I R A M P
11(1), Jan 2020, pp. 40-46

ISSN: 2229-39-44

Quadruple Photoionization of Be

M. S. PINDZOLA[1], J. P. COLGAN[2] AND S. A. ABDEL-NABY[3]

[1]*Department of Physics, Auburn University, Auburn, AL*
[2]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM*
[3]*Department of Physics, American University, Sharjah, UAE*

ABSTRACT: A theoretical method is developed to solve the time-dependent Schrodinger equation on a 4D radial lattice and used to calculate the quadruple photoionization cross sections for the Be atom. At a photon energy of 500 eV cross sections are compared with previous quasiclassical simulations.

1. INTRODUCTION

A theoretical method developed to solve the time-dependent Schrodinger equation on a 3D radial lattice (TDSE-3D) has been used to calculate the triple photoionization of atoms. Total cross sections for the triple photoionization of Li using the TDSE-3D method[1, 2] were found to be in reasonable agreement with experimental measurements[3]. Energy differential cross sections for the triple photoionization of Li[4] at an incident energy of 300 eV were found to be bowl shaped. Angular distributions for the triple photoionization of Li[5] yielded evidence for a T-shaped breakup pattern as predicted by quasiclassical simulations [6, 7]. Energy and angle differential cross sections were also compared for the one-photon and two-photon triple ionization of Li using the TDSE-3D method[8].

Recently a theoretical method developed to solve the time-dependent Schrodinger equation on a 4D radial lattice (TDSE-4D) was used to calculate the triple autoionization of atomic ions[9] in support of experimental measurements[10]. In this paper the TDSE-4D method is furthur developed and used to calculate the quadruple photoionization of atoms. Total cross sections for the quadruple photoionization of Be at an incident energy of 500 eV are compared with quasiclassical simulations[11].

Details of the TDSE-4D method for four electron systems are presented in section 2, quadruple photoionization cross sections for the Be atom are presented in section 3, and a brief summary is given in section 4. Unless otherwise stated, all quantities are given in atomic units.

2. THEORY

For a four-electron target atom, the TDSE-4D equations are given by:

$$
\begin{aligned}
& i\frac{\partial P^{L}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,t)}{\partial t} \\
&= T_{l_1l_2l_3l_4}(r_1,r_2,r_3,r_4)P^{L}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,t) \\
&+ \sum_{l'_1l'_2L'_{12}l'_3L'_{123}l'_4} \sum_{i<j}^{6} V^{L}_{l_1l_2L_{12}l_3L_{123}l_4,l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_i,r_j) \\
&\qquad \times P^{L}_{l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_1,r_2,r_3,r_4,t) \\
&+ \sum_{l'_1l'_2L'_{12}l'_3L'_{123}l'_4} \sum_{i}^{4} W^{LL_0}_{l_1l_2L_{12}l_3L_{123}l_4,l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_i,t) \\
&\qquad \times \bar{P}^{L_0}_{l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_1,r_2,r_3,r_4,\tau\to\infty) \ ,
\end{aligned}
\tag{1}
$$

where $P^{L}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,t)$ is the radial part of the time evolving total wavefunction and $\bar{P}^{L_0}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,\tau\to\infty)$ is the radial part of the initial total wavefunction.

The l_1, l_2, l_3, l_4 are the angular momenta of each of the four electrons, where l_1 coupled to l_2 yields L_{12}, L_{12} coupled to l_3 yields L_{123}, and L_{123} coupled to l_4 yields L for the time evolving total wavefunction and L_0 for the initial total wavefunction. The kinetic and nuclear potential operator, $T_{l_1l_2l_3l_4}(r_1,r_2,r_3,r_4)$, the six electrostatic repulsion operators, $V^{L}_{l_1l_2L_{12}l_3L_{123}l_4,l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_i,r_j)$, and the four dipole radiation field operators, $W^{LL_0}_{l_1l_2L_{12}l_3L_{123}l_4,l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_i,t)$, are all given in the Appendix. The atomic number of the atom is Z, the coupling operator expressions use standard 3j and 6j symbols, in the length gauge $g(r) = r$, and the linearly polarized electric field amplitude $F(t)$ is proportional to cos ω t, where ω is the radiation field frequency.

The radial wavefunctions, $\bar{P}^{L_0}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,\tau)$ are obtained by relaxation of the Schrodinger equation in imaginary time given by:

$$
\begin{aligned}
& -\frac{\partial \bar{P}^{L_0}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,\tau)}{\partial \tau} \\
&= T_{l_1l_2l_3l_4}(r_1,r_2,r_3,r_4)\bar{P}^{L_0}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,\tau) \\
&+ \sum_{l'_1l'_2L'_{12}l'_3L'_{123}l'_4} \sum_{i<j}^{6} V^{L_0}_{l_1l_2L_{12}l_3L_{123}l_4,l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_i,r_j) \\
&\qquad \times \bar{P}^{L_0}_{l'_1l'_2L'_{12}l'_3L'_{123}l'_4}(r_1,r_2,r_3,r_4,\tau) \ .
\end{aligned}
\tag{2}
$$

As an example, for Be($1s^2 2s^2$) the radial wavefunctions at time $\tau = 0$ are given by:

$$
\begin{aligned}
& \bar{P}^{L_0}_{l_1l_2L_{12}l_3L_{123}l_4}(r_1,r_2,r_3,r_4,\tau=0) \\
&= P_{1s}(r_1)P_{1s}(r_2)P_{2s}(r_3)P_{2s}(r_4)\delta_{l_1,0}\delta_{l_2,0}\delta_{l_{12},0}\delta_{l_3,0}\delta_{L_{123},0}\delta_{l_4,0}\delta_{L_0,0} \ ,
\end{aligned}
\tag{3}
$$

where $P_{1s}(r)$ and $P_{2s}(r)$ are radial wavefunctions for Be^{3+}. To prevent relaxation to any state involving three or four electrons with 1s character, a Schmidt orthogonalization is made at every imaginary time step:

$$\begin{aligned}
&\bar{P}^{L_0}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3, r_4, \tau) \\
&= \bar{P}^{L_0}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3, r_4, \tau) \\
&- \int_0^\infty P_{1s}(r_3')\bar{P}^{L_0}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3', r_4, \tau) dr_3' P_{1s}(r_3) \\
&- \int_0^\infty P_{1s}(r_4')\bar{P}^{L_0}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3, r_4', \tau) dr_4' P_{1s}(r_4) \\
&+ \int_0^\infty \int_0^\infty P_{1s}(r_3')P_{1s}(r_4')\bar{P}^{L_0}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3', r_4', \tau) dr_3' dr_4' \\
&\qquad \times P_{1s}(r_3)P_{1s}(r_4) \ .
\end{aligned} \tag{4}$$

The initial condition for the solution of the TDSE-4D equations of Eq.(1) are given by:

$$P^{L}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3, r_4, t = 0) = 0 \ . \tag{5}$$

The quadruple photoionization cross section for $t \to \infty$ is given by:

$$\sigma = \frac{\omega}{I} \sum_{l_1 l_2 L_{12} l_3 L_{123} l_4} \frac{\partial Q^{L}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(t)}{\partial t} \tag{6}$$

where

$$\begin{aligned}
Q^{L}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(t) = &\sum_{k_1}\sum_{k_2}\sum_{k_3}\sum_{k_4} \int_0^\infty dr_1 \int_0^\infty dr_2 \int_0^\infty dr_3 \int_0^\infty dr_4 \\
&\times P_{k_1 l_1}(r_1)P_{k_2 l_2}(r_2)P_{k_3 l_3}(r_3)P_{k_4 l_4}(r_4) \\
&\times P^{L}_{l_1 l_2 L_{12} l_3 L_{123} l_4}(r_1, r_2, r_3, r_4, t) \ .
\end{aligned} \tag{7}$$

Care must be taken in the sums over electron momentum, k_i, to avoid double counting of distinct continuum states. We also restrict the sums over electron momentum so that the conservation of energy:

$$E_{atom} + \omega = \frac{k_1^2}{2} + \frac{k_2^2}{2} + \frac{k_3^2}{2} + \frac{k_4^2}{2} \tag{8}$$

is approximately conserved.

3. RESULTS

We employ a radial grid with a uniform mesh spacing of $\Delta r = 0.10$ with a total number of points N = 360. A complete set of bound and continuum orbitals are generated by matrix diagonalization of a one-electron Hamiltonian. For Be^{3+} using a Coulomb potential we obtain $\in_{1s} = 209.6$ eV and $\in_{2s} = 53.9$ eV.

Upon relaxation of the time-dependent close-coupling equations in imaginary time, using a $(360)^4$ point lattice, it was found that a relaxation time of 10.0 au ($\Delta\tau = 0.01$) yielded a ground state energy of -381.0 eV. The relaxation used one coupled channel for $L_0 = 0$ with $l_1 = 0$, $l_2 = 0$, $l_3 = 0$, $l_4 = 0$, $L_{12} = 0$, $L_{123} = 0$. The experimental Be ground state energy is -399.0 eV.

Once a fully correlated initial state was obtained, the time-dependent close-coupling equations were propagated in real time for 10 periods ($\Delta t = 0.002$) using the same $(360)^4$ point lattice. We note that the TDSE-4D equations for both imaginary time relaxation and real time propagation used a $(360)^4$ point lattice partitioned over 20,736 cores on a massively parallel computer. The propagation used four coupled channels for $L = 1$ with $l_1 = 1$, $l_2 = 0$, $l_3 = 0$, $l_4 = 0$, $L_{12} = 1$, $L_{123} = 1$; $l_1 = 0$, $l_2 = 1$, $l_3 = 0$, $l_4 = 0$, $L_{12} = 1$, $L_{123} = 1$; $l_1 = 0$, $l_2 = 0$, $l_3 = 1$, $l_4 = 0$, $L_{12} = 0$, $L_{123} = 1$; and $l_1 = 0$, $l_2 = 0$, $l_3 = 0$, $l_4 = 1$, $L_{12} = 0$, $L_{123} = 0$. For $\omega = 500.0$ eV, and an excess energy of 119.0 eV, and an intensity $I = 10^{14}$ W/cm^2, we obtained a cross section of 3.2×10^{-28} cm^2. The cross section obtained by Emmanouilidou[11] using a quasiclassical simulation at an excess energy of 120.0 eV is 8.4×10^{-26} cm^2.

4. SUMMARY

In conclusion, we have applied the TDSE-4D method to calculate quadruple photoionization cross sections for the Be atom. Our first calculation with a restricted minimal number of coupled channels at a photon energy of 500 eV yielded a cross section much smaller than that obtained using a quasiclassical simulation method[11]. Larger calculations are necessary before a detailed comparison can be made with previous work[11].

In the future we plan to extend our TDSE-4D calculations for the quadruple photoionization of Be using a smaller uniform mesh spacing Δ r, more mesh points N, and more coupled channels $l_1 l_2 L_{12} l_3 L_{123} l_4 L$. It will take extended runs on massively parallel supercomputers to obtain a fully converged cross section.

Acknowledgments

This work was supported in part by grants from the US Department of Energy. Computational work was carried out at the National Energy Research Scientific Computing Center (NERSC) in Berkeley, California and on the High Performance Computing Center (HLRS) in Stuttgart, Germany.

References

[1] Colgan J, Pindzola M S, and Robicheaux F 2004 Phys. Rev. Letters 93 053201

[2] Colgan J, Pindzola M S, and Robicheaux F 2005 Phys. Rev. A 72 022727

[3] Wehlitz R, Huang M T, DePaola B D, Levin J C, Sellin I A, Nagata T, Cooper J W, and Azuma Y 1998 Phys. Rev. Letters 81 1813

[4] Colgan J and Pindzola M S 2006 J. Phys. B 39 1879

[5] Colgan J, Emmanouilidou A, and Pindzola M S 2013 Phys. Rev. Letters 110 063001

[6] Emmanouilidou A and Rost J M 2006 J. Phys. B 39 4037

[7] Emmanouilidou A, Wang P, and Rost J M 2008 Phys. Rev. Letters 100 063002

[8] Colgan J and Pindzola M S 2015 J. Phys. B 48 181001

[9] Pindzola M S, Abdel-Naby S A, and Colgan J P 2019 J. Phys. B 52 095201

[10] Muller A et al 2015 Phys. Rev. Letters 114 013002

[11] Emmanouilidou A 2007 Phys. Rev. A 76 054701

5. APPENDIX

$$T(r_1, r_2, r_3, r_4) = \sum_i^4 \left(-\frac{1}{2}\frac{\partial^2}{\partial r_i^2} + \frac{l_i(l_i+1)}{2r_i^2} - \frac{Z}{r_i}\right) \tag{9}$$

$$\begin{aligned} V^L(r_1, r_2) = {} & (-1)^{l_1 + l_1' + L_{12}} \delta_{l_3, l_3'} \delta_{l_4, l_4'} \delta_{L_{12}, L_{12}'} \delta_{L_{123}, L_{123}'} \\ & \times \sqrt{(2l_1+1)(2l_1'+1)(2l_2+1)(2l_2'+1)} \\ & \times \sum_\lambda \frac{(r_1, r_2)_<^\lambda}{(r_1, r_2)_>^{\lambda+1}} \begin{pmatrix} l_1 & \lambda & l_1' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & \lambda & l_2' \\ 0 & 0 & 0 \end{pmatrix} \\ & \times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ l_2' & l_1' & \lambda \end{Bmatrix} , \end{aligned} \tag{10}$$

$$V^L(r_1, r_3) = (-1)^{l_2 + L_{123}} \delta_{l_2, l_2'} \delta_{l_4, l_4'} \delta_{L_{123}, L_{123}'}$$

$$
\begin{aligned}
&\times \sqrt{(2l_1+1)(2l'_1+1)(2l_3+1)(2l'_3+1)(2L_{12}+1)(2L'_{12}+1)} \\
&\times (-1)^{\lambda} \sum_{\lambda} \frac{(r_1,r_3)^{\lambda}_{<}}{(r_1,r_3)^{\lambda+1}_{>}} \begin{pmatrix} l_1 & \lambda & l'_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & \lambda & l'_3 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ L'_{12} & \lambda & l'_1 \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ l'_3 & L'_{12} & \lambda \end{Bmatrix} ,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
V^L(r_1,r_4) = (-1)^{l_2+l_3+L_{12}+L'_{12}+L} \delta_{l_2,l'_2}\delta_{l_3,l'_3} & \\
&\times \sqrt{(2l_1+1)(2l'_1+1)(2l_4+1)(2l'_4+1)} \\
&\times \sqrt{(2L_{12}+1)(2L'_{12}+1)(2L_{123}+1)(2L'_{123}+1)} \\
&\times \sum_{\lambda} \frac{(r_1,r_4)^{\lambda}_{<}}{(r_1,r_4)^{\lambda+1}_{>}} \begin{pmatrix} l_1 & \lambda & l'_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_4 & \lambda & l'_4 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ L'_{12} & \lambda & l'_1 \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ L'_{123} & \lambda & L'_{12} \end{Bmatrix} \\
&\times \begin{Bmatrix} L_{123} & l_4 & L \\ l'_4 & L'_{123} & \lambda \end{Bmatrix} ,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
V^L(r_2,r_3) = (-1)^{l_1+l_2+L_{12}+L_{123}+l'_2+L'_{12}} \delta_{l_1,l'_1}\delta_{l_4,l'_4}\delta_{L_{123},L'_{123}} & \\
&\times \sqrt{(2l_2+1)(2l'_2+1)(2l_3+1)(2l'_3+1)(2L_{12}+1)(2L'_{12}+1)} \\
&\times (-1)^{\lambda} \sum_{\lambda} \frac{(r_2,r_3)^{\lambda}_{<}}{(r_2,r_3)^{\lambda+1}_{>}} \begin{pmatrix} l_2 & \lambda & l'_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & \lambda & l'_3 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ \lambda & L'_{12} & l'_2 \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ l'_3 & L'_{12} & \lambda \end{Bmatrix} ,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
V^L(r_2,r_4) = (-1)^{l_1+l'_2+l_3+L} \delta_{l_1,l'_1}\delta_{l_3,l'_3} & \\
&\times \sqrt{(2l_2+1)(2l'_2+1)(2l_4+1)(2l'_4+1)} \\
&\times \sqrt{(2L_{12}+1)(2L'_{12}+1)(2L_{123}+1)(2L'_{123}+1)} \\
&\times \sum_{\lambda} \frac{(r_2,r_4)^{\lambda}_{<}}{(r_2,r_4)^{\lambda+1}_{>}} \begin{pmatrix} l_2 & \lambda & l'_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_4 & \lambda & l'_4 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ \lambda & L'_{12} & l'_2 \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ L'_{123} & \lambda & L'_{12} \end{Bmatrix} \\
&\times \begin{Bmatrix} L_{123} & l_4 & L \\ l'_4 & L'_{123} & \lambda \end{Bmatrix} ,
\end{aligned} \tag{14}
$$

$$
V^L(r_3,r_4) = (-1)^{l_3+l'_3+L_{12}+L_{123}+L'_{123}+L} \delta_{l_1,l'_1}\delta_{l_2,l'_2}\delta_{L_{12},L'_{12}}
$$

$$
\begin{aligned}
&\times \sqrt{(2l_3+1)(2l_3'+1)(2l_4+1)(2l_4'+1)(2L_{123}+1)(2L_{123}'+1)} \\
&\times (-1)^{\lambda} \sum_{\lambda} \frac{(r_3,r_4)_<^{\lambda}}{(r_3,r_4)_>^{\lambda+1}} \begin{pmatrix} l_3 & \lambda & l_3' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_4 & \lambda & l_4' \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ \lambda & L_{123}' & l_3' \end{Bmatrix} \begin{Bmatrix} L_{123} & l_4 & L \\ l_4' & L_{123}' & \lambda \end{Bmatrix}, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
W(r_1,t) = {} & (-1)^{l_2+l_3+l_4+L_{12}+L_{12}'+L_{123}+L_{123}'+L+L_0} \delta_{l_2,l_2'} \delta_{l_3,l_3'} \delta_{l_4,l_4'} \\
&\times \sqrt{(2l_1+1)(2l_1'+1)(2L_{12}+1)(2L_{12}'+1)} \\
&\times \sqrt{(2L_{123}+1)(2L_{123}'+1)(2L+1)(2L_0+1)} \\
&\times g(r_1)F(t) \begin{pmatrix} l_1 & 1 & l_1' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & L_0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} L_{123} & l_4 & L \\ L_0 & 1 & L_{123}' \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ L_{123}' & 1 & L_{12}' \end{Bmatrix} \\
&\times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ L_{12}' & 1 & l_1' \end{Bmatrix} \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
W(r_2,t) = {} & (-1)^{l_1+l_2+l_2'+l_3+l_4+L_{123}+L_{123}'+L+L_0+1} \delta_{l_1,l_1'} \delta_{l_3,l_3'} \delta_{l_4,l_4'} \\
&\times \sqrt{(2l_2+1)(2l_2'+1)(2L_{12}+1)(2L_{12}'+1)} \\
&\times \sqrt{(2L_{123}+1)(2L_{123}'+1)(2L+1)(2L_0+1)} \\
&\times g(r_2)F(t) \begin{pmatrix} l_2 & 1 & l_2' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & L_0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} L_{123} & l_4 & L \\ L_{01} & L_{123}' & \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ L_{123}' & 1 & L_{12}' \end{Bmatrix} \\
&\times \begin{Bmatrix} l_1 & l_2 & L_{12} \\ 1 & L_{12}' & l_2' \end{Bmatrix} \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
W(r_3,t) = {} & (-1)^{l_3+l_3'+l_4+L_{12}+L+L_0} \delta_{l_1,l_1'} \delta_{l_2,l_2'} \delta_{l_4,l_4'} \delta_{L_{12},L_{12}'} \\
&\times \sqrt{(2l_3+1)(2l_3'+1)(2L_{123}+1)(2L_{123}'+1)(2L+1)(2L_0+1)} \\
&\times g(r_3)F(t) \begin{pmatrix} l_3 & 1 & l_3' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & L_0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} L_{123} & l_4 & L \\ L_0 & 1 & L_{123}' \end{Bmatrix} \begin{Bmatrix} L_{12} & l_3 & L_{123} \\ 1 & L_{123}' & l_3' \end{Bmatrix} \qquad (18)
\end{aligned}
$$

$$
W(r_4,t) = (-1)^{l_4+l_4'+L_{123}+1} \delta_{l_1,l_1'} \delta_{l_2,l_2'} \delta_{l_3,l_3'} \delta_{L_{123},L_{123}'}
$$

$$\times \sqrt{(2l_4+1)(2l'_4+1)(2L+1)(2L_0+1)}$$
$$\times\, g(r_4)F(t)\begin{pmatrix} l_4 & 1 & l'_4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & 1 & L_0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \begin{Bmatrix} L_{123} & l_4 & L \\ 1 & L_0 & l'_4 \end{Bmatrix} \tag{19}$$