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EFFECT OF TIME DISCOUNTING AND COST OF RAW MATERIAL ON THE ECONOMIC PRODUCTION MODEL

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ABSTRACT

The classical economic production model does not take into account the effects of time discounting and the costs due to raw material. An inventory model that accounts for the time value of money and studies the effect of the production costs as well as the costs of acquiring and holding the raw material and the finished product is presented. The annual profit function is developed. The optimal production quantity is obtained by maximizing the profit function. A numerical example is given to illustrate the determination of the optimal solution.

1. INTRODUCTION

Production enterprises are faced with the problem of determining the optimal duration of the production run, and consequently the optimal production quantity, and the related problem of controlling and maintaining inventories of physical goods which they produce or which they will need for their production. Most traditional approaches to the these problems consider the different costs involving the finished product without taking into account the costs of raw material and time value of money. These are major factors which affect the length of the optimal duration of the production run and the optimal production quantity. Therefore, ignoring the ordering and holding costs of raw material and the time value of money is not an accurate approach.

Considerable research, with many different assumptions related to time value of money, has already been done on the economic order quantity (EOQ) inventory models and the economic production quantity (EPQ) inventory models. (Bierman and Thomas, 1977), (Misra, 1979), and (Mangiameli *et al.*, 1981), reported on time discounted infinite EOQ models investigated under inflationary trends and cost increases. (Goyal, 1985) considered the EOQ model under condition of permissible delay in payments. (Chung, 1997) presented a procedure to determine the optimal time interval of permissible delay in payment model. (Salameh *et al.*, 1999) investigated the effect of time discounting on the instantaneous replenishment model. (Salameh and El-Kassar, 1999), studied the optimality of the single period inventory model with credit facility. (Salameh *et al.*, 2003), considered the continuous review inventory model with delay in payments. (Salameh and El-Kassar, 2003) examined the effect of time discounting on the EPQ model. (Salameh and El-Kassar, 2007) introduced an EPQ model that accounts for the cost of raw material.

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In this paper, we develop a finite production inventory model that incorporates time value of money and the costs that occur at the different stages of a production process. This model extends the results obtained in both (Salameh and El-Kassar, 2003) and (Salameh and El-Kassar, 2007). In this model, the raw material is purchased and stored for use in the production process, thus incurring ordering and holding costs. The items are produced in lots at the factory, and are sent directly to the warehouse. Part of the produced items is sold and the other part is kept in inventory. Hence, an increase in the inventory level of the finished product occurs and a holding cost is incurred. Using continuous discounting, the present worth of all future cash flows of sales return and of the various costs is calculated in order to develop the profit function. The optimal production quantity is determined by maximizing the profit function.

The different costs that are incurred in this inventory model cover:

- The ordering cost of the raw material.
- The holding cost of raw material which represents the costs for storage, handling, insurance, etc.
- The setup cost which represents the fixed charges of machine preparation, inspection, maintenance, and supervision that are incurred at the beginning of each production run.
- Production cost that includes the cost of operating supervisors and labourers, plus the cost of inspection and rework of the defected items.
- Holding cost which covers insurance, taxes, rent, maintenance, breakage and pilferage at the storage site plus cost of warehouse operation.
- The shortage cost does not affect this model.

The remainder of this paper is organized as follows. In section 2, we formulate the mathematical model. A numerical example illustrating the model is given in section 3. Concluding remarks and future work are given in section 4.

2. MODEL FORMULATION

The following terminology will be used herein:

α annual production rate in units produced
β annual demand rate in units demanded
y production quantity in units per cycle
K_0 ordering cost per order of raw material
K_p setup cost for one production cycle
K sum of K_0 and K_p
C_0 unit cost of raw material
C_p unit production cost
C sum of C_p and C_o
s unit selling price

- I inventory carrying cost rate
- h_o raw material holding cost per unit per year, $h_o = I\, C_0$
- h_p holding cost due to production per unit per year, $h_p = I\, C_p$
- y_m maximum on-hand inventory of the finished product per cycle
- t_0 cycle length
- t_1 production period
- t_2 zero production period
- i real continuous interest rate
- P production cost over the cycle
- H_1 inventory holding cost due to raw material over the inventory cycle
- H_2 inventory holding cost due to production over the production period
- H_3 inventory holding cost due to production over the zero production period
- S sales returns of one cycle
- G profit function

Assumptions:

(1) Demand rate, β, is known and constant.

(2) Production rate, α, is known and constant.

(3) Shortages are not allowed.

(4) Time period is infinite.

(5) $\alpha > \beta$.

(6) Sales revenues are invested and interest is earned.

(7) Continuous discounting at interest rate i is used.

In this model we consider the case where an item is produced and each unit of the finished product requires one unit of raw material. The raw material can be acquired from a supplier and processed into a finished product at a production rate a. At the start of the production cycle, a complete order of size y of raw material is received in one batch and is used to produce y units of the finished product. The raw material is stored and is processed at a rate a until it is depleted at the end of the production period. Therefore, the length of the production cycle is $t_1 = y/\alpha$. The inventory level for the raw material is shown in figure 1a. During the production period, the finished product is produced at the rate a and is sold at the demand rate β. The excess amount of the finished product is accumulated and stored at a rate of α-β. At the end of the production period a maximum inventory level of $y_m = y(\alpha-\beta)/\alpha$ of the finished product is be accumulated. During the zero production period, the finished product is depleted at a rate β so that the length of the zero production period is $t_1 = y(\alpha-\beta)/(\alpha\beta)$ and the total inventory cycle length $t_0 = t_1 + t_2 = y/\beta$. The inventory level for the finished product is shown in figure 1b and the total inventory level, units of raw material and units of finished product, is shown in figure 1c.

During a typical inventory cycle, the cost components involve the ordering, purchasing and holding costs of raw material, as well as the setup, production and holding costs of the finished product. The holding cost due to production is incurred only on finished product on-hand, while the raw material holding cost is incurred for both raw material and finished product on-hand. Therefore, the holding cost per cycle due to production is the average inventory on-hand of the finished product times the inventory cycle length times the holding cost due to production per unit per unit time. The holding cost per cycle incurred on raw material is the average inventory on-hand of raw material plus finished product multiplied by the holding cost due to raw material per unit per unit time multiplied by the inventory cycle length.

If we drop the time discounting assumption, we obtain the result of (Salameh and El-Kassar, 2007) as follows. The total relevant cost per cycle consists of the following components:

(1) Ordering cost of raw material = K_0

(2) Purchasing cost of raw material = $C_0 y$

(3) Holding cost of raw material = $t_0\, h_0 y/2$

(4) Setup cost of the production cycle = K_p

(5) Production cost = $C_p y$

(6) Holding cost of the finished product due to production = $t_0\, h_p\, (1\text{-}\beta/\alpha)\, y\,/2$

The total inventory cost per cycle is

$$TC(y) = K_o + C_0 y + t_0 \frac{h_0}{2} y + K_p + C_p y + t_0 \frac{h_p}{2}(1-\beta/\alpha)y\,. \tag{2.1}$$

Since $K = K_0 + Kp$, $C = C_0 + Cp$, $h_0 = I\, C_0$, and $h_p = IC_p$, we have that

$$TC(y) = K + Cy + t_0 \frac{h_0}{2} y + t_0 \frac{h_p}{2}(1-\beta/\alpha)y\,. \tag{2.2}$$

The total inventory cost per cycle function, *TCU*(y), is obtained by dividing (2.2) by the cycle length t_0. Hence,

$$TCU(y) = \frac{K\beta}{y} + C\beta + \frac{h_0}{2} y + \frac{h_p}{2}\left(1-\frac{\beta}{\alpha}\right) y\,. \tag{2.3}$$

Differentiating *TCU*(*y*) given in (2.3), setting the derivative equal to zero, and solving for *y*, we have that the economic order quantity y^* is

$$y^* = \sqrt{\frac{2K\alpha\beta}{(\alpha-\beta)h_p + \alpha h_0}}\,. \tag{2.4}$$

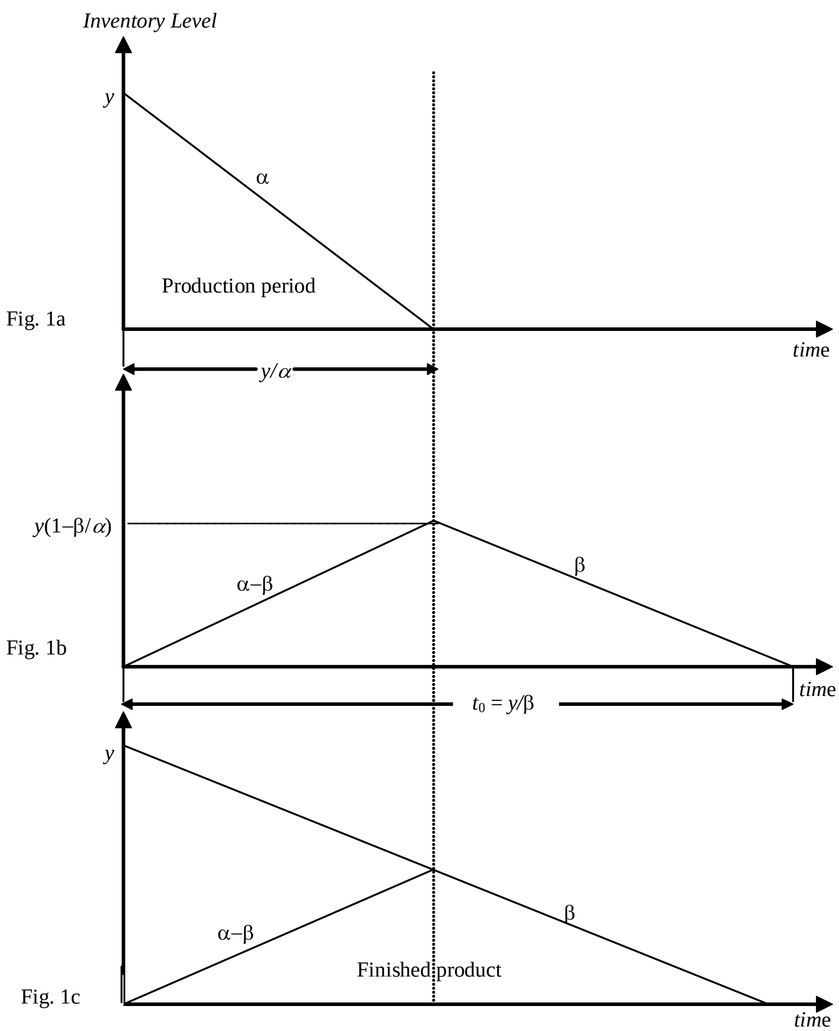
We note that if there is no holding cost incurred on raw material, then the model reduces to the classical EPQ inventory model where (2.4) becomes

$$y^* = \sqrt{\frac{2K\beta}{h_p(1-\beta/\alpha)}}. \tag{2.5}$$

In addition, if the production rate tends to infinity, the model reduces to the classical EOQ model where (2.5) becomes

$$y^* = \sqrt{\frac{2K\beta}{h_p}}. \tag{2.6}$$

Figure 1a: Inventory Level of Raw Material
Figure 1b: Inventory Level of Finished Product
Figure 1c: Total Inventory Level



For continuous compounding at an interest rate i, the present worth PW of a single future cash flow F occurring at time period N is given by $PW = Fe^{-iN}$. For a continuous cash flow per unit time (e.g., per year) represented by a continuous function $f(t)$, the cash flow generated by a typical differential element during an infinitely small time interval dt at any time t is $f(t)dt$. Integrating the present worth of that element, $f(t)e^{-it}dt$, we obtain that the present worth of the continuous cash flow over a period [0, N] is

$$PW = \int_0^N f(t)e^{-it}dt\,. \tag{2.7}$$

Since the interest rate is expressed in terms of a year, the time unit is year and all time units in equivalence calculations must be converted into years. For a uniform cash flow amounting to a sum of A per year, (2.7) gives that the present worth is

$$PW = A\left(\frac{e^{iN}-1}{ie^{iN}}\right) = \frac{A}{i}\left(1-e^{-iN}\right). \tag{2.8}$$

For a linear gradient cash flow with $f(t) = Gt$, the present worth is

$$PW = \frac{G}{i^2}\left(1-e^{-iN}\right) - \frac{G}{i}\left(Ne^{-iN}\right) = \frac{G}{i^2}\left(1-e^{-iN}-iNe^{-iN}\right).. \tag{2.9}$$

From (2.8), we have that a present cash flow PW can be converted to a uniform flow over a time period N by multiplying by the funds-flow capital recovery factor; that is,

$$A = PW\left(\frac{ie^{iN}}{e^{iN}-1}\right) = PW\left(\frac{i}{1-e^{-iN}}\right). \tag{2.10}$$

For the selling returns, the flow during an infinitely small time interval, dt, is $dS = s\beta\, dt$, see figure 2a. The present worth of dS is $PW(dS) = s\,\beta\, e^{-it}dt$. The selling returns flows uniformly over the entire inventory cycle, from $t = 0$ to $t = t_0 = t_1+t_2$. From (2.8), we have that the present worth of S is

$$PW(S) = s\beta\left(\frac{e^{it_0}-1}{ie^{it_0}}\right) = \frac{s\beta}{i}\left(1-e^{-it_0}\right) = \frac{s\beta}{i}\left(1-e^{-iy/\beta}\right). \tag{2.11}$$

As for the cost components, the ordering cost K_0, the setup cost K_p, and the purchasing cost of raw material $C_0 y$ are discrete cash flows occurring at the start of the inventory cycle. The production cost is a uniform continuous flow extending over the production period, from time $t = 0$ to $t = t_1$. The flow during an infinitely small time interval, dt, is $dP = C_p a\, dt$, see figure 2b. From (2.8), we have that the present worth of P is

$$PW(P) = C_p\alpha\left(\frac{e^{it_1}-1}{ie^{it_1}}\right) = \frac{C_p\alpha}{i}\left(1-e^{-it_1}\right) = \frac{C_p\alpha}{i}\left(1-e^{-iy/\alpha}\right). \tag{2.12}$$

Figure 2a: Selling Returns Flow during dt

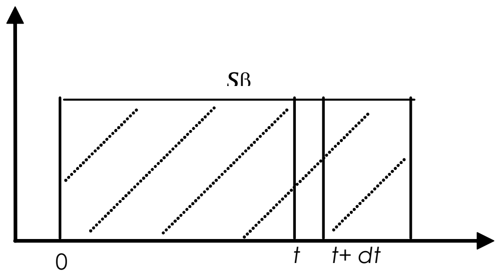
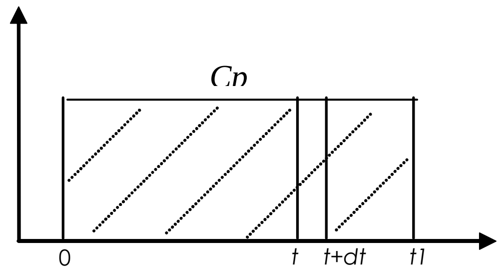


Figure 2b: Production Cost Flow during dt



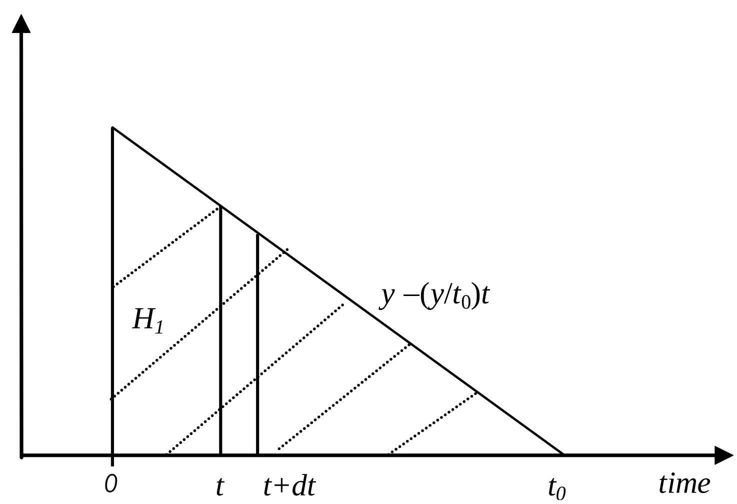
The holding cost component due to raw material is incurred to both raw material and finished product and occurs as a continuous cash flow extending over the entire inventory cycle where the total inventory decreases from a level of y at t = 0 to a level of 0 at t = *to*. Hence, this holding cost during a small time interval *dt*, see figure 3, is

$$dH_1 = h_0\left(y - \frac{y}{t_0}t\right)dt = h_0\left(y - \beta t\right)dt\ .$$

Hence, this flow can be viewed as the difference between a uniform cash flow and a linear gradient both extending over the inventory cycle. From (2.8) and (2.9), we have

$$\begin{aligned} PW(H_1) &= \frac{h_0 y}{i}\left(1 - e^{-it_0}\right) - \frac{h_0 y}{i^2 t_0}\left[1 - e^{-it_0} - it_0 e^{-it_0}\right] \\ &= \frac{h_0 y}{i^2 t_0}\left[-1 + e^{-it_0} + it_0\right] \end{aligned} \quad (2.13)$$

Figure 3: Holding Cost Due to Production during the Production Period



The holding cost component due to production is made up of the sum of two continuous cash flows, H_1 and H_2. The first extends over the production period, from $t = 0$ to $t = t_1$, where the inventory starts at a zero level and increases at the rate of α-β until it reaches a maximum level of y_m. Accordingly, the holding cost increases with the finished products in inventory. The second cash flow, H_2, extends over the zero production period, from $t = t_1$ to $t = t_0$, where the inventory starts at a maximum level of y_m and decreases at the rate of β until it reaches a zero level. The two cash flows, H_1 and H_2, are shown in figure 4.

During the time interval $[0 , t_1]$, the number of items in inventory at time t is

$$\frac{y_m}{t_1}t = (\alpha - \beta)t$$

Hence, the holding cost during a small time interval dt is

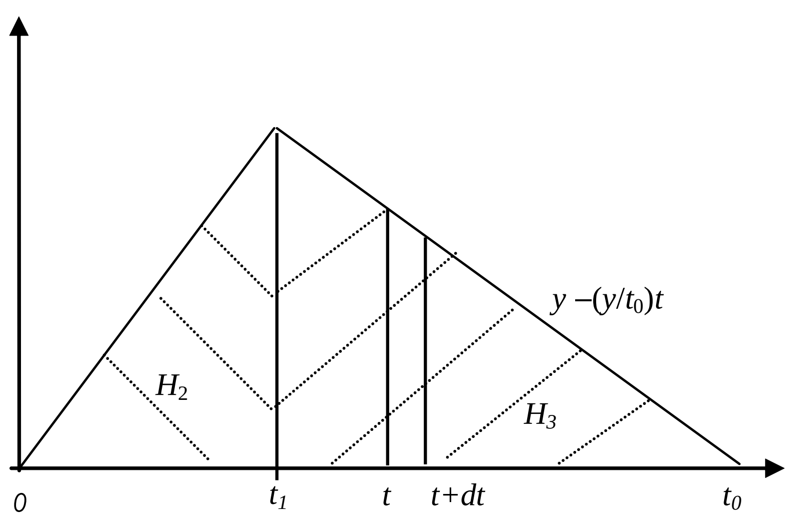
$$dH_2 = \frac{h_p y_m}{t_1} tdt = h_p(\alpha - \beta)tdt\ .$$

From (2.9), the present worth of H_2 is

$$PW(H_2) = \frac{h_p y_m}{i^2 t_1}\left(1 - e^{-it_1} - it_1 e^{-it_1}\right) = \frac{h_p(\alpha - \beta)}{i^2}\left(1 - e^{-it_1} - it_1 e^{-it_1}\right). \qquad (2.14)$$

As for the second cash flow, the number of finished product in stock at time t, $t_1 \le t \le t_0$, is $y - \frac{y}{t_0}t.$

Figure 4: Holding Cost Due to Production during the Zero Production Period



The cost during a small time interval dt is

$$dH_3 = h_p\left[y - \frac{y}{t_0}t\right]dt.$$

Hence, H_3 is the difference between a uniform continuous cash flow, of $h_p y$, and a continuous linear gradient cash flow, of $(y/t_0)t$, both extending over the zero production period of length t_2, from $t = t_1$ to $t = t_0$. Applying (2.8) to the uniform cash flow and (2.9) to the linear gradient cash flow and taking the difference, we obtain the equivalent of H_3 as a single discrete cash flow at time $t = t_1$. Therefore, the present worth of H_3 is

$$\begin{aligned} PW(H_3) &= \left[\frac{h_p y}{i}\left(1-e^{-it_2}\right) - \frac{h_p y}{i^2 t_0}\left(1-e^{-it_2} - it_2 e^{-it_2}\right)\right]e^{-it_1} \\ &= \frac{h_p y}{i^2 t_0}\left[it_0\left(e^{-it_1} - e^{-it_0}\right) - \left(e^{-it_1} - e^{-it_0} - it_2 e^{-it_0}\right)\right] \\ &= \frac{h_p y}{i^2 t_0}\left[it_0 e^{-it_1} - it_1 e^{-it_0} - e^{-it_1} + e^{-it_0}\right]. \end{aligned} \tag{2.15}$$

Adding the present worth of H_1, H_2 and H_3 of equations (2.13), (2.14) and (2.15), we obtain the present worth of the total holding cost

$$\begin{aligned} PW(H) = {} & \frac{h_0 y}{i^2 t_0}\left[-1 + e^{-it_0} + it_0\right] \\ & + \frac{h_p y_m}{i^2 t_1}\left(1 - e^{-it_1} - it_1 e^{-it_1}\right) \\ & + \frac{h_p y}{i^2 t_0}\left[it_0 e^{-it_1} - it_1 e^{-it_0} - e^{-it_1} + e^{-it_0}\right]. \end{aligned} \tag{2.16}$$

Using (2.11), (2.12) and (2.16), we obtain the present worth of the profit per inventory cycle function as

$$\begin{aligned} PW = {} & \frac{s\beta}{i}\left(1-e^{-it_0}\right) - K_0 - K_p - C_0 y - \frac{Cp\alpha}{i}\left(1-e^{-it_1}\right) - \frac{h_0 y}{i^2 t_0}\left[-1 + e^{-it_0} + it_0\right] \\ & - \frac{h_p y_m}{i^2 t_1}\left(1 - e^{-it_1} - it_1 e^{-it_1}\right) - \frac{h_p y}{i^2 t_0}\left[it_0 e^{-it_1} - it_1 e^{-it_0} - e^{-it_1} + e^{-it_0}\right]. \end{aligned}$$

Multiplying the last expression by the funds-flow capital recovery factor of (2.10), we obtain

$$G(y)=s\beta-\frac{(K+C_0y)i+Cp\alpha(1-e^{-it_1})}{\left(1-e^{-it_0}\right)}-\frac{h_0y}{it_0}\left[-1+\frac{it_0}{1-e^{-it_0}}\right]$$
$$-\frac{h_py_m}{it_1}\left(\frac{1-e^{-it_1}-it_1e^{-it_1}}{1-e^{-it_0}}\right)-\frac{h_py}{it_0}\left[\frac{it_0e^{-it_1}-it_1e^{-it_0}-e^{-it_1}+e^{-it_0}}{1-e^{-it_0}}\right].$$

This is the equivalent uniform flow of profit over the time period t_0. Substituting $t_1 = y/\alpha$, $t_0 = y/\beta$, and $y_m = [(\alpha-\beta)/\alpha)]\, y$ in the last equation, we get

$$G(y)=s\beta-\frac{(K+C_0y)i+Cp\alpha(1-e^{-iy/\alpha})}{\left(1-e^{-iy/\beta}\right)}-\frac{h_0}{i}\left[-\beta+\frac{iy}{1-e^{-iy/\beta}}\right]$$
$$-\frac{h_p(\alpha-\beta)}{i\alpha}\left(\frac{\alpha-\alpha e^{-iy/\alpha}-iye^{-iy/\alpha}}{1-e^{-iy/\beta}}\right)$$
$$-\frac{h_p}{\alpha i}\left[\frac{i\alpha ye^{-iy/\alpha}-i\beta ye^{-iy/\beta}-\alpha\beta e^{-iy/\alpha}+\alpha\beta e^{-iy/\beta}}{1-e^{-iy/\beta}}\right].$$

Simplifying the above expression, we get

$$G(y)=s\beta-\frac{(K+C_0y)i+Cp\alpha(1-e^{-iy/\alpha})}{\left(1-e^{-iy/\beta}\right)}-\frac{h_0}{i}\left[-\beta+\frac{iy}{1-e^{-iy/\beta}}\right]$$
$$-\frac{h_p}{i\alpha}\left[\frac{i\beta y\left(e^{-iy/\alpha}-e^{-iy/\beta}\right)+\alpha^2\left(1-e^{-iy/\alpha}\right)-\alpha\beta\left(1-e^{-iy/\beta}\right)}{\left(1-e^{-iy/\beta}\right)}\right]. \qquad (2.17)$$

This is the equivalent of the annual profit function. The economic production quantity y^* is obtained by maximizing equation (2.17). This can done using any one-dimensional search method or by using a computer software package. The uniqueness of the optimal is not demonstrated mathematically. However, all numerical experiments revealed a unique optimal solution.

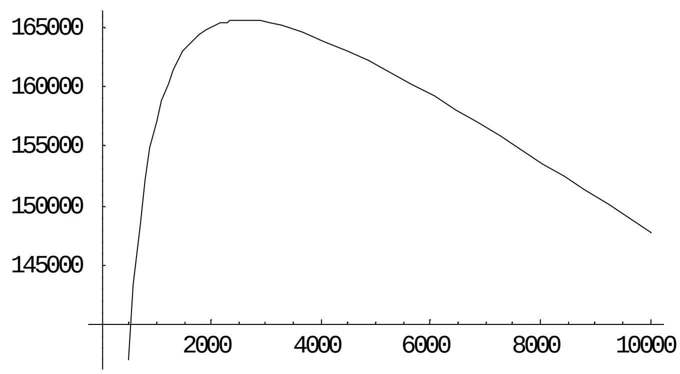
3. NUMERICAL EXAMPLE

Consider the situation where the daily demand for an item is 100 and the daily production rate is 300. The ordering cost is $500 and the setup cost is $100. The purchasing cost per item is $10, the production cost per item is $5 and the carrying cost rate is 20% per year. Then, α = 109500 items per year, β = 36500 items per year, $C_0 = 10$, $C_p = 5$, $I = 0.2$, $h_0 = 10(0.2) = \$2$ per item per year, and $h_p = 5(0.2) = \$1$ per item per year, $K_0 = \$500$, $K_p = \$100$. Then,

$$G(y)=\frac{1}{-1+e^{0.00000685y}}\left(-1168000+985500\,e^{-.0000046y}+182350\,e^{0.00000685y}\right.$$
$$\left.+(y/3)(1-e^{0.00000454y})-4.5e^{0.0000068y}\right)$$

The optimal quantity is $y^* = 2600$. The corresponding optimal inventory cycle length is $t_0^* = 26$ days, the production cycle is $t_1^* = 8.67$ days and the maximum annual profit is $G(y^*) =$ \$ 165,606. The plot of the $G(y)$ is shown in figure 4.

Figure 5: The Annual Profit Function



4. CONCLUSION

The economic production quantity model presented in this paper incorporates the effect of time discounting and the costs that occur at the different stages of a production process, including the cost due to raw material. These factors are usually ignored in the traditional EPQ models. This model extends two previously developed EPQ models. One of the two models deals only with the effect of time discounting on the classical EPQ model, and the other accounts for the cost of raw material in the EPQ model without considering the time value of money. The mathematical model was derived and an expression for the profit function was obtained. The optimal production quantity can be obtained by maximizing the profit function. Numerical example was given to illustrate the model. For future work, we suggest investigating the uniqueness of the optimal solution. In other direction, we suggest incorporating credit facility for this model.

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