{P}(B_1) \le (1-\theta)^2$ and $\mathbb{P}(B_k) \le (1-\theta)^{k+1}$ for all integer $k \ge 3$. This implies $\sum_{k=0}^{\infty} \mathbb{P}(B_k) < \infty$ since $\theta > 0$. So $\sum_{k=0}^{\infty} \mathbb{I}_{B_k}(\cdot) < \infty$ with probability 1. This implies that with probability 1, $\mathbb{I}_{B_k} = 0$ for all large $k > 1$. That is, for all $x \in S$, $\mathbb{P}_x(X_{\tau_k} \in A$ for some $k < \infty) = 1$. Hence, for all $x \in S$, $\mathbb{P}_x(\tau_A < \infty) = 1$. □

Proof of Theorem 3.3. In view of Definition 3.2, it is enough to prove (3.4) to show $\{X_n\}$ is Harris recurrent. The proof of (3.4) follows from Lemma 3.4 and 3.5. Now from Lemma 2.2.5 of [2], it follows that X_n is regenerative. □

Theorem 3.3 provides sufficient conditions on (ρ_1, ϵ_1) so that the sequence X_n becomes Harris recurrent and hence regenerative. These sufficient conditions are fairly general and hold for large class of distribution of (ρ_1, ϵ_1). Here are some examples where (3.4) and (3.5) hold.

Example 3.6. ϵ_1 is a standard normal, $N(0,1)$ random variable, ρ_1 has bounded support with $\mathbb{E}\log|\rho_1| < 0$ and ϵ_1, ρ_1 are independent.

Example 3.7. ϵ_1 is a Uniform $(-1, 1)$ random variable, ρ_1 has bounded support with $\mathbb{E}\log|\rho_1| < 0$ and ϵ_1, ρ_1 are independent.

In both the cases hypothesis of Theorem 2.1 hold and X_∞ is of the form $(\epsilon_1 + \rho_1\tilde{X}_\infty)$ where $\tilde{X}_\infty$ has the same distribution as X_∞ and independent of X_∞. One can show in both above cases that for some $c < 0 < d$, $|c|$ and d sufficiently small, conditions (3.4) and (3.5) hold.

In Theorem 3.3, growth sequence $\{\rho_n\}$ has no mass at zero and the regeneration property of X_n is established by showing Harris recurrence of the sequence. When $\mathbb{P}(\rho_1 = 0) > 0$, then the regenerative property of X_n can be shown more easily.

Theorem 3.8. *Let $\{X_n\}_{n\geq 0}$ be a RCAR(1) sequence as in* (1.1). *If $\mathbb{P}(\rho_1 = 0) \equiv \alpha > 0$, then $\{X_n\}_{n\geq 0}$ is a positive recurrent regenerative sequence.*

Proof. Let $\tau_0 = 0$ and $\tau_{j+1} = \min\{n : n \geq \tau_j + 1, \rho_n = 0\}$ for $j \geq 0$. We need to show that

$$\begin{aligned} &\mathbb{P}(\tau_{j+1} - \tau_j = k_j, X_{\tau_j + l} \in A_{l,j}, 0 \leq l < k_j, 1 \leq j \leq r) \\ &= \prod_{j=1}^{r} \mathbb{P}(\tau_2 - \tau_1 = k_j, X_{\tau_1 + l} \in A_{l,j}, 0 \leq l < k_j) \end{aligned} \tag{3.10}$$

for all $k_1, k_2, \ldots, k_r \in \mathbb{N}$ and $A_{l,j} \in \mathcal{B}(\mathbb{R})$, $0 \leq l < k_j, j = 1, 2, \ldots, r$, $r = 1, 2, \ldots$.

Since $\{(\rho_n, \epsilon_n)\}_{n\geq 1}$ are i.i.d. and $\mathbb{P}(\rho_1 = 0) = \alpha > 0$, it follows that $\{\tau_{j+1} - \tau_j, j \geq 0\}$ are i.i.d. with jump distribution

$$\mathbb{P}(\tau_{j+1} - \tau_j = k) = (1-\alpha)^{k-1}\alpha, \quad \text{for} \quad k = 1, 2, \ldots,$$

that is, geometric with "success" parameter α. Next, since $\{(\rho_n, \epsilon_n)\}_{n\geq 1}$ are i.i.d. (3.10) follows. Further since $\mathbb{E}(\tau_2 - \tau_1) < \infty$, the sequence $\{X_n\}$ is positive recurrent regenerative. □

Remark 3.9. When $\mathbb{P}(\rho_1 = 0) = \alpha > 0$ and the joint distribution of (ρ_1, ϵ_1) is discrete, then the limiting distribution π of X_∞ is a discrete probability distribution, that is, there exists a countable set A_0 in $\mathbb{R}^2$ such that $\pi(A_0) = 1$. This is in contrast to Theorem 2.2 which provides a sufficient condition for X_∞ to have a non atomic distribution.

4. Estimation of Transition Function and Characteristic Functions of ρ_1 and ϵ_1

The transition function $\mathbb{P}(x, A)$ of the Markov chain $\{X_n\}_{n\geq 0}$, defined by (1.1), is precisely equal to $\mathbb{P}(\rho_1 x + \epsilon_1 \in A)$. The following result determines the joint distribution of (ρ_1, ϵ_1) in terms of the transition function, $\mathbb{P}(\cdot, \cdot)$.

Theorem 4.1. *If the distribution of $\rho_1 x + \epsilon_1$ is known for all x of the form $\frac{t_1}{t_2}$ where $t_2 \neq 0$ and (t_1, t_2) is dense in $\mathbb{R}^2$ then the distribution of (ρ_1, ϵ_1) is determined.*

Proof. For any $(t_1, t_2) \in \mathbb{R}^2$, the characteristic function of (ρ_1, ϵ_1) is

$$\psi_{(\rho_1,\epsilon_1)}(t_1, t_2) = \mathbb{E}(e^{i(t_1\rho_1 + t_2\epsilon_1)}) = \mathbb{E}(e^{it_2(\rho_1 \frac{t_1}{t_2} + \epsilon_1)}) = \phi_{\frac{t_1}{t_2}}(t_2)$$

where $\phi_x(t) = \mathbb{E}(e^{it(\rho_1 x + \epsilon_1)})$ for all $x, t \in \mathbb{R}$. If $\phi_x(\cdot)$ is known for all x of the form $\frac{t_1}{t_2}$ where (t_1, t_2) is dense in $\mathbb{R}^2$, then $\psi_{(\rho_1,\epsilon_1)}(t_1, t_2)$ is determined for all such (t_1, t_2) and hence by continuty for all $(t_1, t_2) \in \mathbb{R}^2$. Hence the distribution of (ρ_1, ϵ_1) is determined completely. □

Theorem 4.1 implies that if the transition function $\mathbb{P}(x, A)$ of $\{X_n\}_{n\geq 0}$ can be determined from observing the sequence sequence $\{X_n\}$, then the distribution of (ρ_1, ϵ_1) can also be determined. We now estimate the transition probability function $\mathbb{P}(x, (-\infty, y])$, for $x, y \in \mathbb{R}^2$ from the data $\{X_i\}_{i=0}^n$. In the following theorems, we show that the estimator $F_{n,h}(x, y)$, given in (4.1) below, is a strongly consistent estimator for $\mathbb{P}(X_1 \leq y|X_0 = x)$.

Theorem 4.2. *Let $\{X_n\}$ satisfies the hypothesis of Theorem 3.3. For $n \geq 1$, $h > 0$, $x, y \in \mathbb{R}$, let*

$$F_{n,h}(x,y) = \begin{cases} \frac{\frac{1}{nh}\sum_{i=0}^{n-1}\mathbb{I}(x\leq X_i\leq x+h, X_{i+1}\leq y)}{\frac{1}{nh}\sum_{i=0}^{n}\mathbb{I}(x\leq X_i\leq x+h)} & \text{if } \mathbb{I}(x \leq X_i \leq x+h) \neq 0 \text{ for some } i \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\mathbb{I}(A)$ denotes the indicator function of the event A.

(a) Then with probability 1, for each $x, y \in \mathbb{R}$

$$\lim_{n\to\infty} F_{n,h}(x,y) \equiv \psi(x,y,h) = \frac{\int_x^{x+h} G(u,y)\mathbb{P}(X_\infty \in du)}{\mathbb{P}(X_\infty \in (x, x+h])}, \tag{4.2}$$

where $G(u, y) = \mathbb{P}(x, (-\infty, y]) = \mathbb{P}(X_1 \leq y|X_0 = x)$.

(b) In addition, let $G(x, y)$ and the random variable X_∞ satisfy

$$\lim_{h\to 0} \frac{\int_x^{x+h} G(u,y)\mathbb{P}(X_\infty \in du)}{\mathbb{P}(x < X_\infty \leq x+h)} = G(x,y), \quad \text{for } \ x,y \in \mathbb{R}^2. \tag{4.3}$$

Then for $x, y \in \mathbb{R}$

$$\lim_{h\to 0}\lim_{n\to 0} F_{n,h}(x,y) = \mathbb{P}(X_1 \leq y|X_0 = x), \text{ with probability } 1. \tag{4.4}$$

Proof. Since $\{X_i\}_{i\geq 0}$ is regenerative and positive recurrent, the vector sequence $\{(X_i, X_{i+1})\}_{i\geq 0}$ is also regenerative and positive recurrent Markov chain. The numerator in (4.1) converges to $\int_x^{x+h} G(u, y)\mathbb{P}(X_\infty \in du)$ with probability 1 by using Theorem 9.2.10 of [3]. Similarly denominator converges to $\mathbb{P}(X_\infty \in (x, x+h])$ with probability 1. This completes the proof of part (a).

The proof of part (b) follows from (4.2) and (4.3). □

Remark 4.3. A sufficient condition for (4.3) to hold is that the distribution of X_∞ is absolutely continuous with strictly positive and continuous density function and the function $G(x, y)$ is continuous in x for fixed y.

The following result is similar to that of Theorem 4.2.

Theorem 4.4. *Fix $x, t, h \in \mathbb{R}$. Let*

$$\phi_{n,h,x}(t) = \begin{cases} \frac{\frac{1}{nh}\sum_{j=0}^{n-1} e^{itX_{j+1}}\mathbb{I}(x<X_j\leq x+h)}{\frac{1}{nh}\sum_{j=0}^{n-1}\mathbb{I}(x<X_j\leq x+h)} & \text{if } \mathbb{I}(x \leq X_i \leq x+h) \neq 0 \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\lim_{h\to 0}\lim_{n\to\infty} \phi_{n,h,x}(t) = \mathbb{E}(e^{it(\rho_1 x+\epsilon_1)}) \text{ with probability } 1,$$

provided

$$\lim_{h\to 0} \frac{\frac{1}{h}\int_x^{x+h} \mathbb{E}(e^{it(\rho_1 x+\epsilon_1)})\mathbb{P}(X_\infty \in du)}{\frac{1}{h}\int_x^{x+h} \mathbb{P}(X_\infty \in du)} = \mathbb{E}(e^{it(\rho_1 x+\epsilon_1)}). \tag{4.5}$$

Proof. Proof of this theorem is similar to the proof of Theorem 4.2 and hence omitted. □

Remark 4.5. A sufficient condition for (4.5) to hold is that the distribution of X_∞ is absolutely continuous with strictly positive and continuous density function on $(-\infty, \infty)$ and the function $\mathbb{E}(e^{it(\rho_1 x+\epsilon_1)})$ is continuous in x for fixed t.

Let $\{\rho_1\}$ and $\{\epsilon_1\}$ are independent random variables.Then

$$\phi_x(t) \equiv \mathbb{E}_x(e^{itX_1}) = \mathbb{E}(e^{it(\rho_1 x+\epsilon_1)}) = \psi_\rho(tx)\psi_\epsilon(t),$$

where $\psi_\rho(t) = \mathbb{E}(e^{it\rho})$ and $\psi_\epsilon(t) = \mathbb{E}(e^{it\epsilon})$. Also, note that

$$\psi_\epsilon(t) = \phi_0(t) \ \text{ and } \ \psi_\rho(tx) = \frac{\phi_x(t)}{\phi_0(t)}, \text{ when } \ \psi_\epsilon(t) \neq 0.$$

This yields the following corollary of Theorem 4.4.

Corollary 4.6. *Let ρ_1 and ϵ_1 be independent and conditions of Theorem 4.4 holds. Then*

(a) $\lim_{h\to 0}\lim_{n\to\infty} \phi_{n,h,0}(t) = \psi_\epsilon(t)$ *for all* $t \in \mathbb{R}$ *with probability* 1.
(b) Let $\psi_\epsilon(t) \neq 0$ *for all* $t \in \mathbb{R}$*, then for all* $x \neq 0$

$$\lim_{h\to 0}\lim_{n\to\infty} \frac{\phi_{n,h,x}(t/x)}{\phi_{n,h,0}(t/x)} = \psi_\rho(t) \text{ for all } t \in \mathbb{R} \text{ with probability } 1.$$

5. Appendix

Proof of Theorem 2.1: Choose $\epsilon > 0$ such that $\mathbb{E}(\log|\rho_1|) + \epsilon < 0$. Now, by the strong law of large number,

$$\mathbb{E}(\log|\rho_1|) < 0 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} \log|\rho_i| \leq \mathbb{E}(\log|\rho_1|) + \epsilon,$$

for sufficiently large n, with probability 1. Hence

$$|\rho_1\rho_2 \dots \rho_n| \leq e^{-n\lambda}, \tag{5.1}$$

where $0 < \lambda \equiv -(\mathbb{E}(\log|\rho_1| + \epsilon) < \infty$, for all large n, with probability 1.

Also $\mathbb{E}(\log|\epsilon_1|)^+ < \infty$ implies that for any $\mu > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(\log|\epsilon_1| > n\mu) < \infty$ and hence $\sum_n \mathbb{P}(\log|\epsilon_n| > n\mu) < \infty$. By Borel Cantelli lemma, $|\epsilon_n| \leq e^{n\mu}$ for all n large enough, with probability 1.

Now choose $0 < \mu < \lambda$. Then for sufficiently large n, with probability 1,

$$|\epsilon_{n+1}\rho_1\rho_2 \dots \rho_n| \leq e^{-n\lambda}e^{(n+1)\mu}.$$

Therefore $\sum_n |\epsilon_{n+1}||\rho_1\rho_2 \dots \rho_n| < \infty$ with probability 1. Hence $\tilde{X}_\infty = \epsilon_1 + \rho_1\epsilon_2 + \rho_1\rho_2\epsilon_3 + \ldots + \rho_1 \ldots \rho_n\epsilon_{n+1} + \ldots$ is well defined.

Observe that

$$\begin{aligned} X_n &= \rho_n(\rho_{n-1}X_{n-2} + \epsilon_{n-1}) + \epsilon_n \\ &= \rho_n\rho_{n-1}\cdots\rho_1 X_0 + \rho_n\rho_{n-1}\cdots\rho_2\epsilon_1 + \cdots + \rho_n\epsilon_{n-1} + \epsilon_n \end{aligned}$$

and which has the same distribution as

$$\epsilon_1 + \rho_1\epsilon_2 + \cdots + \rho_1\rho_2\cdots\rho_{n-1}\epsilon_n + \rho_1\rho_2\cdots\rho_n X_0. \tag{5.2}$$

Now by using (5.1) and above, we have $|\rho_1\rho_2\cdots\rho_n X_0|$ converges to zero with probability 1. Thus, from (5.2), as $n \to \infty$, we have

$$X_n \stackrel{d}{\to} \tilde{X}_\infty,$$

where $\stackrel{d}{\to}$ stands for convergence in distribution. □

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