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SOME PROPERTIES OF $\mathcal{B}$-MULTIPLICATIVE FUNCTIONS OF ONE VARIABLE AND TWO VARIABLES

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Abstract: By a basic sequence B we mean a set of pairs (a, b) of positive integers with the properties

1. $(a, b) \in \mathcal{B} \Leftrightarrow (b, a) \in \mathcal{B}$
2. $(a, bc) \in \mathcal{B} \Leftrightarrow (a, b) \in \mathcal{B}$ and $(a, c) \in \mathcal{B}$
3. $(1, k) \in \mathcal{B}$ for $k = 1, 2, 3, \ldots\ldots\ldots\ldots$

In this paper we define $\mathcal{B}$ - multiplicative functions.

Definition: An arithmetical function f is said to be $\mathcal{B}$ - multiplicative if f is not identically zero and $f(m\,n) = f(m) f(n)$ for all $(m, n) \in \mathcal{B}$.

For Example,

Our $\mathcal{B}$ - multiplicative function is the generalization of multiplicative and completely multiplicative functions.

Definition : ($\mathcal{B}$ - multiplicative function of two variables)

$\mathcal{B}$ - multiplicative function in two variables is defined as

$f(mm', nn') = f(m, n) f(m', n')$ for all $(m, n) \in \mathcal{B}$ and $(m', n') \in \mathcal{B}$.

In this paper we have shown the following:

(i) If f and g are $\mathcal{B}$ - multiplicative functions, then their Dirichlet product $f * g$ is also a $\mathcal{B}$ - multiplicative function.

i.e. $(f * g)(m\,n) = (f * g)(m)(f * g)(n)$ for all $(m, n) \in \mathcal{B}$.

(ii) If f and g are $\mathcal{B}$ - multiplicative functions, then their Unitary product $f \text{ x } g$ is also a B - multiplicative function.

i.e. $(f \text{ x } g)(m\,n) = (f \text{ x } g)(m)(f \text{ x } g)(n)$.

(iii) If f and g are $\mathcal{B}$ - multiplicative functions of two variables m , n then $f o_{\mathcal{B}} g$ is also a $\mathcal{B}$ - multiplicative function of two variables.

$(f\, 0_{\mathcal{B}} g)(mm', nn') = (f\, 0_{\mathcal{B}} g)(m, n)(f\, 0_{\mathcal{B}} g)(m', n')$ for all $(m, n) \in \mathcal{B}$ and $(m', n') \in \mathcal{B}$

We also have shown some more properties of B - multiplicative functions.

Key wards and Phrases: Multiplicative Functions, Completely Multiplicative Functions, Basic Sequence, $\mathcal{B}$ - Multiplicative Functions.

1. INTRODUCTION

A real or complex valued function defined on the set of all positive integers is called an arithmetical function.

An arithmetical function f is said to be a multiplicative function if f is not identically zero and $f(m\,n) = f(m) f(n)$ when ever $(m, n) = 1$, f is said to be completely multiplicative function if $f(mn) = f(m) f(n)$ for all m, n.

1.1 Definition : A set of pairs $(a\ ,\ b)$ of positive integers is said to be a Basic Sequence $\mathcal{B}$, if

1. $(a, b) \in \mathcal{B} \Leftrightarrow (b, a) \in \mathcal{B}$
2. $(a, bc) \in \mathcal{B} \Leftrightarrow (a, b) \in \mathcal{B}$ and $(a, c) \in \mathcal{B}$
3. $(1, k) \in \mathcal{B}$, $k = 1, 2, 3, \ldots\ldots$

Examples : 1. The set $\mathcal{L}$ of all pairs of positive integers forms a basic sequence.

2. The set $\mathcal{M}$ of all pairs of relative prime positive integers forms a basic sequence.

1.2 Definition : An arithmetical function f is said to be a $\mathcal{B}$- multiplicative function if f is not identically zero and $f(m\,n) = f(m) f(n)$ for all $(m, n) \in \mathcal{B}$.

1.3 Remark : If we take $\mathcal{L}$ as the basic sequence, then our $\mathcal{B}$ - multiplicative function becomes completely multiplicative function and if we take $\mathcal{M}$ as the basic sequence, then our $\mathcal{B}$ - multiplicative function becomes multiplicative function.

Therefore our $\mathcal{B}$ - multiplicative function is the generalization of multiplicative and completely multiplicative functions.

1.4 Definition : An arithmetical function f of two variables is said to be a $\mathcal{B}$ - multiplicative function of two variables if

$$f(mm', nn') = f(m, n)\, f(m', n') \quad \text{for all } (m, n) \in \mathcal{B} \text{ and } (m', n') \in \mathcal{B}.$$

1.5 Definition : If f and g are two arithmetical functions, then Donald L. Goldsmith has defined their convolution over $\mathcal{B}$ as

$$\left(f\, 0_{\mathcal{B}}\, g\right)(n) = \sum_{\substack{d\delta = n \\ (d, \delta) \in \mathcal{B}}} f(d)\, g(\delta) \qquad \ldots. (1)$$

1.6 Definition : If $f(m, n)$ and $g(m, n)$ are two $\mathcal{B}$ - multiplicative functions of two variables m and n then their $\mathcal{B}$ - convolution, written as $(f0_{\mathcal{B}}g)$ is defined by

$$(f\,0_{\mathcal{B}}\,g)(m,n)=\sum_{\substack{d_1|m\\ \left(d_1,\frac{m}{d_1}\right)\in\mathcal{B}}}\ \sum_{\substack{d_2|n\\ \left(d_2,\frac{n}{d_2}\right)\in\mathcal{B}}} f\left(\frac{m}{d_1},\frac{n}{d_2}\right)g(d_1,d_2) \text{ for all } (m,n)\in\mathcal{B} \qquad (2)$$

1.**7 Definition :** If *f* and *g* are two arithmetical functions then their Dirichlet product denoted by $f * g$ and defined as

$$(f*g)(n)=\sum_{d|n} f(d)\,g\left(\frac{n}{d}\right) \text{ for all } n \qquad \text{.... (3)}$$

1.8 Note : If *f* and *g* are multiplicative functions, then their Dirichlet product is also a multiplicative function.

1.9 Definition : If *f* and *g* are arithmetical functions then their unitary product denoted by *f* x *g* and defined as

$$(f\times g)(n)=\sum_{d\|n} f(d)\,g\left(\frac{n}{d}\right) \text{ for all } n \qquad \text{.... (4)}$$

[Here $d\|n$ means *d* is unitary divisor on *n.* i.e. $d|n$ and $\gcd\left(d,\frac{n}{d}\right)=1$]

1.10 Note : If *f* and *g* are multiplicative functions, then their unitary product is also a multiplicative function.

2. In this section we have proved some properties of $\mathcal{B}$ - multiplicative functions of one variable.

2.1 Theorem : If *f* and *g* are $\mathcal{B}$ - multiplicative, then their Dirichlet product $f * g$ is also a $\mathcal{B}$ - multiplicative.

i.e. $(f * g)(m\,n) = (f * g)(m)\,(f * g)(n)$ for all $(m, n) \in \mathcal{B}$

Proof : Write $h = f * g$

To show *h* is $\mathcal{B}$ - multiplicative we have to prove

$h(m\,n) = h(m)\,h(n)$, for all $(m, n) \in \mathcal{B}$

If one of m and n is 1, then the proof is clear.

i.e. $h\,(m\ 1) = h\,(m)\,h\,(1)$ and $h\,(1\ n) = h\,(1)\,h\,(n)$

Now, suppose that $m > 1$ and $n > 1$

We have $(m, n) \in \mathcal{B}$

Take a divisor b of n such that $b\,b' = n$,

so $(m, b\,b') \in \mathcal{B} \Leftrightarrow (m, b) \in \mathcal{B}$ and $(m, b') \in \mathcal{B}$ or $\left(m\ ,\ \frac{n}{b}\right) \in \mathcal{B}$.

Now $(m, b) \in \mathcal{B} \Leftrightarrow (b, m) \in \mathcal{B}$

Take a divisor a of m such that $aa' = m$,

so $(b, m) \in \mathcal{B}$ or $(b, a\,a') \in \mathcal{B} \Leftrightarrow (b, a) \in \mathcal{B}$ and $(b, a') \in \mathcal{B}$

Now, $$(b, a) \in \mathcal{B} \Leftrightarrow (a, b) \in \mathcal{B}. \qquad \ldots.\ (5)$$

Also $$\left(m\ ,\ \frac{n}{b}\right) \in \mathcal{B} \Leftrightarrow \left(\frac{n}{b}\ ,\ m\right) \in \mathcal{B} \text{ or} \left(\frac{n}{b}, aa'\right) \in \mathcal{B}$$

$$\left(\frac{n}{b}\ ,\ aa'\right) \in \mathcal{B} \Leftrightarrow \left(\frac{n}{b}\ ,\ a\right) \in \mathcal{B} \text{ and} \left(\frac{n}{b}\ ,\ a'\right) \in \mathcal{B}$$

so, $$\left(\frac{n}{b}\ ,\ a'\right) \in \mathcal{B} \text{ or} \left(\frac{n}{b}\ ,\ \frac{m}{a}\right) \in \mathcal{B} \Leftrightarrow \left(\frac{m}{a}\ ,\ \frac{n}{b}\right) \in \mathcal{B} \qquad \ldots.\ (6)$$

Therefore, by definition

$$h(mn) = \sum_{ab|mn} f(ab)\ g\left(\frac{mn}{ab}\right)$$

$$= \sum_{ab|mn} f(a) f(b)\, g\left(\frac{m}{a}\right) g\left(\frac{n}{b}\right) \text{ (from 5 \& 6)}$$

$$= \sum_{a|m} f(a)\, g\left(\frac{m}{a}\right) \sum_{b|n} f(b)\ g\left(\frac{n}{b}\right)$$

$$= h\,(m)\,h\,(n)$$

Hence $h\ (m\ n) = h\ (m)\ h\ (n)$, for all $(m, n) \in \mathcal{B}$

i.e. $(f * g)\ (mn) = (f * g)\ (m)\ (f * g)\ (n)$, for all $(m, n) \in \mathcal{B}$

Therefore the theorem is proved.

2.2 Theorem : If f and g are $\mathcal{B}$ - multiplicative then their unitary product f x g is also $\mathcal{B}$ - multiplicative.

i.e. $(f \text{ x } g)\ (mn) = (f \text{ x } g)\ (m)\ (f \text{ x } g)\ (n)$, for all $(m, n) \in \mathcal{B}$

Proof : Write $u = f \text{ x } g$

Now we show that $u\ (m\ n) = u\ (m)\ u\ (n)$ for all $(m, n) \in \mathcal{B}$

Take a divisor b of n such that $b\ b' = n$ and $(b, b') = 1$

We have $(m\ , n) \in \mathcal{B}$

So $(m, b\ b') \in \mathcal{B} \Leftrightarrow (m, b) \in \mathcal{B}$ and $(m, b') \in \mathcal{B}$ also $(m, b) \Leftrightarrow (b, m) \in \mathcal{B}$

Take a divisor a of m such that $a\ a' = m$ and $(a, a') = 1$

As $(b, m) \in \mathcal{B}$ or $(b, a\ a')$ µ $\mathcal{B} \Leftrightarrow (b, a) \in \mathcal{B}$ and $(b, a') \in \mathcal{B}$

We have, $(b, a) \in \mathcal{B} \Leftrightarrow (a, b) \in \mathcal{B}$.

Also $(m, b') \in \mathcal{B} \Leftrightarrow (b', m) \in \mathcal{B}$ or $(b', a\ a') \in \mathcal{B}$

$(b', a\ a') \in \mathcal{B} \Leftrightarrow (b'\ , a) \in \mathcal{B}$ and $(b', a') \in \mathcal{B}$

$(b', a') \in \mathcal{B} \Leftrightarrow (a', b') \in \mathcal{B}$

Therefore, by definition,

$$u\ (m\ n) = \sum_{\substack{a\ b\ a'\ b' = m\ n \\ (a\ b\ ,\ a'\ b') = 1}} f(a\ b)\ g(a'\ b')$$

$$= \sum_{\substack{a\ b\ a'\ b' = m\ n \\ (a\ b\ ,\ a'\ b') = 1}} f(a)\ f(b)\ g(a')\ g(b')$$

$$= \sum_{\substack{a\ a' = m \\ (a\ ,\ a') = 1 \\ b\ b' = n \\ (b\ ,\ b') = 1}} f(a)\ f(b)\ g(a')\ g(b')$$

$$= \sum_{\substack{a\,a' = m \\ (a\,,\,a') = 1}} f(a)\, g(a') \sum_{\substack{b\,b' = n \\ (b\,,\,b')=1}} f(b)\, g(b')$$

$$= u\,(m)\; u\,(n)$$

Hence $u\,(m\,n) = u\,(m)\,u\,(n)$ for all $(m, n) \in \mathcal{B}$

i.e. $(f \times g)\,(m\,n) = (f \times g)\,(m)\,(f \times g)\,(n)$ for all $(m, n) \in \mathcal{B}$

which gives $f \times g$ is a $\mathcal{B}$ - multiplicative.

2.3 Theorem : If f and g are $\mathcal{B}$ - multiplicative then $f\,0_{\mathcal{B}}g$ is also a $\mathcal{B}$ - multiplicative.

i.e. $(f\,0_{\mathcal{B}}\,g)\,(m\,n) = (f\,0_{\mathcal{B}}\,g)\,(m)\,(f\,0_{\mathcal{B}}\,g)\,(n)$, for all $(m\,,n) \in \mathcal{B}$

Proof : The proof of this result follows at once from the theorems **2.1** and **2.2.**

2.4 Theorem : Suppose that f is a $\mathcal{B}$ - multiplicative function and $f\,(k) \neq 0$. If g is defined by,

$$g(n) = \frac{f(k\,n)}{f(k)} \text{ for all } (k, n) \in \mathcal{B},$$

then g is a $\mathcal{B}$ - multiplicative function.

Proof : To prove that g is $\mathcal{B}$ - multiplicative function, we have to show that

$$g\,(m\,n) = g\,(m)\,g\,(n) \text{ for all } (m, n) \in \mathcal{B}$$

By definition of g,

$$g\,(m\,n) = \frac{f(k\,m\,n)}{f(k)}$$

If $(k, n) \in \mathcal{B}$, we have $f\,(k\,n) = f\,(k)\,f\,(n)$

Now $g\,(m\,n) = \dfrac{f\,(k)\,f\,(m)}{f\,(k)}$ [since $m\,n$ is a positive integer and $(k, m\,n) \in \mathcal{B}$]

$$= \frac{f\,(k)\,f\,(m)\,f\,(n)}{f\,(k)} \text{ [since } f \text{ is } \mathcal{B} \text{ – multiplicative]}$$

$$= \frac{f(k)\, f(m)\, f(n)}{f(k)} \cdot \frac{f(k)}{f(k)} \quad [\text{since } f(k) \neq 0]$$

$$= \frac{f(k)\, f(m)\, f(k)\, f(n)}{f(k)\, f(k)}$$

$$= \frac{f(k)\, f(m)}{f(k)} \;\; \frac{f(k)\, f(n)}{f(k)}$$

$$= \frac{f(k\; m)}{f(k)} \;\; \frac{f(k\; n)}{f(k)}$$

$$= g(m)\, g(n)$$

Hence $g\,(m\,n) = g\,(m)\, g\,(n)$

Thus g is a B - multiplicative.

3. In this section we have proved some properties of B -multiplicative functions of two variables.

3.1 Theorem : If $f\,(m, n)$ and $g\,(m, n)$ are two $\mathcal{B}$ - multiplicative functions of m, n then $f\, 0_{\mathcal{B}}\, g$ is also $\mathcal{B}$ - multiplicative function of two variables.

Proof : To prove $f\, 0_{\mathcal{B}}\, g$ is a $\mathcal{B}$ -multiplicative function we have to show that

$(f\, 0_{\mathcal{B}}\, g)\,(m\, m', n\, n') = (f\, 0_{\mathcal{B}}\, g)\,(m, n)\,(f\, 0_{\mathcal{B}}\, g)\,(m', n')$ for all $(m, n) \in \mathcal{B}$ and $(m', n') \in \mathcal{B}$

Given that f and g are $\mathcal{B}$ -multiplicative functions of m and n.

i.e. $\quad f\,(m\, m', n\, n') = f\,(m, n)\, f\,(m', n')$

and $\quad g\,(m\, m', n\, n') = g\,(m, n)\, g\,(m', n')$

for all $(m, n) \in \mathcal{B}$ and $(m', n') \in \mathcal{B}$

Consider

$$(f\, 0_B\, g)\,(m\, m', n\, n') = \sum_{\substack{d_1 d_1' | mm' \\ (d_1 d_1',\, \frac{mm'}{d_1 d_1'}) \in \mathcal{B} \\ d_1 | m\,,\; (d_1\,,\, \frac{m}{d_1}) \in \mathcal{B} \\ d_1' | m',\; (d_1',\, \frac{m'}{d_1'}) \in \mathcal{B}}} \;\; \sum_{\substack{d_2 d_2' | nn' \\ (d_2 d_2',\, \frac{nn'}{d_2 d_2'}) \in \mathcal{B} \\ d_2 | n\,,\; (d_2,\, \frac{n}{d_2}) \in \mathcal{B} \\ d_2' | n',\; (d_2',\, \frac{n'}{d_2'}) \in \mathcal{B}}} f\left(\frac{mm'}{d_1 d_1'}\,,\, \frac{nn'}{d_2 d_2'}\right) g(d_1 d_1'\,,\; d_2 d_2'\,)$$

$$= \sum_{\substack{d_2|n\ ,\ (d_2,\ \frac{n}{d_2}) \in \mathcal{B} \\ d_2^{'}|n^{'},\ (d_2^{'},\ \frac{n^{'}}{d_2^{'}}) \in \mathcal{B}}} f\left(\frac{m}{d_1}\ ,\ \frac{n}{d_2}\right) f\left(\frac{m^{'}}{d_1^{'}}\ ,\ \frac{n^{'}}{d_2^{'}}\right) g(d_1\ ,\ d_2)\ g(d_1^{'}\ ,\ d_2^{'})$$

[since f and g are B -multiplicative functions]

$$= \sum_{\substack{d_1|m \\ (d_1\ ,\ \frac{m}{d_1}) \in \mathcal{B}}} \sum_{\substack{d_2|n \\ (d_2\ ,\ \frac{n}{d_2}) \in \mathcal{B}}} f\left(\frac{m}{d_1}\ ,\ \frac{n}{d_2}\right) g(d_1\ ,\ d_2)$$

$$= \sum_{\substack{d_1^{'}|m^{'} \\ (d_1^{'},\ \frac{m^{'}}{d_1^{'}}) \in \mathcal{B}}} \sum_{\substack{d_2^{'}|n^{'} \\ (d_2^{'},\ \frac{n^{'}}{d_2^{'}}) \in \mathcal{B}}} f\left(\frac{m^{'}}{d_1^{'}}\ ,\ \frac{n^{'}}{d_2^{'}}\right) g\,(d_1^{'}\ ,\ d_2^{'})$$

$= (f\ 0_{\mathcal{B}}\, g)\ (m\ ,n)\ (f\ 0_{\mathcal{B}}\, g)\ (m^{'},\ n^{'})$ [by definition]

Therefore

$$(f\ 0_{\mathcal{B}}\, g)\ (m\ m^{'},\ n\ n^{'}) = (f\ 0_{\mathcal{B}}\, g)\ (m,\ n)\ (f\ 0_{\mathcal{B}}\, g)\ (m^{'},\ n^{'})$$

Hence $f\ 0_{\mathcal{B}}\, g$ is $\mathcal{B}$ -multiplicative in $m,\ n$ and the proof of the theorem is completed.

3.2 Theorem : If $f\ (m,\ n) = g\ (m)\ h\ (n)$ where g and h are $\mathcal{B}$ -multiplicative functions of m and n respectively, then $f\ (m,\ n)$ is $\mathcal{B}$ -multiplicative function of m and n.

Proof : To prove $f\ (m,\ n)$ is $\mathcal{B}$ -multiplicative in two variables m and n,

we have to show that

$$f\ (m\ m^{'},\ n\ n^{'}) = f\ (m,\ n)\ f\ (m^{'},\ n^{'}) \text{ for all } (m,\ n) \in \mathcal{B} \text{ and } (m^{'},\ n^{'}) \in \mathcal{B}$$

In the hypothesis, given that g and h are $\mathcal{B}$ -multiplicative functions of m and n respectively

i.e $g\ (m\ m^{'}) = g\ (m)\ g\ (m^{'})$ for all $(m,\ m^{'}) \in \mathcal{B}$

and

$h\ (n\ n^{'}) = h\ (n)\ h\ (n^{'})$ for all $(n,\ n^{'}) \in \mathcal{B}$

Consider

$$f(m\,m', n\,n') = g(m\,m')\,h(n\,n') \text{ [by the hypothesis]}$$
$$= g(m)\,g(m')\,h(n)\,h(n')$$

Since g and h are $\mathcal{B}$ -multiplicative functions

$$= g(m)\,h(n)\,g(m')\,h(n')$$
$$= f(m, n)\,f(m', n') \text{ [by the hypothesis]}$$

Therefore $f(m\,m', n\,n') = f(m, n)\,f(m', n')$

Thus f is $\mathcal{B}$ -multiplicative function of two variables m and n.

Hence completes the proof of the theorem.

3.3 Remark : $f^{-1}(m, n) = g^{-1}(m)\,h^{-1}(n)$

4. In this section we prove the following theorems.

4.1 Theorem : A $\mathcal{B}$ - multiplicative function $f(m, n)$ can be written as

$$f(m, n) = \sum_{\substack{d_1|m \\ (d_1,\, \frac{m}{d_1}) \in \mathcal{B}}} \sum_{\substack{d_2|n \\ (d_2,\, \frac{n}{d_2}) \in \mathcal{B}}} f\left(\frac{m}{d_1}, 1\right) f\left(1, \frac{n}{d_2}\right) u(d_1, d_2) \quad \ldots. (7)$$

where $u(m, n)$ is given by

$$u(m, n) = \sum_{\substack{d_1|m \\ (d_1,\, \frac{m}{d_1}) \in \mathcal{B}}} \sum_{\substack{d_2|n \\ (d_2,\, \frac{n}{d_2}) \in \mathcal{B}}} f^{-1}\left(\frac{m}{d_1}, 1\right) f^{-1}\left(1, \frac{n}{d_2}\right) f(d_1, d_2) \quad \ldots. (8)$$

Proof : By the **Theorem 3.2,** we have $f(m, n)$ is $\mathcal{B}$ - multiplicative in m, n. Because writing the function

$$f(m, n) = f(m, 1)\,f(1, n) \text{ where } (m, 1) \in \mathcal{B} \text{ and } (1, n) \in \mathcal{B},$$

this gives $f(m, 1)$ and $f(1, n)$ are $\mathcal{B}$ - multiplicative functions.

Also we have

$$f^{-1}(m, n) = f^{-1}(m, 1)\,f^{-1}(1, n)$$

Therefore by the above observations the result follows.

4.2 Remark : By definition we have

$f(m, n) = f(m, 1)\, f(1, n)$

since $(1, n) \in \mathcal{B}$, we have $f(1, n)$ is $\mathcal{B}$ - multiplicative function.

since $(m, 1) \in \mathcal{B}$, we have $f(m, 1)$ is $\mathcal{B}$ - multiplicative function.

so $f(m, n)$ is $\mathcal{B}$ - multiplicative in m, n for all $(m, n) \in \mathcal{B}$.

Also we have $f^{-1}(m, n) = f^{-1}(m, 1)\, f^{-1}(1, n)$

Applying the same argument as above we have $f^{-1}(m, n)$ is also $\mathcal{B}$ - multiplicative in m, n for all $(m, n) \in \mathcal{B}$.

4.3 Theorem : Suppose f is a $\mathcal{B}$ - multiplicative of a single variable.

Define $f(m, n) = f(m\, n)$ for all $(m, n) \in \mathcal{B}$

then $f(m\, n)$ can be defined as

$$f(m\,n) = \sum_{\substack{d_1|m \\ \left(d_1,\, \frac{m}{d_1}\right) \in \mathcal{B}}} \; \sum_{\substack{d_2|n \\ \left(d_2,\, \frac{n}{d_2}\right) \in \mathcal{B}}} f\left(\frac{m}{d_1}\right) f\left(\frac{n}{d_2}\right) f^{-1}(d_1, d_2)\, \lambda(d_1, d_2) \qquad \text{.... (9)}$$

$$\text{with} \quad \lambda(m, n) = \begin{cases} (-1)^r, & \text{if } \omega(m) = \omega(n) = r \\ 0, & \text{otherwise} \end{cases}$$

for all $(m, n) \in \mathcal{B}$.

(Here $\omega(m)$ is the number of distinct prime factors of m).

Proof : Given that $f(m\, n)$ is $\mathcal{B}$ - multiplicative function.

By Theorem **4.1,** it suffices to prove that $u(m, n)$ is $\mathcal{B}$ - multiplicative function which is defined in the Theorem **4.1.**

Here $u(m, n)$ can be written as

$$u(m, n) = f^{-1}(m\, n)\; \lambda(m, n) \text{ for all } (m, n) \in \text{B} \;\ldots. \qquad (10)$$

$$= \begin{cases} (-1)^r f^{-1}(mn) \ , & \text{if } \omega(m) = \omega(n) = r \\ 0 & , \text{ otherwise} \end{cases}$$

Again by the Theorem **4.1,** $u\ (m,\ n)$ defined by (10) is ℬ-multiplicative.

There fore by all the above observations it suffices to prove (10) in case of taking prime powers of m and n writing m and n in canonical form.

i.e ., it suffices to prove

$$u\ (p^i,\ p^j) = (-1)^r f^{-1}(p^{i+j}) \qquad \text{.... (11)}$$

where i , j are indices and for all $(p^i,\ p^j) \in$ ℬ.

This can be proved by using induction on i and j.

The proof follows in the same lines as in the case of single variable.

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