International Review of Pure and Applied Mathematics
Volume 14 • Number 2 • July-December 2018

INVERSE QUARTIC SPLINE INTERPOLATION USING POLYNOMIAL ITERATION METHOD

Dhanya Ramachandran, R. Sreekumar and V. Madhukar Mallayya

Abstract: By applying polynomial iteration method inverse of a quartic spline interpolation is derived and is illustrated using example.

Keywords: Quartic Spline, polynomial iteration, Natural spline , Spline Interpolation

Subject Classification (2000)

65D07

1. PRELIMINARIES

DEFINITION - 1.1 [5]

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the set of n data points. The process of constructing a function say $y = f(x)$ which fits the given data points is called interpolation. If the interpolated function is a polynomial it is called a polynomial interpolation.

DEFINITION – 1.2 [1]

A function consisting of polynomial bits joined together with certain smoothness conditions is known as a Spline function.

Consider an interval $I = [a, b]$ composed of n sub intervals $I_i = [x_{i-1}, x_i]$, $i = 2, 3, \ldots\ldots, n$ with $a = x_1 < x_2 < --- < x_n = b$. A spline $S(x)$ of degree m on the interval I is a polynomial having continuous derivatives up to order $m - 1$ and coinciding with $y = f(x)$ on each subinterval $I_i = [x_{i-1}, x_i]$, $i = 2,3,\ldots\ldots, n$ such that

$$S(x) = \begin{cases} S_1(x), x_1 < x < x_2 \\ S_2(x), x_2 < x < x_3 \\ , \\ , \\ , \\ S_n(x), x_{n-1} < x < x_n \end{cases}$$

Let $S'(a) = f'(a), S'(b) = f'(b)$ and $S''(x_i) = M_i$, $i = 1,2,...n$. If $m = 4$, the corresponding spline is called quartic spline interpolation.

DEFINITION – 1.3 [4]

If y is a single valued function of x say $y = f(x)$, the process of finding the value of dependent variable x is known as inverse interpolation.

2. POLYNOMIAL ITERATION METHOD [2], [3]

Let the cubic polynomial be $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ (2.1)

Hence
$$y - a_0 = a_1 x + a_2 x^2 + a_3 x^3$$

$$a_1 x = (y - a_0) - a_2 x^2 - a_3 x^3 \tag{2.2}$$

From (2.2) we get the value of x,

$$x = \frac{(y - a_0)}{a_1} - \frac{a_2}{a_1} x^2 - \frac{a_3}{a_1} x^3 \tag{2.3}$$

Neglecting higher powers of x from (2.3), we get the first approximation of x as:

$$x^{(1)} = \frac{y - a_0}{a_1} \tag{2.4}$$

Substituting the value x from (2.4) to (2.3), we get the second approximation of x as:

$$x^{(2)} = \frac{y - a_0}{a_1} - \frac{a_2}{a_1}\left[\frac{y - a_0}{a_1}\right]^2 - \frac{a_3}{a_1}\left[\frac{y - a_0}{a_1}\right]^3$$

$$= \frac{y}{a_1} - \frac{a_0}{a_1} - \frac{a_2 y^2}{a_1^3} + \frac{2a_0 a_2 y}{a_1^3} - \frac{a_2 a_0^2}{a_1^3} - \frac{a_3 y^3}{a_1^4} + \frac{3a_3 y^2 a_0}{a_1^4} - \frac{3a_3 y a_0^2}{a_1^4} + \frac{a_3 a_0^3}{a_1^4}$$

$$x = \frac{-a_3}{a_1^4} y^3 + \left(\frac{3a_0 a_3}{a_1^4} - \frac{a_2}{a_1^3}\right) y^2 + \left(\frac{1}{a_1} + \frac{2a_0 a_2}{a_1^3} - \frac{3a_0^2 a_3}{a_1^4}\right) y + \left(\frac{a_0^3 a_3}{a_1^4} - \frac{a_0}{a_1} - \frac{a_2 a_0^2}{a_1^3}\right)$$

Hence the inverse of the cubic polynomial in (2.1) is

$$x = \frac{-a_3}{a_1^{\;4}} y^3 + \left(\frac{3a_0 a_3}{a_1^{\;4}} - \frac{a_2}{a_1^{\;3}} \right) y^2 + \left(\frac{1}{a_1} + \frac{2a_0 a_2}{a_1^{\;3}} - \frac{3a_0^{\;2} a_3}{a_1^{\;4}} \right) y + \left(\frac{a_0^{\;3} a_3}{a_1^{\;4}} - \frac{a_0}{a_1} - \frac{a_2 a_0^{\;2}}{a_1^{\;3}} \right) \tag{2.5}$$

3. DERIVATION OF QUARTIC SPLINE INTERPOLATION FORMULA[6]

Let $a = x_0 < x_1 < \ldots\ldots < x_n = b$ be a partition of $[a, b]$. Let $Q_i(x)$ be the quartic spline which interpolates at the knots. [30]

Let $x_{i+1} - x_i = h_i$, $i = 0, 1, 2, \ldots\ldots, (n-1)$

Since $Q_i(x)$ is a quartic spline its third derivative should be linear and continuous in x.

Hence $$Q_i'''(x) = M_{i+1} \frac{x - x_i}{h_i} + M_i \frac{x_{i+1} - x}{h_i} \quad , i = 0,1,2,\ldots\ldots, n-1 \tag{3.1}$$

Integrating (3.1) three times we get,

$$Q_i(x) = \frac{M_{i+1}}{24h_i}(x - x_i)^4 - \frac{M_i}{24h_i}(x_{i+1} - x)^4 + C_i(x - x_i)(x_{i+1} - x) + D_i(x - x_i) + E_i(x_{i+1} - x) \quad , i = 0,1,2,\ldots\ldots n-1 \tag{3.2}$$

Since the spline always interpolates at the nodes we have,

$$\begin{aligned} Q_i(x_i) &= y_i \\ Q_i(x_{i+1}) &= y_{i+1} \end{aligned} \quad i = 0,1,2,\ldots\ldots, n-1 \tag{3.3}$$

Since $Q_i(x)$ is a quartic spline its first three derivatives are continuous and hence we have

$$\begin{aligned} Q_i''(x_{i+1}) &= Q_{i+1}''(x_{i+1}) \\ Q_i'(x_{i+1}) &= Q_{i+1}'(x_{i+1}) \quad i = 0,1,2,\ldots\ldots, n-2 \\ Q_i'''(x_{i+1}) &= Q_{i+1}'''(x_{i+1}) \end{aligned} \tag{3.4}$$

The spline function has a chance to oscillate at large abscissa value, so to avoid this we take

$$Q_{n-1}''(x_n) = 0 \tag{3.5}$$

Applying (3.3) to (3.2) we get the constants D_i and E_i as

$$D_i = -\frac{M_{i+1}h_i^2}{24} + \frac{y_{i+1}}{h_i}$$

$$E_i = \frac{M_i h_i^2}{24} + \frac{y_i}{h_i} \quad i = 0,1,2,......., n-1 \tag{3.6}$$

Differentiating (3.2) we get,

$$Q_i'(x) = \frac{M_{i+1}}{6h_i}(x - x_i)^3 + \frac{M_i}{6h_i}(x_{i+1} - x)^3 + C_i(x_{i+1} + x_i - 2x) + D_i - E_i \tag{3.7}$$

Substituting the value of D_i and E_i in (3.7) and applying the continuity condition $Q_i'(x_{i+1}) = Q_{i+1}'(x_{i+1})$ we get,

$$\frac{h_{i-1}^2}{24}M_{i-1} + \frac{h_i^2 - h_{i-1}^2}{8}M_i - \frac{h_i^2}{24}M_{i+1} + C_i h_i + C_{i-1}h_{i-1} = \delta_{i-1} - \delta_i, i = 1,2,....n-1 \tag{3.8}$$

Where $\delta_i = \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i} = \dfrac{y_{i+1} - y_i}{h_i}$, $i = 0,1,........,n-1$

Differentiating (3.7),

$$Q_i''(x) = \frac{M_{i+1}}{2h_i}(x - x_i)^2 - \frac{M_i}{2h_i}(x_{i+1} - x)^2 - 2C_i \ , \ i = 0,1,2,......., n-1 \tag{3.9}$$

Applying the continuity condition $Q_i''(x_{i+1}) = Q_{i+1}''(x_{i+1})$ in (3.9) we get,

$$C_{i-1} = C_i + \frac{(h_i + h_{i-1})}{4}M_i \ , \ i = 1,2,......, n-1 \tag{3.10}$$

Substituting (3.10) in (3.8) and grouping the terms,

$$\frac{h_{i-1}^{2}}{3}M_{i-1}+\chi_i^{2}M_i-\frac{h_i^{2}}{3}M_{i+1}+8\chi_i C_i=8[\delta_{i-1}-\delta_i],i=1,2,.....n-1 \tag{3.11}$$

Where $$\chi_i=h_i+h_{i-1}=x_{i+1}-x_{i-1}\ ,\ i=1,2,.......n-1 \tag{3.12}$$

Hence the quartic spline is given by (3.2) where the constants D_i, E_i and C_i are given by (3.6) and (3.10)

4. DERIVATION OF INVERSE QUARTIC SPLINE INTERPOLATION FORMULA USING POLYNOMIAL ITERATION METHOD

The quartic spline interpolation formula is

$$Q_i(x)=\frac{(M_{i+1}-M_i)}{24h_i}X_i^{4}+\frac{M_i}{6}X_i^{3}-\left[\frac{M_ih_i}{4}+C_i\right]X_i^{2}+\left[\frac{M_ih_i^{2}}{6}+C_ih_i+D_i-E_i\right]X_i$$
$$+E_ih_i-\frac{M_ih_i^{3}}{24}$$

The above equation is also written as

$$y=l_i+m_iX_i+n_iX_i^{2}+o_iX_i^{3}+u_iX_i^{4},i=0,1,2,...n-1$$

Where

$$\left.\begin{aligned} l_i&=E_ih_i-\frac{M_ih_i^{3}}{24}\\ m_i&=\left[\frac{M_ih_i^{2}}{6}+C_ih_i+D_i-E_i\right]\\ n_i&=-\left[\frac{M_ih_i}{4}+C_i\right]\\ o_i&=\frac{M_i}{6}\ \ u_i=\frac{(M_{i+1}-M_i)}{24h_i}\end{aligned}\right\} \tag{4.1}$$

In the above equation polynomial iteration method is applied as follows:

$$y-l_i = m_i X_i + n_i X_i^{\,2} + o_i X_i^{\,3} + u_i X_i^{\,4}, \quad i = 0,1,2,..n-1$$

$$m_i X_i = (y-l_i) - n_i X_i^{\,2} - o_i X_i^{\,3} - u_i X_i^{\,4}, \quad i = 0,1,2,..n-1$$

$$X_i = \frac{(y-l_i)}{m_i} - \frac{n_i}{m_i} X_i^{\,2} - \frac{o_i}{m_i} X_i^{\,3} - \frac{u_i}{m_i} X_i^{\,4} \tag{4.2}$$

The first approximation for X_i is obtained by neglecting higher powers of X_i

Hence
$$X_i^{(1)} = \frac{y-l_i}{m_i} \tag{4.3}$$

The second approximation is obtained by substituting the value of (4.3) in (4.2)

$$X_i^{(2)} = \frac{y-l_i}{m_i} - \frac{n_i}{m_i}\left[\frac{y-l_i}{m_i}\right]^2 - \frac{o_i}{m_i}\left[\frac{y-l_i}{m_i}\right]^3 - \frac{u_i}{m_i}\left[\frac{y-l_i}{m_i}\right]^4$$

$$X_i^{(2)} = \frac{y-l_i}{m_i} - \frac{n_i}{m_i}\left[\frac{y^2 - 2yl_i + l_i^{\,2}}{m_i^{\,2}}\right] - \frac{o_i}{m_i}\left[\frac{y^3 - 3y^2 l_i + 3yl_i^{\,2} - l_i^{\,3}}{m_i^{\,3}}\right] - \frac{u_i}{m_i}\left[\frac{y^4 - 4y^3 l_i + 6y^2 l_i^{\,2} - 4yl_i^{\,3} + l_i^{\,4}}{m_i^{\,4}}\right]$$

Grouping y^4, y^3, y^2, y and constant terms we get,

$$X_i = \frac{-u_i}{m_i^{\,5}} y^4 + \left(-\frac{o_i}{m_i^{\,4}} + \frac{4l_i u_i}{m_i^{\,5}}\right) y^3 + \left(-\frac{n_i}{m_i^{\,3}} - \frac{6l_i^{\,2} u_i}{m_i^{\,5}} + \frac{3l_i o_i}{m_i^{\,4}}\right) y^2 + \left(\frac{1}{m_i} + \frac{4l_i^{\,3} u_i}{m_i^{\,5}} + \frac{2n_i l_i}{m_i^{\,3}} - \frac{3o_i l_i^{\,2}}{m_i^{\,4}}\right) y + \left[\frac{-l_i}{m_i} - \frac{n_i l_i^{\,2}}{m_i^{\,3}} + \frac{l_i^{\,3} o_i}{m_i^{\,4}} - \frac{l_i^{\,4} u_i}{m_i^{\,5}}\right]$$

Since $x - x_i = X_i$ we get $x = x_i + X_i$

Thus the inverse quartic spline is $Q_i^{-1}(y) = x_i + X_i$ where $y \in [y_i, y_{i+1}]$, i = 0,1,…, n-1

Hence

$$Q_i^{-1}(y) = x_i + \frac{-u_i}{m_i^5} y^4 + \left(-\frac{o_i}{m_i^4} + \frac{4l_i u_i}{m_i^5} \right) y^3 + \left(-\frac{n_i}{m_i^3} - \frac{6l_i^2 u_i}{m_i^5} + \frac{3l_i o_i}{m_i^4} \right) y^2$$

$$+ \left(\frac{1}{m_i} + \frac{4l_i^3 u_i}{m_i^5} + \frac{2n_i l_i}{m_i^3} - \frac{3o_i l_i^2}{m_i^4} \right) y + \left[\frac{-l_i}{m_i} - \frac{n_i l_i^2}{m_i^3} + \frac{l_i^3 o_i}{m_i^4} - \frac{l_i^4 u_i}{m_i^5} \right] \quad (4.4)$$

5. EXAMPLE

Consider the data points be (-1 , -0.762), (-0.5, -0.462), (0, 0), (0.5, 0.462), (1, 0.762)

Since $h_i = x_{i+1} - x_i$, $i = 0,1,........,(n-1)$, we get

$h_0 = h_1 = h_2 = h_3 = 0.5$

$\delta_0 = 0.6$, $\delta_1 = 0.924$, $\delta_2 = 0.924$, $\delta_3 = 0.6$

$\chi_1 = \chi_2 = \chi_3 = 1$

Hence we get the following set of equation

$$\frac{h_0^2}{3} M_0 + \chi_1^2 M_1 - \frac{h_1^2}{3} M_2 + 8\chi_1 C_1 = 8[\delta_0 - \delta_1]$$

$$\frac{h_1^2}{3} M_1 + \chi_2^2 M_2 - \frac{h_2^2}{3} M_3 + 8\chi_2 C_2 = 8[\delta_1 - \delta_2]$$

$$\frac{h_2^2}{3} M_2 + \chi_3^2 M_3 - \frac{h_3^2}{3} M_4 + 8\chi_3 C_3 = 8[\delta_2 - \delta_3]$$

Hence
$$\begin{aligned} 0.0833M_0 + M_1 - 0.0833M_2 + 8C_1 &= -2.592 \\ 0.0833M_1 + M_2 - 0.0833M_3 + 8C_2 &= 0 \\ 0.0833M_2 + M_3 - 0.0833M_4 + 8C_3 &= 2.592 \end{aligned} \quad (5.1)$$

FOR NATURAL SPLINE

$$C_{n-1} = 0 \Rightarrow C_3 = 0$$

$$\left.\begin{aligned} C_2 &= C_3 + \frac{h_3 + h_2}{4} M_3 = 0.25 M_3 \\ C_1 &= C_2 + \frac{h_2 + h_1}{4} M_2 = 0.25[M_2 + M_3] \\ C_0 &= C_1 + \frac{h_1 + h_0}{4} M_1 = 0.25[M_1 + M_2 + M_3] \end{aligned}\right\} \quad (5.2)$$

Substituting the values of (5.2) into (5.1),

$$\left.\begin{aligned} &0.0833 M_0 + M_1 + 1.917 M_2 + 2 M_3 = -2.592 \\ &0.0833 M_1 + M_2 + 1.917 M_3 = 0 \\ &0.0833 M_2 + M_3 - 0.0833 M_4 = 2.592 \end{aligned}\right\} \quad (5.3)$$

For natural spline $M_0 = M_4 = 0$ and solving (5.3) we get,

$$M_1 = 3.109\ ,\ M_2 = -6.217\ ,\ M_3 = 3.108$$

From (5.2), $C_0 = 0$, $C_1 = -0.7773$, $C_2 = 0.777$, $C_3 = 0$

Using (3.6), (4.2.1), (4.2.4) we get

$D_0 = -0.9564$	$D_1 = 0.065$	$D_2 = 0.892$	$D_3 = 1.524$
$E_0 = -1.524$	$E_1 = -0.892$	$E_2 = -0.065$	$E_3 = 0.9564$
$m_0 = 0.5676$	$m_1 = 0.698$	$m_2 = 1.086$	$m_3 = 0.697$
$n_0 = 0$	$n_1 = 0.389$	$n_2 = 0.00013$	$n_3 = -0.389$
$o_0 = 0$	$o_1 = 0.518$	$o_2 = -1.036$	$o_3 = 0.518$
$u_0 = 0.259$	$u_1 = -0.777$	$u_2 = 0.777$	$u_3 = -0.259$
$l_0 = -0.762$	$l_1 = -0.462$	$l_2 = -0.0001$	$l_3 = 0.462$

Using (4.4), the inverse natural spline in each interval is

$$Q^{-1}(y) = \begin{cases} -4.396 y^4 - 13.399 y^3 - 15.316 y^2 - 6.019 y - 1.139,\ y \in [-0.762, -0.462] \\ 4.689 y^4 + 6.484 y^3 + 1.837 y^2 + 0.828 y - 0.084,\ y \in [-0.462, 0] \\ -0.5144 y^4 + 0.745 y^3 + 0.0001 y^2 + 0.921 y + 0.0001,\ y \in [0, 0.462] \\ 1.574 y^4 - 5.104 y^3 + 6.2072 y^2 - 1.653 y + 0.371,\ y \in [0.462, 0.762] \end{cases}$$

REFERENCES

[1] Anthony Ralston and Philip Rabinowitz, A First Course in Numerical Analysis, Second Edition, Dover Publications, INC, 2001.

[2] Dhanya Ramachandran and Dr.V. MadhukarMallayya, Inverse Cubic Spline, IJMMS, vol.12 , No.1 (January- June 2016) pp : 41-52, Serial Publications.

[3] Comparison of inverse cubic spline formulae, Bulletin of Kerala Mathematics Association Vol.15, No.2, (2017 December) 229-233.

[4] Madhumangal Pal, Numerical Analysis for Scientists and Engineers, Theory and C- Programs, Narosa Publishing House, 2007.

[5] P. Thangaraj , Computer Oriented Numerical Methods, Prentice Hall of India Private Limited, 2008.

[6] Rafael E. Banchs, Natural Quartic spline, *www.rbanchs.com/documents/THFEL-PR07.pdf*

Dhanya Ramachandran
Assistant Professor (Mathematics),
Mar Baselios College of Engineering & Technology
Nalanchira, Trivandrum, *E-mail: dhanyar@gmail.com*

R Sreekumar
Associate Professor & Head (Mathematics)
Sanatana Dharma College, Sanathanapuram P.O, Alappuzha
E-mail: dr.r.sreekumar@gmail.com

V. Madhukar Mallayya
Professor & Head (Mathematics)
Mohandas College of Engineering & Technology
Anad, Trivandrum, *E-mail: mmallayyav@gmail.com*