International Review of Pure and Applied Mathematics
(January-June) 2017, Volume 13, No. 1, pp. 1–9

Pseudo-Differential Operators Associated with the Bessel Type Operators-II

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Abstract: In this paper we define a symbol $a(x, y)$ by using inverse Hankel type transform. The pseudo differential type operator $G(x, D)$ associated with the Bessel type operator $\Delta_{\alpha,\beta}$. in terms of this symbol is also defined and it is shown that the pseudo differential type operator is bounded in a certain Sobolev type space associated with the Hankel type transform. Finally some properties of symbols are discussed.

AMS Subject Classification: Primary 46E30, 46F12, Secondary 44A15, 44A35.

Keywords: Bessel type operator, Hankel type transform, pseudo differential type operator, Sobolev type spaces, and Schwartz's theory of Fourier transformation.

1. Introduction

In recent years many authors have extended Hankel transformation

$$(h_\mu \phi)(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\phi(y)\,dy \tag{1.1}$$

to distributions belonging to H'_μ, the dual of the test function space H_μ satisfying certain condition on I = (0, ∞). Zemanian [12] has considered these transformations in his monograph. Waphare [9] has investigated Hankel type transformation

$$(h_{\alpha\beta}\phi)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)\,dy \tag{1.2}$$

and has been extended to distributions belonging to the dual space $H'_{\alpha,\beta}$ consisting of all complex valued infinitely differentiable functions ϕ defined on $I = (0, \infty)$ satisfying

$$\rho_{m,k}^{\alpha,\beta}(\phi) = \sup_{x \in I} \left| x^m (x^{-1}D_x)^k \, (x^{2\beta-1}\phi(x)) \right| < \infty \tag{1.3}$$

for every $m, k, \in \mathbb{N}_0$, where $D_x = \dfrac{d}{dx}$.

Zaidman [11] studied a class of pseudo differential operators using Schwartz's theory of Fourier transformation. Pseudo differential type operators associated to a numerical valued symbol $a(x, y)$ were studied by Waphare [9]. One formula for such an operators appears as follows:

$$(h_{\alpha,\beta,a}u)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, a(x, y)\, U_{\alpha,\beta}(y)\, dy \tag{1.4}$$

where

$$U_{\alpha,\beta}(y) = (h_{\alpha,\beta,a}u)(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, u(x)\, dx;\ (\alpha-\beta) \geq -1/2 \tag{1.5}$$

The symbol $a(x, y)$ is defined to be the complex valued infinitely differentiable functions on $I \times I$. which satisfies

$$(1 + x)^q \,|(x^{-1}D_x)^i (y^{-1}D_y)^p\, a(x, y)| \leq K_{p,i,m,q}(1 + y)^{m-p} \tag{1.6}$$

for all $q, i, p \in \mathbb{N}_0$ and m is a fixed real number, and where $D_y = \dfrac{d}{dy}$.

The class of all such symbols is denoted by H^m. If $a(x, y)$ satisfies (1.6) with $q = 0$, then the symbol class will be denoted by H_0^m. Clearly $H^m \subset H_0^m$.

2. Notations and Terminology

We shall use the notation and terminology of Waphare [9]. The differential operators $\Delta_{\alpha,\beta}$ is defined by

$$\begin{aligned}\Delta_{\alpha,\beta} = \Delta_{\alpha,\beta,x} &= x^{2\beta-1} D_x x^{4\alpha} D_x x^{2\beta-1}\\ &= (2\beta-1)(4\alpha+2\beta-2)x^{4(\alpha+\beta-1)} + 2(2\alpha+2\beta-1)x^{4\alpha+4\beta-3} D_x + x^{2(2\alpha+2\beta-1)} D_x^2\end{aligned} \tag{2.1}$$

Notice that for $\alpha = \dfrac{1}{4} + \dfrac{\mu}{2}, \beta = \dfrac{1}{4} - \dfrac{\mu}{2}$, (2.1) reduces to

$$x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} D_x x^{-\mu-\frac{1}{2}} = S_\mu = D_x^2 + \frac{(1-4\mu^2)}{4x^2}$$

which is a differential operator studied by Zemanian [12] and many authors later on.

From Waphare [9], we know that for any $\phi \in H_{\alpha,\beta}$

$$h_{\alpha,\beta}(\Delta_{\alpha,\beta}\phi) = -y^2 h_{\alpha,\beta}\phi \tag{2.2}$$

$$(x^{-1}D_x)^k(x^{2\beta-1}\theta\phi) = \sum_{i=0}^{k}\binom{k}{i}(x^{-1}D_x)^i\theta(x^{-1}D_x)^{k-i}(x^{2\beta-1}\phi) \tag{2.3}$$

$$\Delta^r_{\alpha,\beta,x} = \sum_{j=0}^{r} b_j x^{2j+2\alpha}(x^{-1}D_x)^{r+j}(x^{2\beta-1}\phi(x)) \tag{2.4}$$

where b_j are constants depending only on $(\alpha - \beta)$.

We also need a lemma due to Haimo [1] for the Hankel type convolution transform.

Lemma 2.1: Let $\Delta(x, y, z)$ be the area of a triangle with sides x, y, z if such a triangle exists. For fixed $(\alpha-\beta)\geq -\frac{1}{2}$, set

$$D(x, y, x) = 2^{2(\alpha-2\beta)}(\pi)^{-1/2}(\Gamma(3\alpha+\beta)^2[\Gamma(2\alpha)]^{-1}(xyz)^{-2(\alpha-\beta)}[\Delta(x, y, z]^{-4\beta}. \tag{2.5}$$

if Δ exists and zero otherwise.

We note that $D(x, y, z) \geq 0$ and that it is symmetric in x, y, z and from Waphare [9, equation (2.10)], we have

$$\int_0^\infty i(zt)D(x, y, z)d\mu(z) = i(xt)i(yt) \tag{2.6}$$

where

$$d\mu(x) = \frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}x^{4\alpha}dx \tag{2.7}$$

and

$$i(x) = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)x^{-(\alpha-\beta)}J_{\alpha-\beta}(x) \tag{2.8}$$

Refering to Pathak and Pathak [6, p. 311, (1.14)], we can obtain

$$J_{\alpha-\beta}(x\xi)J_{\alpha-\beta}(x\lambda) = \frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\int_0^\infty (x\lambda\xi)^{\alpha-\beta}z^{-(\alpha-\beta)}J_{(\alpha-\beta)}(zx)D(\xi,\lambda,z)d\mu(z) \tag{2.9}$$

Now we define the space $L^p_{\alpha,\beta}(I), 1\leq p<\infty,$ as the space of those real valued measurable functions on I for which

$$\|f\|_p = \left[\int_0^\infty |f(x)|^p\, d\mu(x)\right]^{\frac{1}{p}} < \infty \tag{2.10}$$

$L^{\infty}_{\alpha,\beta}$ is the space of all real valued measurable functions on I for which

$$\| f \|_{\infty} = \underset{0<x<\infty}{ess\,\sup} \left| f(x) \right| < \infty \tag{2.11}$$

If $f \in L^{1}_{\alpha,\beta}(I)$ then its associated function $f(x, y)$ is defined by

$$f(x, y) = \int_{0}^{\infty} f(z) D(x, y, z) d\,\mu(z),\ 0 < x, y < \infty \tag{2.12}$$

We shall require a Lemma due to Haimo.

Lemma 2.2 (Haimo): Let f and g be functions of $L^{1}_{\mu}(I)$ and let

$$f \# g(x) = \int_{0}^{\infty} f(x, y)\, g(y) d\,\mu(y),\ 0 < x < \infty \tag{2.13}$$

Then the integral defining $f \# g(x)$ converges for almost all x, $0 < x < \infty$ and

$$\| f \# g \|_{L^1} \le \| f \|_{L^1} \| g \|_{L^1} \tag{2.14}$$

3. Pseudo-Differential Type Operator *G*(*x*, *D*)

The symbol $a(x, y)$ is defined as the Hankel type transform

$$a(x, y) = x^{2\beta-1} \int_{0}^{\infty} (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda) W(\lambda, y)\, d\lambda \tag{3.1}$$

where $W(\lambda, y)$ is a complex valued measurable function on $I \times I$. such that $W(\lambda, y)$ is λ-measurable for all $y \in I$ and

$$| W(\lambda, y) | \le k(\lambda) \quad \text{for all} \quad y \in I,\ \lambda \in I \tag{3.2}$$

where

$$k(\lambda) \in L^{1}_{\alpha,\beta},\ (\alpha - \beta) \ge -\frac{1}{2}$$

Note that as

$$| x^{\alpha+\beta} J_{\alpha-\beta}(x) | \le A_{\alpha,\beta} \quad \text{for} \quad (\alpha - \beta) \ge -\frac{1}{2}$$

the integral (3.1) exists under the assumption (3.2).

Let G denote the set of all $a(x, y)$ on $I \times I$. such that (3.1) and (3.2) hold.

For $a(x, y) \in G$, define the pseudo-differential type operator

$$G(x, D)\,u(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\,a(x, y)\,U_{\alpha,\beta}(y)\,dy \tag{3.3}$$

where

$$U_{\alpha,\beta}(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\,u(x)\,dx,\ (\alpha-\beta) \geq -\frac{1}{2} \tag{3.4}$$

Now for $|x^{\alpha+\beta} J_{\alpha-\beta}(x)| \leq A_{\alpha,\beta}$, using (3.1), we have

$$|G(x, D)u(x)| = \left|\int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\left[x^{2\beta-1}\int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda) W(\lambda, y) d\lambda\right] U_{\alpha,\beta}(y) dy\right|$$

$$= \left|x^{-1+2\beta}\right| \left|\int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda) d\lambda \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) W(\lambda, y) U_{\alpha,\beta}(y) dy\right|$$

$$\leq \int_0^\infty \int_0^\infty A^2_{\alpha,\beta} x^{2\beta-1} \left|W(\lambda, y)\right| \left|U_{\alpha,\beta}(y)\right| d\lambda dy$$

$$\leq x^{2\beta-1} A^2_{\alpha,\beta} \int_0^\infty \left|k(\lambda)\right| d\lambda \int_0^\infty \left|U_{\alpha,\beta}(y)\right| dy < \infty$$

because $k(\lambda) \in L^1(I)$ and $U_{\alpha,\beta}(y) \in H_{\alpha,\beta}(I)$

Definition 3.1 (Sobolev Type Space): The space $G^s_{\alpha,\beta,p}(I) \in \mathbb{R}$, $(\alpha-\beta) \in \mathbb{R}$ is defined as the set of all those elements $u \in H'_{\alpha,\beta}(I)$ which satisfy

$$\|u\|_{G^s_{\alpha,\beta,p}} = \|\eta^{s+2\beta-1}\, h_{\alpha,\beta} u\|_p < \infty \tag{3.5}$$

Theorem 3.2 (Boundedness for *G(x, D)*): Let $(\alpha-\beta) \geq -\frac{1}{2}$. Then

$$\|G(x, D)\,u\|_{G^0_{\alpha,\beta,1}} \leq \|K\|_{L^1} \|u\|_{G^0_{\alpha,\beta,1}},\ u \in H_{\alpha,\beta}(I) \tag{3.6}$$

Proof: By using (3.3) and (1.5) we have

$$G(x, D)\,u(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\,a(x, y)\,(h_{\alpha,\beta} u)(y)\,dy$$

where
$$a(x, y) = x^{2\beta-1} \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)\,W(\lambda, y)\,d\lambda$$

Now we make use of Fubini's theorem to change the order of integration and obtain

$$G(x, D)\,u(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\left[x^{2\beta-1}\int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)W(\lambda, y)d\lambda\right]U_{\alpha,\beta}(y)dy$$

$$= \int_0^\infty \int_0^\infty x^{2\beta-1}(xy)^{\alpha+\beta}(x\lambda)^{\alpha+\beta}\, W(\lambda, y)\, U_{\alpha,\beta}(y)\, C_{\alpha,\beta}$$

$$\times \int_0^\infty (x\lambda y)^{\alpha-\beta} z^{-(\alpha-\beta)} J_{\alpha-\beta}(zx)\, D(y, \lambda, z)\, d\,\mu(z)\, d\lambda\, dy$$

where

$$C_{\alpha,\beta} = \frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}$$

$$G(x, D)\; u(x) = \int_0^\infty \int_0^\infty x^{2\beta-1}(xy)^{\alpha+\beta}(x\lambda)^{\alpha+\beta}\, W(\lambda, y)\, U_{\alpha,\beta}(y)$$

$$\times\, C_{\alpha,\beta} \int_0^\infty (x\lambda y)^{\alpha-\beta} z^{-(\alpha-\beta)} J_{\alpha-\beta}(zx)\, D(y, \lambda, z)\; C_{\alpha,\beta}\, z^{4\alpha}\, dz\, d\lambda\, dy$$

$$= (C_{\alpha,\beta})^2 \int_0^\infty (zx)^{\alpha-\beta} J_{\alpha-\beta}(zx)\left[\int_0^\infty\int_0^\infty y^{2\alpha}U_{\alpha,\beta}(y)\lambda^{2\alpha}W(\lambda,y)D(y,\lambda,z)d\lambda dy\right]z^{2\alpha}dz$$

Now by inverse Hankel type transform, we have

$$\int_0^\infty (zx)^{\alpha+\beta} J_{\alpha-\beta}(zx)\, G(x, D)\, u(x)\, dx$$

$$= z^{2\alpha}(C_{\alpha,\beta})^2 \int_0^\infty \int_0^\infty y^{2\alpha}\, U_{\alpha,\beta}(y)\lambda^{2\alpha}\, W(\lambda, y)\, D(y, \lambda, z)\, d\lambda\, dy$$

Thus we obtain

$$h_{\alpha,\beta}[G(x, D)u(x)](z) = z^{2\alpha}\int_0^\infty \int_0^\infty \lambda^{2\beta-1} y^{2\beta-1} W(\lambda, y)D(y, \lambda, z)U_{\alpha,\beta}(y)d\,\mu(\lambda)d\,\mu(y)$$

Now we can use (3.2) to obtain

$$|h_{\alpha,\beta}[G(x,D)u(x)](z)|$$

$$\leq z^{2\alpha}\int_0^\infty\int_0^\infty \lambda^{2\beta-1}k(\lambda)D(y,\lambda,z)\,y^{2\beta-1}|U_{\alpha,\beta}(y)|d\mu(y)d\mu(\lambda)$$

$$= z^{2\alpha}(K\,\#\,V_{\alpha-\beta})(z)$$

where

$$K(\lambda) = \lambda^{2\beta-1}k(\lambda)$$

and

$$V_{\alpha-\beta}(y) = y^{2\beta-1}\,|U_{\alpha,\beta}(y)|$$

Hence

$$\int_0^\infty z^{2\beta-1}|h_{\alpha,\beta}[G(x,D)u(x)](z)|d\mu(z)$$

$$\leq \int_0^\infty (K\,\#\,V_{\alpha-\beta})(z)d\mu(z)$$

Finally applying (3.1) and (2.14) we get

$$\|G(x,D)u(x)\|_{G^0_{\alpha,\beta,1}} \leq \|K\|_{L^1}\|u\|_{G^0_{\alpha,\beta,1}}$$ □

4. Symbol Related Properties

A. Assume that $x^{2\alpha}a(x,y)\in L^1(I)$ for fixed $y\in(0,\infty)$ and $(\alpha-\beta)\geq -\frac{1}{2}$

Let us define

$$W(\lambda,y) = \int_0^\infty (x\lambda)^{\alpha+\beta}J_{\alpha-\beta}(x\lambda)x^{2\alpha}a(x,y)\,dx$$

$W(\lambda,y)$ is the same as $a_\eta(y)$ in Waphare [9]. Now from Waphare [9, equation (5.2)] we know that for $(\alpha-\beta)\geq -\frac{1}{2}$, $r\in\mathbb{N}_0$,

$$|W(\lambda,y)| \leq C_{r,m,q}(1+y)^m\lambda^{2\alpha}(1+\lambda^{2r})^{-1}$$

Now if $a(x, y) \in H^0$, then

$$|W(\lambda, y)| \le k(\lambda), \quad \text{for all} \quad y \in I \quad \text{and for all } \lambda \in I. \tag{4.1}$$

where

$$k(\lambda) = C_{r,q}\, \lambda^{2\alpha}(1 + \lambda^{2r})^{-1} \in L^1(I) \tag{4.2}$$

Conversely suppose that

$$k(\lambda) \in L^1(I) \quad \text{and} \quad |W(\lambda, y)| \le k(\lambda) \tag{4.3}$$

Then by inverse Hankel type transform the symbol $a(x, y)$ is defined by

$$a(x, y) = x^{2\beta-1} \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)\, W(\lambda, y)\, d\lambda \tag{4.4}$$

The integral (4.4) exists under the assumption (4.3) because

$$|x^{\alpha+\beta} J_{\alpha-\beta}(x)| \le A_{\alpha,\beta}, \quad \text{for} \quad (\alpha - \beta) \ge -\frac{1}{2}$$

B. Variable Separable form of the Symbol

Let $a(x, y) = a(x)\, b(y)$, where $\lambda^{2a} h_{\alpha,\beta}[a(x)](\lambda) \in L^1$ and $b(.)$ is bounded measurable function on I.

$$|b(y)| \le M, \quad y \in I.$$

Since

$$x^{2\alpha} a(x) = \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)\, h_{\alpha,\beta}[x^{2\alpha} a(x)](\lambda)\, d\lambda$$

and

$$\lambda^{2\alpha} h_{\alpha,\beta}[a(x)(\lambda) \in L^1(I),$$

therefore

$$x^{2\alpha} a(x, y) = \int_0^\infty (x\lambda)^{\alpha+\beta} J_{\alpha-\beta}(x\lambda)\, [h_{\alpha,\beta}\, x^{2\alpha} a(x)](\lambda)\, b(y)\, d\lambda$$

Thus we get

$$W(\lambda, y) = h_{\alpha,\beta}[x^{2\alpha} a(x, y)](\lambda) = h_{\alpha,\beta}[x^{2\alpha} a(x)](\lambda)\, b(y) \tag{4.5}$$

Since

$$|b(y)| \le M,$$

and

$$|W(\lambda, y)| \leq M\,|h_{\alpha,\beta}[x^{2\alpha}\,a(x)](\lambda)| \in L^1(I),$$

therefore $W(\lambda, y)$ in (4.5) is measurable function on $I \times I$, for all $y \in I$.

Therefore (4.3) is clearly verified with

$$K(\lambda) = M\,|h_{\alpha,\beta}(x^{2\alpha}\,a(x))(\lambda)|$$

Thus by Theorem 3.2, we have

$$\|G(x, D)\,u(x)\|_{G^0_{\alpha,\beta,1}} \leq \|K\|_{L^1}\,\|u\|_{G^0_{\alpha,\beta,1}}$$

where

$$\begin{aligned} K(\lambda) &= \lambda^{2\beta-1}\,k(\lambda) \\ &= \lambda^{2\beta-1}\,M\,h_{\alpha,\beta,1}\,(x^{2a}\,a(x))(\lambda). \end{aligned}$$

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