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EXPONENTIAL INEQUALITIES FOR EXIT TIMES FOR STOCHASTIC NAVIER-STOKES EQUATIONS AND A CLASS OF EVOLUTIONS

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Abstract. Exponential estimates for exit from a ball of radius r by time T for solutions of the two-dimensional stochastic Navier-Stokes equations are first derived, and then studied in the context of Freidlin-Wentzell type large deviations principle. The existence of a similar estimate is discussed for a suitable class of stochastic evolution equations with multiplicative noise.

1. Introduction

The stochastic Navier-Stokes system has been an important and active area of research, and has received considerable attention in recent years. The introduction of randomness in the Navier-Stokes equations arises from a need to understand (i) the velocity fluctuations observed in wind tunnels under identical experimental conditions, and (ii) the onset of turbulence. Random body forces also arise as control terms, or from random disturbances such as structural vibrations that act on the fluid. It was originally the idea of Kolmogorov (see Vishik and Fursikov [12]) to introduce white noise in the Navier-Stokes system in order to obtain an invariant measure for the system.

The two-dimensional stochastic Navier-Stokes equation perturbed by an additive noise (driven by a Wiener process) can be written as an abstract evolution equation as follows:

$$d\mathbf{u}\,(t) + [\nu \mathbf{A}\,\mathbf{u}\,(t) + \mathbf{B}(\mathbf{u}\,(t))]dt = \mathbf{f}(t)dt + \Sigma_t\, d\mathbf{W}(t) \tag{1.1}$$

with viscosity coefficient, $\nu > 0$, external body force, $\mathbf{f}$, and initial data, $\mathbf{u}(0)$, specified, and operators $\mathbf{A}, \mathbf{B}$ are defined in Section 2. The objectives of this paper consist in obtaining (i) exponential estimates on certain exit times associated with the solution $\mathbf{u}$, and (ii) their optimality from the standpoint of large deviations principle. Specifically, for any fixed $r > 0$, define

$$\tau_r := \inf\{t \in [0, T] : |\mathbf{u}(t)| > r\},$$

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namely, the first time of exit for the solution from the r-ball in a suitable Hilbert space. We also consider the time of exit from the r-ball for enstrophy of the solution when the energy of $\mathbf{u}$ stays bounded.

In general, exponential estimates for exit times for a class of stochastic evolution equations were obtained systematically by Chow and Menaldi [2]. Inspired by their work, we find a connection to such estimates and a Freidlin-Wentzell type large deviations result (in the small noise asymptotics) for the stochastic Navier-Stokes system (1.1). It should be noted that equation (1.1) does not have a small parameter $\sqrt{\epsilon}$ in front of the noise term; however, we prove that the solution is a continuous function of a scaleable process, and use the continuous mapping theorem to obtain the large deviations result. The latter result provides us a method to measure the optimality of the exponential estimates for the solution.

The content of this article is organized as follows. The functional analytic setup of stochastic Navier-Stokes equations as an abstract evolution equation is given in Section 2. Exponential estimates for exit times are presented in Section 3 based on the energy equality. Section 4 establishes the connection of the exponential estimates to the large deviation principle. In Section 5 the exponential estimates for exit times are derived for a class of SPDEs with multiplicative noise.

2. Stochastic Navier-Stokes Equation

In this section, we express the Navier-Stokes equation using appropriate function spaces. Let G be a bounded open domain in $\mathbb{R}^2$ with a smooth boundary ∂G. For $t \in [0, T]$, consider the stochastic Navier-Stokes equation for a viscous incompressible flow with no-slip condition at the boundary:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) + \Sigma_t \frac{d\mathbf{W}(t)}{dt} \tag{2.1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

with initial and boundary data

$$\mathbf{u}(t, x) = 0 \ \forall \ x \in \partial G, \text{and} \ \forall \ t \geq 0$$
$$\mathbf{u}(0, x) = \mathbf{u}_0(x) \ \ \forall \ x \in G$$

where $\mathbf{u}$ is the two-dimensional velocity vector field, $\nu > 0$ is the viscosity coefficient and p denotes the pressure field and is a scalar-valued function. The function $\mathbf{f}$ is an external body force, and $\mathbf{W}$ is a Wiener process taking values in a suitable function space. Later, we will provide more details on $\mathbf{W}$ and the noise coefficient Σ.

To study the stochastic Navier-Stokes system (2.1), (2.2), we first write the stochastic partial differential equation in the abstract (variational, or evolution) form on suitable function spaces. For the functional analytic setup and the mathematical details, we refer the reader to Ladyzhenskaya [6] and Temam [10]. Let $\mathcal{V}$ be the space of two-dimensional vector functions $\mathbf{u}$ on G which are infinitely differentiable with compact support strictly contained in G, satisfying $\nabla \cdot \mathbf{u} = 0$.

For any fixed $\alpha \in \mathbb{R}$, we can define the restriction of the standard Sobolev space $W^{\alpha,2}$ to those divergence-free 2D-vectors by letting V_α denote the closure of $\mathcal{V}$ in $W^{\alpha,2}$.

We will use the shorthand notations $H := V_0$, $V := V_1$, and $L^2(G)$, $W_0^{1,2}(G)$, etc. to denote two-dimensional vector functions on G with each coordinate in the scalar versions of $L^2(G)$, $W_0^{1,2}(G)$, etc. For instance, we simply have

$$W_0^{1,2}(G) = \{\mathbf{u} \in L^2(G, \mathbb{R}^2) : \nabla \mathbf{u} \in L^2(G, \mathbb{M}_2(\mathbb{R})),\ \mathbf{u}|_{\partial G} = 0\}$$

where $\mathbb{M}_2(\mathbb{R})$ denotes the space of 2×2 real matrices.

Denoting by $\mathbf{n}$ the outward normal vector on ∂G, the following characterizations of spaces H and V are well-known, and will be convenient:

$$H := \{\mathbf{u} \in L^2(G) : \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \mathbf{n}|_{\partial G} = 0\},$$
$$V := \{\mathbf{u} \in W_0^{1,2}(G) : \nabla \cdot \mathbf{u} = 0\}.$$

Let V' be the dual of V. We will denote the norm in H by $|\cdot|$, and its inner product by $(\cdot,\cdot)$. By [10] we have the dense, continuous and compact embedding:

$$V \subset_{\rightarrow} H \equiv H' \subset_{\rightarrow} V'.$$

Let $\mathcal{D}(\mathbf{A}) = W^{2,2}(G) \cap V$. Define the linear operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to H$ by $\mathbf{A}\mathbf{u} = -\Pi\Delta\mathbf{u}$, where Π denotes the Leray projection of $L^2(G)$ into H. Since $V = \mathcal{D}(\mathbf{A}^{1/2})$, we can endow V with the norm $\|\mathbf{u}\| = \left|\mathbf{A}^{1/2}\mathbf{u}\right|$ which is equivalent to the $W^{1,2}$-norm by the Poincaré inequality. From this point on, $\|\cdot\|$ will denote the V-norm. The pairing between V and its dual V' will be denoted by $\langle \cdot, \cdot \rangle$. The operator $\mathbf{A}$ is known as the Stokes operator and is positive, self-adjoint with compact resolvent. The eigenvalues of $\mathbf{A}$ will be denoted by $0 < \lambda_1 < \lambda_2 \leq \cdots$, and the corresponding eigenfunctions by $e_1, e_2, \cdots$ form a complete orthonormal system for H. It is known (cf. [6]) that there are values $c, c' > 0$ such that

$$\lim_{j\to\infty} \frac{j}{\lambda_j} = c > 0 \text{ and } \left\|e_j\right\|_{L^4(G)} \leq c\lambda_j^{1/4} \text{ for all } j.$$

Define $b(\cdot,\cdot,\cdot) : V \times V \times V \to \mathbb{R}$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_G u_i(x) \frac{\partial v_j}{\partial x_i}(x)\, w_j(x)\, dx.$$

This allows us to define $\mathbf{B} : V \times V \to V'$ as a continuous bilinear operator such that

$$\left\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \right\rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \text{ for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

Note that $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$. We will denote $\mathbf{B}(\mathbf{u}, \mathbf{u})$ by $\mathbf{B}(\mathbf{u})$ which satisfies the following estimate:

$$\left\|\mathbf{B}(\mathbf{u})\right\|_{V'} \leq 2|\mathbf{u}|\,||\mathbf{u}|| \tag{2.3}$$

by setting a constant that depends on the domain G as 1. Let U be a real separable Hilbert space. We assume that $\mathbf{u}_0$ is H-valued, and is independent of

W, a U-valued Wiener process with nuclear covariance operator Q. Then space $U_0 = Q^{1/2}U$ is a Hilbert space with inner product,

$$(\mathbf{u}, \mathbf{v})_0 = (Q^{-1/2}\mathbf{u}, Q^{-1/2}\mathbf{v})_U \ \forall \ \mathbf{u}, \mathbf{v} \in U_0. \tag{2.4}$$

Let $|\cdot|_0$ denote the norm in U_0. Clearly, the imbedding of U_0 in U is Hilbert-Schmidt since Q is a trace class operator.

Let L_Q be the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from U to H. Define the norm on space L_Q by

$$|S|^2_{L_Q} := \left\|SQ^{1/2}\right\|^2_{L_2} = \mathrm{tr}\,(SQS^*)$$

where L_2 stands for the Hilbert Schmidt norm of the operator. The noise coefficient $\Sigma : \Omega \times [0,T] \to L_Q$ is assumed to be predictable with

$$\mathbb{E}\int_0^T \mathrm{tr}\{\Sigma_t Q \Sigma_t^*\}dt < \infty. \tag{2.5}$$

We also assume that $\mathbf{f}(s)$ is V'-valued for each $s \in [0,T]$, and $\int_0^T \left\|\mathbf{f}(s)\right\|^2_{V'} ds < \infty$.

By applying projection Π to each term of the Navier-Stokes system, and invoking the Leray decomposition of $L^2(G)$ into divergence free and irrotational components, we write the system (2.1) and (2.2) as

$$d\mathbf{u}\,(t) + [\nu\mathbf{A}\,\mathbf{u}\,(t) + \mathbf{B}(\mathbf{u}\,(t))]\,dt = \mathbf{f}(t)dt + \Sigma_t\,d\mathbf{W}(t). \tag{2.6}$$

This is to be understood in the integral form

$$\mathbf{u}\,(t) = \mathbf{u}\,(0) - \nu\int_0^t \mathbf{A}\,\mathbf{u}\,(s)\,ds - \int_0^t \mathbf{B}(\mathbf{u}\,(s))ds + \int_0^t \Sigma_s d\mathbf{W}(s) + \int_0^t f(s)ds. \tag{2.7}$$

The existence and uniqueness of solutions of the stochastic Navier-Stokes equation (2.6) has been studied under a variety of hypotheses and levels of generality by several authors (e.g. [1, 3, 4, 7, 9]). We base our work on the following theorem due to Viot [11].

Theorem 2.1. *(Viot) Let $E(|\mathbf{u}_0|^2) < \infty$ and $\mathbf{f} \in L^2(0,T;V')$. Suppose that there exists a positive constant C such that $|\Sigma_s Q^{1/2}|^2_{L_2} \leq C$ for all $s \in [0,T]$. Then there exists a solution of the martingale problem posed by (2.7), and the solution is pathwise unique.*

The martingale solution guaranteed by the above theorem is a weak solution of (2.7) in the sense of stochastic analysis and partial differential equations. Since the solution is pathwise unique, a result of Yamada and Watanabe (Proposition 5.3.20 in [5]) allows us to conclude that the solution $\mathbf{u}$ is a strong solution in the sense of stochastic analysis.

The solution $\mathbf{u}$ lies in $L^2(\Omega; C(0,T;H)) \cap L^2(\Omega \times (0,T); V)$, and if the initial value $\mathbf{u}(0)$ satisfies $\mathbb{E}|\mathbf{u}(0)|^4 < \infty$, and $\mathbf{f} \in L^4([0,T];V')$, then there exists a constant $C(T,\nu) > 0$ that depends on T and ν such that

$$\mathbb{E}\left(\sup_{0\le s\le T}|\mathbf{u}(s)|^4 + \int_0^T |\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^2\, ds\right)$$
$$\le C(T,\nu)\left(1+\mathbb{E}|\mathbf{u}(0)|^4 + \int_0^T \|\mathbf{f}(s)\|_{V'}^4\, ds\right) \tag{2.8}$$

(cf. Proposition 2.3 of [9]).

3. Exponential Inequalities

We begin by proving an exponential inequality for the energy of the solution $\mathbf{u}$ of equation (2.7) when it exceeds a threshold $r > 0$ by time T. The first result is quite in the spirit of the work of Chow and Menaldi [2].

Proposition 3.1. *Assume that there exists a positive constant C such that*

(i) *for all $s \in [0,T]$, $\left\|\Sigma_s Q^{1/2}\right\|_{L_2}^2 \le C$, and*

(ii) $\int_0^T \|\mathbf{f}(s)\|_{V'}^2\, ds \le C$.

Then, for any given $r > 0$, the solution $\mathbf{u}$ of (2.6) satisfies

$$P\{\sup_{0\le t\le T}|\mathbf{u}(t)| > r\} \le \exp\{K + CT - r^2\exp\{-2CT\}\} \tag{3.1}$$

where $K := |\mathbf{u}(0)|^2 + \frac{C}{8\nu}$.

Proof. By the Itô formula,

$$\begin{aligned}|\mathbf{u}(t)|^2 + 2\nu\int_0^t \|\mathbf{u}(s)\|^2\, ds &= |\mathbf{u}(0)|^2 + 2\int_0^t \langle \mathbf{f}(s),\mathbf{u}(s)\rangle ds + 2\int_0^t(\mathbf{u}(s),\Sigma_s dW_s)\\ &\quad + \int_0^t \mathrm{tr}\,(\Sigma_s Q\Sigma_s^*)ds\\ &\le |\mathbf{u}(0)|^2 + \frac{1}{8\nu}\int_0^t \|\mathbf{f}(s)\|_{V'}^2\, ds + 2\nu\int_0^t\|\mathbf{u}(s)\|^2\, ds + \eta_t\\ &\quad + 2\int_0^t \left\|Q^{1/2}\Sigma_s^*\mathbf{u}(s)\right\|_{U_0}^2 ds + \int_0^t \mathrm{tr}\,(\Sigma_s Q\Sigma_s^*)ds \end{aligned} \tag{3.2}$$

where we have used the notation

$$\eta_t := 2\int_0^t(\mathbf{u}(s),\Sigma_s dW_s) - 2\int_0^t\left\|Q^{1/2}\Sigma_s^*\mathbf{u}(s)\right\|_{U_0}^2 ds.$$

By the hypotheses, $\left\|Q^{1/2}\Sigma_s^*\mathbf{u}(s)\right\|_{U_0}^2 \le \left\|\Sigma_s Q^{1/2}\right\|_{L(U;H)}^2 |\mathbf{u}(s)|^2 \le C|\mathbf{u}(s)|^2$ yielding thereby

$$|\mathbf{u}(t)|^2 \le K + \eta_t + 2C\int_0^t |\mathbf{u}(s)|^2 ds + Ct \tag{3.3}$$

where $K := |\mathbf{u}(0)|^2 + \frac{C}{8\nu}$. By Gronwall inequality, we write,

$$\begin{aligned} |\mathbf{u}(t)|^2 &\leq K + \eta_t + Ct + \int_0^t (K + \eta_s + Cs) 2C \exp\{2C(t-s)\} ds \\ &\leq \left[K + \sup_{0\leq s\leq t} \eta_s + Ct \right] \exp\{2Ct\}. \end{aligned} \tag{3.4}$$

Hence, for any fixed $r > 0$, we obtain,

$$\begin{aligned} P\{\sup_{0\leq t\leq T} |\mathbf{u}(t)| > r\} &\leq P\{(K + \sup_{0\leq t\leq T} \eta_t + CT) \exp\{2CT\} > r^2\} \\ &= P\{\sup_{0\leq t\leq T} \eta_t > r^2 e^{-2CT} - K - CT\} \\ &= P\{\sup_{0\leq t\leq T} \exp\{\eta_t\} > \exp\{r^2 e^{-2CT} - K - CT\}\} \\ &\leq \exp\{K + CT - r^2 \exp\{-2CT\}\} \end{aligned}$$

by the basic martingale inequality, and the proof is complete. □

Exponential estimates for the maximum of enstrophy of the solution $\mathbf{u}$ over the interval $[0, T]$ are difficult to obtain under our general setup. Here we content ourselves with the following related estimate on the supremum of $s\|\mathbf{u}(s)\|^2$ over $[0, T]$ when $\int_0^T (|\mathbf{u}(s)|^2 + 1)\ \|\mathbf{u}(s)\|^2\, ds$ remains bounded. It is worthwhile to note that $\mathbb{E} \int_0^T (|\mathbf{u}(s)|^2 + 1)\|\mathbf{u}(s)\|^2\, ds$ is finite, and hence $\int_0^T (|\mathbf{u}(s)|^2 + 1)\ \|\mathbf{u}(s)\|^2\, ds$ remains bounded with a large probability. The following result once again illustrates the simplicity and the wide applicability of the methods of Chow and Menaldi [2].

Proposition 3.2. *Suppose there exists a finite constant $C > 0$ such that*

(i) *for all $s \in [0, T]$, $\left\|\Sigma_s Q^{1/2}\right\|_{L_2}^2 \leq \frac{\nu}{T}$, and*

(ii) $\int_0^T \|\mathbf{f}_s\|_{V'}^2\, ds \leq C$.

Then, for any given $K > 0$ and $r > 0$, there exists a constant $C_1 > 0$ such that

$$P\left\{ \int_0^T \alpha(s) ds \leq K, \sup_{0\leq s\leq T} \frac{s}{2} \|\mathbf{u}(s)\|^2 > r^2 \right\} \leq \exp\left(C_1 - r^2 e^{-K} \right). \tag{3.5}$$

where $\alpha(s) = 2C_\nu\, (|\mathbf{u}(s)|^2 + 1)\, \|\mathbf{u}(s)\|^2$ with $C_\nu = \frac{27}{4\nu^3}$.

Proof. From equation (2.6), one obtains,

$$\begin{aligned} &\sqrt{t}\mathbf{u}(t) + \int_0^t \sqrt{s}\{\nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s))\} ds \\ &= \int_0^t \sqrt{s}\mathbf{f}(s) ds + \int_0^t \sqrt{s}\Sigma_s dW_s + \int_0^t \frac{\mathbf{u}(s)}{2\sqrt{s}} ds. \end{aligned}$$

By the Itô formula, one obtains

$$\begin{aligned}
&\frac{t}{2}\|\mathbf{u}(t)\|^2 + \nu\int_0^t s|\mathbf{A}\mathbf{u}(s)|^2 ds + \int_0^t s\langle \mathbf{B}(\mathbf{u}(s)), \mathbf{A}\mathbf{u}(s)\rangle ds \\
&= \int_0^t s\langle \mathbf{f}(s), \mathbf{A}\mathbf{u}(s)\rangle ds + \int_0^t s(\mathbf{A}\mathbf{u}(s), \Sigma_s dW_s) + \frac{1}{2}\int_0^t s\,\mathrm{tr}(\Sigma_s Q \Sigma_s^*) \mathrm{ds} \\
&\quad + \frac{1}{2}\int_0^t (\mathbf{u}(s), \mathbf{A}\mathbf{u}(s)) ds. \qquad (3.6)
\end{aligned}$$

Note that

$$\begin{aligned}
|\langle \mathbf{B}(\mathbf{u}(s)), \mathbf{A}\mathbf{u}(s)\rangle| &= |b(\mathbf{u}(s), \mathbf{u}(s), \mathbf{A}\mathbf{u}(s))| \\
&\le |\mathbf{u}(s)|_{L^4(G)} |\nabla \mathbf{u}(s)|_{L^4(G)} |\mathbf{A}\mathbf{u}(s)| \\
&\le |\mathbf{u}(s)|^{1/2} \|\mathbf{u}(s)\|\, |\mathbf{A}\mathbf{u}(s)|^{3/2}.
\end{aligned}$$

Using this estimate in equation (3.6) gives

$$\begin{aligned}
&\frac{t}{2}\|\mathbf{u}(t)\|^2 + \nu\int_0^t s|\mathbf{A}\mathbf{u}(s)|^2 ds \\
&\le \int_0^t s|\mathbf{u}(s)|^{1/2}\|\mathbf{u}(s)\|\, |\mathbf{A}\mathbf{u}(s)|^{3/2} ds + \int_0^t s|\mathbf{f}(s)||\mathbf{A}\mathbf{u}(s)| ds + \int_0^t (s\mathbf{A}\mathbf{u}(s), \Sigma_s dW_s) \\
&\quad + \frac{1}{2}\int_0^t s\,\mathrm{tr}(\Sigma_s Q \Sigma_s^*) \mathrm{ds} + \frac{1}{2}\int_0^{\mathrm{t}} \|\mathbf{u}(\mathrm{s})\|^2 \,\mathrm{ds} \\
&\le \frac{\nu}{2}\int_0^t s|\mathbf{A}\mathbf{u}(s)|^2 ds + C_\nu \int_0^t s|\mathbf{u}(s)|^2 \|\mathbf{u}(s)\|^4 ds + \frac{2}{\nu}\int_0^t s|\mathbf{f}(s)|^2 ds \\
&\quad + \int_0^t (s\mathbf{A}\mathbf{u}(s), \Sigma_s dW_s) + \frac{1}{2}\int_0^t s\left\|\Sigma_s Q^{1/2}\right\|_{L_2}^2 ds + \frac{1}{2}\int_0^t \|\mathbf{u}(s)\|^2 ds. \qquad (3.7)
\end{aligned}$$

Thus, by taking $C_\nu > 1/2$, we have,

$$\begin{aligned}
&\frac{t}{2}\|\mathbf{u}(t)\|^2 + \frac{\nu}{2}\int_0^t s|\mathbf{A}\mathbf{u}(s)|^2 ds \\
&\le C_\nu \int_0^t \{s|\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^2 + 1\}\|\mathbf{u}(s)\|^2 ds + \int_0^t \{\frac{2}{\nu}s|\mathbf{f}(s)|^2 \\
&\quad + \frac{1}{2}s\left\|\Sigma_s Q^{1/2}\right\|_{L_2}^2\} ds + \int_0^t (s\mathbf{A}\mathbf{u}(s), \Sigma_s dW_s). \qquad (3.8)
\end{aligned}$$

Define the processes $A_t := \int_0^t \{C_\nu \|\mathbf{u}(s)\|^2 + \frac{2}{\nu}s|\mathbf{f}(s)|^2 + \frac{1}{2}s\left\|\Sigma_s Q^{1/2}\right\|_{L_2}^2\} ds$, and $\eta_t := \int_0^t (s\mathbf{A}\mathbf{u}(s), \Sigma_s dW_s) - \frac{1}{2}\int_0^t s^2|Q^{1/2}\Sigma_s^* \mathbf{A}\mathbf{u}(s)|^2 ds$. From these definitions, we write (3.8) as

$$\begin{aligned}
&\frac{t}{2}\|\mathbf{u}(t)\|^2 + \frac{\nu}{2}\int_0^t s|\mathbf{A}\mathbf{u}(s)|^2 ds \\
&\le A_t + \eta_t + \int_0^t \frac{s}{2}\alpha(s)\|\mathbf{u}(s)\|^2 ds + \frac{1}{2}\int_0^t s^2|Q^{1/2}\Sigma_s^* \mathbf{A}\mathbf{u}(s)|^2 ds \qquad (3.9)
\end{aligned}$$

with $\alpha(s) = 2C_\nu\,(|\mathbf{u}(s)|^2 + 1)\ \|\mathbf{u}(s)\|^2$.

Using the condition $|Q^{1/2}\Sigma_s^*|^2 \leq \frac{\nu}{T}$ for all $s \leq T$, we obtain

$$\frac{t}{2}\|\mathbf{u}(t)\|^2 \leq A_t + \eta_t + \int_0^t \frac{s}{2}\alpha(s)\|\mathbf{u}(s)\|^2\, ds \tag{3.10}$$

and by a Gronwall argument, we conclude that

$$\begin{aligned}\frac{t}{2}\|\mathbf{u}(t)\|^2 &\leq A_t + \eta_t + \int_0^t e^{\int_s^t \alpha(r)dr}(A_s + \eta_s)\alpha(s)ds \\ &\leq \sup_{s\leq t}(A_s + \eta_s)\left(1 + e^{\int_0^t \alpha(r)dr}\int_0^t e^{-\int_0^s \alpha(r)dr}\alpha(s)ds\right) \\ &\leq \sup_{s\leq t}(A_s + \eta_s)e^{\int_0^t \alpha(r)dr}. \end{aligned}\tag{3.11}$$

Hence, we conclude that

$$\begin{aligned}&P\left\{\int_0^T \alpha(s)ds \leq K, \sup_{0\leq s\leq T}\frac{s}{2}\|\mathbf{u}(s)\|^2 > r^2\right\} \\ &\leq P\left\{\sup_{0\leq s\leq T}(A_s + \eta_s)e^K > r^2\right\} \\ &\leq P\left\{A_T + \sup_{0\leq s\leq T}\eta_s > r^2e^{-K}\right\} \\ &= P\left\{\sup_{0\leq s\leq T}e^{\eta_s} > \exp\{r^2e^{-K} - A_T\}\right\} \\ &\leq \exp\left\{C_1 - r^2e^{-K}\right\}, \end{aligned}\tag{3.12}$$

where $C_1 = C_\nu K + T(\frac{2C}{\nu} + \frac{\nu}{4})$. □

4. Connection to Large Deviations

Here, we study exit times of solutions of stochastic Navier-Stokes equations from the $r-$ball by using small noise asymptotics provided by large deviations theory. It is worthwhile to point out that the analysis is carried out despite the fact that the stochastic equations do not have a small parameter in the noise term.

Consider the unique solution $\mathbf{z}(t)$ of

$$d\mathbf{z} + \mathbf{A}\mathbf{z}dt = \Sigma_t dW_t, \tag{4.1}$$

with $\mathbf{z}(0) = 0$. Define $\mathbf{v} := \mathbf{u} - \mathbf{z}$, and notice that

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial \mathbf{u}}{\partial t} - \frac{\partial \mathbf{z}}{\partial t} \\ &= (-\mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u}) + f(t) + \Sigma_t\frac{dW}{dt}) - (-\mathbf{A}\mathbf{z} + \Sigma_t\frac{dW}{dt}) \\ &= -\mathbf{A}(\mathbf{u} - \mathbf{z}) - \mathbf{B}(\mathbf{u}) + f(t) = -\mathbf{A}\mathbf{v} - \mathbf{B}(\mathbf{v} + \mathbf{z}) + \mathbf{f}\end{aligned}$$

Therefore, with $\mathbf{z}$ given, solving for $\mathbf{u}-\mathbf{z}$ would be equivalent to solving for $\mathbf{v}$ in

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}\mathbf{v} + \mathbf{B}(\mathbf{v}+\mathbf{z}) + \mathbf{f} = 0 \tag{4.2}$$

with initial data $\mathbf{v}(0) = \mathbf{u}_0 \in H$. Note that equation (4.2) is a non-random, nonlinear partial differential equation and is solved for each ω, where ω enters the equation through $\mathbf{z}(\omega)$.

From a priori bounds, one can easily show that (similar to Proposition 2.3 in [9]),

$$\mathbb{E}\left(\sup_{0\le t\le T} |\mathbf{z}(t)|^2\right) + \mathbb{E}\left(\int_0^T \left\|\mathbf{z}(t)\right\|^2 dt\right) \le K(\nu, T, C)$$

where $K(\nu, T, C)$ is a finite constant that depends on ν, T, and C that appear in the hypotheses made in Proposition 3.1. Hence, one obtains that almost surely, $\mathbf{z} \in C_0([0,T]; H) \cap L^2(0,T;V)$.

Lemma 4.1. *Given a function $\varphi \in C_0([0,T];H) \cap L^2(0,T;V)$, let map $\Lambda : \varphi \mapsto \mathbf{v}_\varphi$ be defined by*

$$\frac{\partial \mathbf{v}_\varphi}{\partial t} + \mathbf{A}\mathbf{v}_\varphi + \mathbf{B}(\mathbf{v}_\varphi + \varphi) + \mathbf{f} = 0 \tag{4.3}$$

for $t \in [0,T]$, with $\mathbf{v}_\varphi(0) = \mathbf{u}(0)$. Then Λ is a continuous map from $C_0([0,T];H) \cap L^2(0,T;V)$ to the space $C([0,T];H) \cap L^2(0,T;V)$.

Proof. Consider functions φ_1 and φ_2 in $C_0([0,T];H) \cap L^2(0,T;V)$, and denote the corresponding solutions of equation (4.3) as $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Let

$$\mathbf{w}_i := \mathbf{v}_i + \varphi_i \ \text{ for } \ i = 1, 2.$$

Then, by the energy equality,

$$\begin{aligned} &|\mathbf{v}_1(t) - \mathbf{v}_2(t)|^2 + 2\nu \int_0^t \left\|\mathbf{v}_1(s) - \mathbf{v}_2(s)\right\|^2 ds \\ &= 2\int_0^t \langle \mathbf{B}(\mathbf{w}_1(s)) - \mathbf{B}(\mathbf{w}_2(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle ds. \end{aligned} \tag{4.4}$$

By the basic properties of the bilinear operator $\mathbf{B}$, we have,

$$\begin{aligned} \langle \mathbf{B}(\mathbf{w}_1(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle &= \langle \mathbf{B}(\mathbf{w}_1(s), \mathbf{w}_2(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle \\ &\quad + \langle \mathbf{B}(\mathbf{w}_1(s), \varphi_1 - \varphi_2), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle \end{aligned}$$

which enables us to write the integrand on the right side of (4.4) (suppressing the time parameter s) as

$$\begin{aligned} &\langle \mathbf{B}(\mathbf{w}_1) - \mathbf{B}(\mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle \\ &= \langle \mathbf{B}(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle + \langle \mathbf{B}(\mathbf{w}_1, \varphi_1 - \varphi_2), \mathbf{v}_1 - \mathbf{v}_2\rangle \\ &= \langle \mathbf{B}(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle + \langle \mathbf{B}(\varphi_1 - \varphi_2, \mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle \\ &\quad + \langle \mathbf{B}(\mathbf{w}_1, \varphi_1 - \varphi_2), \mathbf{v}_1 - \mathbf{v}_2\rangle. \end{aligned} \tag{4.5}$$

Thus the integral on the right side of (4.4) can be split into three integrals, each of which is bounded as follows: First, consider

$$\begin{aligned}
&\left|\int_0^t \langle \mathbf{B}(\mathbf{v}_1(s)-\mathbf{v}_2(s),\mathbf{w}_2(s)),\mathbf{v}_1(s)-\mathbf{v}_2(s)\rangle ds\right| \\
&\le \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2\,ds + \frac{3}{\nu}\int_0^t |\mathbf{v}_1(s)-\mathbf{v}_2(s)|^2\|\mathbf{w}_2(s)\|^2\,ds \qquad (4.6)
\end{aligned}$$

by applying the properties of $\mathbf{B}$ and Young's inequality.

Next, consider the expression

$$\begin{aligned}
&\left|\int_0^t \langle \mathbf{B}(\varphi_1(s)-\varphi_2(s),\mathbf{w}_2(s)),\mathbf{v}_1(s)-\mathbf{v}_2(s)\rangle ds\right| \\
&\le \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2\,ds + \frac{3}{\nu}\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2_{L^4(G)}\,\|\mathbf{w}_2(s)\|^2_{L^4(G)}\,ds \\
&\le \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2\,ds + \frac{3}{2\nu}\int_0^t |\varphi_1(s)-\varphi_2(s)|^2\|\mathbf{w}_2(s)\|^2\,ds \\
&\quad + \frac{3}{2\nu}\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2\,|\mathbf{w}_2(s)|^2 ds \\
&\le \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2\,ds + \frac{3}{2\nu}\Bigg[\sup_{0\le s\le T}|\varphi_1(s)-\varphi_2(s)|^2\int_0^t \|\mathbf{w}_2(s)\|^2\,ds \\
&\quad + \sup_{0\le s\le T}|\mathbf{w}_2(s)|^2\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2\,ds\Bigg]. \qquad (4.7)
\end{aligned}$$

Finally, by the same reasoning employed in obtaining (4.7), we have

$$\begin{aligned}
&\left|\int_0^t \langle \mathbf{B}(\mathbf{w}_1(s),\varphi_1(s)-\varphi_2(s)),\mathbf{v}_1(s)-\mathbf{v}_2(s)\rangle ds\right| \\
&\le \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2\,ds + \frac{3}{2\nu}\Bigg[\sup_{0\le s\le T}|\varphi_1(s)-\varphi_2(s)|^2\int_0^t \|\mathbf{w}_1(s)\|^2\,ds \\
&\quad + \sup_{0\le s\le T}|\mathbf{w}_1(s)|^2\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2\,ds\Bigg]. \qquad (4.8)
\end{aligned}$$

Using bounds (4.6), (4.7) and (4.8) in equation (4.4), we obtain upon simplification,

$$\begin{aligned}
&|\mathbf{v}_1(t)-\mathbf{v}_2(t)|^2 + \nu\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2\,ds \\
&\le \frac{6}{\nu}\int_0^t |\mathbf{v}_1(s)-\mathbf{v}_2(s)|^2\|\mathbf{w}_2(s)\|^2\,ds \\
&\quad + \frac{3}{\nu}\Big(\sup_{0\le s\le T}|\varphi_1(s)-\varphi_2(s)|^2\Big)\int_0^t (\|\mathbf{w}_1(s)\|^2 + \|\mathbf{w}_2(s)\|^2) ds \\
&\quad + \frac{3}{\nu}\Big(\sup_{0\le s\le T}|\mathbf{w}_1(s)|^2 + \sup_{0\le s\le T}|\mathbf{w}_2(s)|^2\Big)\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2\,ds. \qquad (4.9)
\end{aligned}$$

Dropping the second term on the left, and applying the Gronwall inequality, we obtain

$$
\begin{aligned}
&|\mathbf{v}_1(t) - \mathbf{v}_2(t)|^2 \\
&\leq \frac{3}{\nu}\Big(\sup_{0\leq s\leq T} |\varphi_1(s) - \varphi_2(s)|^2\Big) \\
&\qquad \times \int_0^t (\|\mathbf{w}_1(s)\|^2 + \|\mathbf{w}_2(s)\|^2) \exp\Big(\frac{6}{\nu}\int_s^t \|\mathbf{w}_2(r)\|^2\, dr\Big) ds \\
&\quad + \frac{3}{\nu}\Big(\sup_{0\leq s\leq T} |\mathbf{w}_1(s)|^2 + \sup_{0\leq s\leq T} |\mathbf{w}_2(s)|^2\Big) \\
&\qquad \times \int_0^t \|\varphi_1(s) - \varphi_2(s)\|^2 \exp\Big(\frac{6}{\nu}\int_s^t \|\mathbf{w}_2(r)\|^2\, dr\Big) ds.
\end{aligned}
\tag{4.10}
$$

If $\varphi_n \to \varphi$ in $C_0([0,T];H) \cap L^2(0,T;V)$, as $n \to \infty$, it is simple to obtain an upper bound uniform in n for $\sup_{0\leq t\leq T} |\mathbf{w}_n(t)|$ and $\int_0^T \|\mathbf{w}_n(s)\|\, ds$, where $\mathbf{w}_n := \mathbf{v}_n + \varphi_n$. Hence, (4.10) allows us to conclude that $\mathbf{v}_n - \mathbf{v} \to 0$ in $C_0([0,T];H)$, and we use this result to estimate (4.9) to justify that $\mathbf{v}_n - \mathbf{v} \to 0$ in $L^2(0,T;V)$ as well. The continuity of the map Λ has thus been proven.

□

For each $h \in L^2(0,T;U_0)$, we will use the notation $\mathcal{G}^0(\int_0^\cdot h(s)ds)$ to denote the set of all solutions of the equation

$$dx(t) + \mathbf{A}x(t)dt = \Sigma_t h(t)dt$$

with $x(0) = 0$.

For each $\epsilon > 0$, let $\mathbf{z}^\epsilon$ denote the solution of

$$d\mathbf{z}^\epsilon(t) + \mathbf{A}\mathbf{z}^\epsilon(t)dt = \sqrt{\epsilon}\Sigma_t dW_t$$

for $0 \leq t \leq T$ with $\mathbf{z}^\epsilon(0) = 0$. Then $\mathbf{z}^\epsilon(t) = \sqrt{\epsilon}\int_0^t S_{t-s}\Sigma_s dW_s$ where S is the semigroup generated by $\mathbf{A}$. It is well-known (cf. [9]) that the large deviations rate function for the family $\{\mathbf{z}^\epsilon\}$ is given by,

$$I(x) = \inf_{\{h\in L^2(0,T;U_0):x\in\mathcal{G}_0(\int_0^\cdot h(s)ds)\}} \frac{1}{2}\int_0^T |h(s)|_0^2 ds.$$

Define the map Γ from $C_0([0,T];H) \cap L^2(0,T;V)$ to $C([0,T];H) \cap L^2(0,T;V)$ by

$$\Gamma(\mathbf{z}) = \mathbf{z} + \Lambda(\mathbf{z}).$$

Then Γ is continuous by Lemma 4.1, and $\mathbf{u}^\epsilon = \Gamma(\mathbf{z}^\epsilon)$ for all $\epsilon > 0$. Hence, by the contraction mapping principle, $\{\mathbf{u}^\epsilon\}$ satisfies the large deviation principle large deviations principle with rate function

$$J(A) = \inf_{x\,\in\,\Gamma^{-1}(A)} I(x)$$

for any Borel set A in $C([0,T];H) \cap L^2(0,T;V)$, and in particular,

$$\limsup_{\epsilon\to 0} \epsilon \log P\{\mathbf{u}^\epsilon \in B_r^c\} \leq -J(B_r^c). \tag{4.11}$$

Thus, for any given $\delta > 0$, there exists an $\epsilon_1 > 0$ such that for all $0 < \epsilon \leq \epsilon_1$,

$$P\{\mathbf{u}^\epsilon \in B_r^c\} \leq \exp\{-\frac{1}{\epsilon}(J(B_r^c) - \delta)\}.$$

That is,

$$P\{\mathbf{z} \in \frac{1}{\sqrt{\epsilon}}\Gamma^{-1}(B_r^c)\} \leq \exp\{-\frac{1}{\epsilon}(J(B_r^c) - \delta)\}. \tag{4.12}$$

Let A denote the set $\Gamma(\frac{1}{\sqrt{\epsilon}}\Gamma^{-1}(B_r^c))$. Then (4.12) can be written as

$$P\{\mathbf{u} \in A\} \leq \exp\{-\frac{1}{\epsilon}(J(B_r^c) - \delta)\}.$$

We have thus proved the following theorem:

Theorem 4.2. *For any given $r > 0$ and $\delta > 0$, there exists a large positive constant ρ_0, such that for all $\rho \geq \rho_0$ if we define the set $A_\rho := \Gamma(\rho\Gamma^{-1}(B_r^c))$, then solution $\mathbf{u}$ of equation* (2.6) *satisfies*

$$P\{\mathbf{u}(t) \in A_\rho\} \leq \exp\{-\rho(J(B_r^c) - \delta)\} \tag{4.13}$$

where $B_r = \{h \in C([0,T]; H) : \sup_{0\leq t\leq T} |h(t)|^2 < r\}$,

$$J(B_r^c) = \inf_{x\in\Gamma^{-1}(B_r^c)} I(x),$$

and

$$I(x) = \inf_{\{h\in L^2(0,T;U_0):x \in \mathcal{G}^0(\int_0^{\cdot} h(s)ds)\}} \frac{1}{2}\int_0^T |h(s)|_0^2 ds.$$

Remark 4.3. (i) In case $\rho_0 = 1$, A_1 coincides with B_r^c, and the theorem gives the rate of decay as $J(B_r^c)$. Also, if we can ascertain the existence of an R such that $B_R^c \subseteq A_{\rho_0}$, the above result leads to a simpler inequality.

(ii) Since we know, by Proposition 3.1 that the rate of decay is of the order of r^2, we can follow the above procedure by considering the set

$$F_r = \{x : J(x) \leq r^2\}$$

for $r > 0$ and define the set G_r as any open neighborhood of F_r. Then given any $\delta > 0$, there exists an $\epsilon_1 > 0$ such that for all $\epsilon < \epsilon_1$, we have

$$\begin{aligned} P\{\mathbf{u}^\epsilon \in G_r^c\} &\leq \exp\{-\frac{1}{\epsilon}(J(G_r^c) - \delta)\} \\ &\leq \exp\{-\frac{1}{\epsilon}(r^2 - \delta)\} \end{aligned}$$

by the definition of G_r. Thus, we can conclude that

$$P\{\mathbf{u} \in \Gamma(\frac{1}{\sqrt{\epsilon}}\Gamma^{-1}(G_r^c))\} \leq \exp\{-\frac{1}{\epsilon}(r^2 - \delta)\}. \tag{4.14}$$

5. Exponential Inequalities for a Class of Evolution Equations

This section is devoted to the study of exponential inequalities for class of stochastic evolution equations. Here, the equation and the functional analytic setup differ from the previous sections. The class of equations that we consider below requires strong hypotheses on the operators, and hence, do not include the Navier-Stokes system.

Let $(H_0, (\cdot,\cdot)_0)$ be a real separable real Hilbert space and $(H, (\cdot,\cdot))$ a Hilbert space containing H_0 such that the embedding $i : H_0 \to H$ is Hilbert Schmidt. Consider the equation

$$dX(t) + [LX(t) - G(t, X(t))]dt = \Sigma(t, X(t))dW(t), \quad X(0) = x,$$

where $x \in H_0$ and $W(t)$ is a cylindrical Wiener process in H_0. The above equation is to be understood in its mild form

$$X(t) = T_t x + \int_0^t T_{t-s}\Sigma(s, X(s))dW_s + \int_0^t T_{t-s}G(s, X(s))ds, \tag{5.1}$$

where $x \in H_0$.

The following assumptions are made on the operators L, G and Σ.

(1) L is a densely defined linear operator on H_0 such that L^{-1} is a bounded self-adjoint operator with discrete spectrum, and $T_t = e^{-tL}$ is a contraction semigroup on H_0.

(2) The maps $G : [0,T] \times H_0 \to H_0$ and $\Sigma : [0,T] \times H_0 \to \mathcal{L}(H_0, H_0)$ have the following smoothness and growth properties. Let $\{\phi_k\}$ be a complete orthonormal system of eigenvectors of L with eigenvalues $\{\lambda_k\}$. Then there exist real sequences $\{a_k\}$, $\{b_k\}$ such that for all $k \geq 1$, $t \in [0,T]$ and $h, h_1, h_2 \in H_0$:

$$\begin{aligned} |(G(t,h), \phi_k)_0|^2 &\leq a_k^2(1 + \|h_0\|_0^2), \\ \|\Sigma^*(t,h)\phi_k\|_0^2 &\leq b_k^2(1 + \|h\|_0^2), \\ |(G(t,h_1) - G(t,h_2), \phi_k)_0| &\leq a_k\|h_1 - h_2\|_0, \\ \|\Sigma^*(t,h_1) - \Sigma^*(t,h_2))\phi_k\|_0 &\leq b_k\|h_1 - h_2\|_0, \end{aligned}$$

and

$$\sum_{k=1}^{\infty} a_k^2\lambda_k^{-1} = C_1 < \infty, \quad \sum_{k=1}^{\infty} b_k^2\lambda_k^{\gamma} = C_2 < \infty,$$

for some $0 < \gamma < 1$.

Suppose $\mathbb{E}\exp(\sup_{0\leq t\leq T}\|X(t)\|_0) < \infty$. It is clear that for $r > 0$,

$$\begin{aligned} P\{\sup_{0\leq t\leq T}\|X(t)\|_0^2 > r\} &= P\{\exp(\sup_{0\leq t\leq T}\|X(t)\|_0^2) > e^r\} \\ &\leq \frac{1}{e^r}\mathbb{E}\exp(\sup_{0\leq t\leq T}\|X(t)\|_0^2). \end{aligned} \tag{5.2}$$

The above is an exponential inequality, and thus the study of exponential inequality for the $X(t)$ itself can be transformed into the study of the expectation of $\exp(X(t))$. If $\mathbb{E}\exp(X(t)) < \infty$, then $X(t)$ is called exponentially integrable.

We investigate the exponential integrablilty of the function $\sup_{0\leq t\leq T}\|X(t)\|_0^2$ in this section.

In the proof of Theorem 5.2, we employ a result in [8], and we state here for the convenience of the reader.

Theorem 5.1 (Theorem 1.3 in [8]). *Assume that there exist a measurable function $k:[0,T]\to\mathbb{R}_+$ and a number $\gamma\in(0,1]$ such that*

$$\|T_{t-s}\|_{L_2(H)}\leq k(t-s),\quad 0\leq s<t\leq T, \tag{5.3}$$

and

$$\kappa:=\int_0^T s^{-\gamma}k^2(s)ds<\infty. \tag{5.4}$$

Then there exist constant $\tilde{K}<\infty$ and $\tilde{\lambda}>0$ such that

$$\mathbb{E}\exp\Big(\frac{\tilde{\lambda}}{|\psi|_\infty^2}\sup_{0\leq t\leq T}\|\int_0^T T_{t-s}\psi(s)dW_s\|^2\Big)\leq\tilde{K}$$

holds for every ψ satisfying

$$|\psi|_\infty:=\sup_{(s,\omega)\in[0,T]\times\Omega}\|\psi(s,\omega)\|_{\mathcal{L}(H_0,H)}<\infty. \tag{5.5}$$

Theorem 5.2. *Assume that*

(i) there is a $\gamma\in(0,1]$ such that $\sum_{k=1}^\infty\lambda_k^{\gamma-1}<\infty$, and
(ii) $\sup_{(s,\omega)\in[0,T]\times\Omega}\|\Sigma(s,\omega)\|_{\mathcal{L}(H_0,H)}<\infty$.

Then, for any $r>0$, the solution $X(t)$ to (5.1) satisfies (5.2).

Proof. Set $k(s):=\sum_{i=1}^\infty e^{-2\lambda_i s}$, and $\psi(s,\omega):=\Sigma(s,\omega)$. It is not hard to see that (5.5) is satisfied automatically.

Let us verify (5.4). For $\gamma\in(0,1]$, consider the integral

$$\begin{aligned}\int_0^T s^{-\gamma}k^2(s)ds&=\int_0^T s^{-\gamma}(\sum_{i=1}^\infty e^{-2\lambda_i s})^2ds\\&\leq\sum_{i=1}^\infty\int_0^T s^{-\gamma}e^{4\lambda_i s}ds.\end{aligned}$$

Taking $t=4\lambda_i s$, we have

$$\begin{aligned}\int_0^T s^{-\gamma}k^2(s)ds&\leq\sum_{i=1}^\infty\int_0^T s^{-\gamma}e^{4\lambda_i s}ds\\&\leq\sum_{i=1}^\infty(4\lambda_i)^{\gamma-1}\int_0^\infty t^\gamma e^{-t}dt\\&=\sum_{i=1}^\infty(4\lambda_i)^{\gamma-1}\Gamma(\gamma+1)\end{aligned}$$

is finite by assumption and $\Gamma(\gamma+1)$ is finite for $\gamma\in(0,1]$. Moreover,

$$\|T_{t-s}\|_{L_2(H)}^2=\sum_{i=1}^\infty\|T_{t-s}\phi_i\|^2\leq\sum_{i=1}^\infty e^{-2\lambda_i(t-s)}\|\phi_i\|^2=\sum_{i=1}^\infty e^{-2\lambda_i(t-s)}=k(t-s).$$

Thus, (5.3) and (5.4) are satisfied. Hence, the theorem implies

$$\mathbb{E}\exp\Big(\frac{\tilde{\lambda}}{|\Sigma|_\infty^2}\sup_{0\le t\le T}\|\int_0^t T_{t-s}\Sigma(s,X_s)dW_s\|_0^2\Big)\le \tilde{K}. \tag{5.6}$$

In order to estimate the third term in (5.1), consider

$$\begin{aligned}\Big\|\int_0^t T_{t-s}G(s,X_s)ds\Big\|_0^2 &\le \int_0^t \|T_{t-s}G(s,X_s)\|_0^2 ds\\ &= \int_0^t \sum_{k=1}^\infty (T_{t-s}G(s,X_s),e_k)_0^2 ds\\ &= \int_0^t \sum_{k=1}^\infty e^{-2\lambda_k(t-s)}(G(s,X_s),e_k)_0^2 ds\\ &\le \int_0^t \sum_{k=1}^\infty e^{-2\lambda_k(t-s)}a_k^2(1+\|X_s\|_0^2)ds\end{aligned}$$

Since $X_s\in C_0([0,T];H_0)$, $\sup_s\|X_s\|_0^2\le C$. Therefore,

$$\begin{aligned}\Big\|\int_0^t T_{t-s}G(s,X_s)ds\Big\|_0^2 &\le \int_0^t \sum_{k=}^\infty e^{-2\lambda_k(t-s)}a_k^2(1+\|X_s\|_0^2)ds\\ &\le C\int_0^t \sum_{k=1}^\infty e^{-2\lambda_k(t-s)}a_k^2 ds &(5.7)\\ &= C\sum_{k=1}^\infty a_k^2\frac{1}{\lambda_k}(e^{-2\lambda_k(0)}-e^{-2\lambda_k t}) &(5.8)\end{aligned}$$

is finite. In view of (5.6) and (5.8), one infers from (5.1) that

$$\begin{aligned}\|X_t\|_0^2 &\le 3\|T_t x\|_0^2+3\Big\|\int_0^t T_{t-s}G(s,X_s)ds\Big\|_0^2+3\Big\|\int_0^f T_{t-s}\Sigma(s,X_s)dW_s\Big\|_0^2\\ &\le C_1+C_2+3\sup_{0\le t\le T}\Big\|\int_0^t T_{t-s}\Sigma(s,X_s)dW_s\Big\|_0^2.\end{aligned}$$

It follows from the convexity of exponential function that

$$\exp(\lambda\sup_t\|X_t\|_0^2)\le\frac{1}{2}e^{2\lambda(C_1+C_2)}+\frac{1}{2}\exp\Big(6\lambda\sup_{0\le t\le T}\Big\|\int_0^t T_{t-s}G(s,X_s)dW_s\Big\|_0^2\Big).$$

Taking expectation, one has

$$\begin{aligned}&\mathbb{E}\exp(\lambda\sup_{0\le t\le T}\|X_t\|_0^2)\\ &\le\frac{1}{2}e^{2\lambda(C_1+C_2)}+\frac{1}{2}\mathbb{E}\exp\left(6\lambda\sup_{0\le t\le T}\Big\|\int_0^t T_{t-s}G(s,X_s)dW_s\Big\|_0^2)\right)\le C,\end{aligned}$$

where C denotes a generic constant. Also, the above inequality implies that $\sup_{0\le t\le T}\|X_t\|_0^2$ is exponentially integrable, which completes the proof. □

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