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Qualitative Properties of Discrete Version of Generalized Kneser's and Arzela-Ascoli's Theorems

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Abstract : In this paper, the authors extend and discuss the oscillatory behavior of the discrete version of Kneser's theorem and Arzela-Ascoli's theorem according to the generalized difference operator.
Keywords : Decreasing function, Generalized difference equation, Increasing function, Oscillation.

1. INTRODUCTION

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [1].

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k), k \in \mathbb{N} = \{0,1,2,\ldots\}.$$

Even though many authors [1, 8, 9] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), k \in [0,\infty), \ell \in (0,\infty), \tag{1}$$

and no significant progress took place on this line. But in [3], took up the definition of Δ as given in (1) and developed the theory of difference equations in a different direction. For convenience, they labeled the operator Δ defined by (1) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, the formulae for sum of higher powers of arithmetic progressions, sum of consecutive terms of arithmetic progressions and sum of arithmetic-geometric progressions using the Stirling numbers of first kind and second kind respectively in the field of Numerical methods were obtained. By extending theory of Δ_ℓ to complex function, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were established for the solutions of difference equations involving Δ_ℓ. Also, a method to find a formula for sum of n^{th} power of arithmetico-geometric progression using the generalized Bernoulli polynomial $B_{n+1}(k, -\ell)$, and solution to the generalized difference equation. The results obtained can be found in [3-7].

Hence, in this paper, we derive the oscillatory behavior of the discrete version of generalized Kneser's theorem and Arzela-Ascolis theorems.

Throughout this paper we make use of the following assumptions :

Consider the partial difference equation

$$\Delta^h_{m(\ell)}\Delta^r_{k(\ell)}(u(m,k) - cu(m-n, k-s)) + f(m,k,u(m-\tau\ell, k-\sigma\ell)) = 0, m,k \in K_0,$$

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where $c \neq 0$ is a real constant, $h, r \in \mathbb{N}$, $n, s \in \mathrm{K}_1$, $\tau, \sigma \in \mathrm{K}$, Δ_ℓ is the generalized forward difference operator defined by

$$\Delta_{\ell(m)} u(m,k) = u(m+\ell,k) - u(m,k),$$

$$\Delta_{\ell(k)} u(m,k) = u(m,k+\ell) - u(m,k)$$

and

$$\Delta^h_{\ell(m)} u(m,k) = \Delta_{\ell(m)}(\Delta^{h-1}_{\ell(m)} u(m,k)),\ \Delta^0_{\ell(m)} u(m,k) = u(m,k),$$

$$\Delta^r_{\ell(k)} u(m,k) = \Delta_{\ell(k)}(\Delta^{r-1}_{\ell(k)} u(m,k)),\ \Delta^0_{k(\ell)} u(m,k) = u(m,k),\ f \in \mathrm{C}(\mathrm{K}_0 \times \mathrm{K}_0 \times \mathrm{R}, \mathrm{R}).$$

2. PRELIMINARIES

Definition 2.1.

[1] A solution $u(k)$ of (2) is said to be oscillatory if for every $k_1 > 0$ there exists a real $k \geq k_1$ such that $u(k)u(k+\ell) \leq 0$. Otherwise it is nonoscillatory. Equation (2) is said to be oscillatory if all its solutions are oscillatory.

Definition 2.2.

Let U be the linear space of all bounded real function $u = \{u(m,k)\}, m \geq \mathrm{M}, k \geq \mathrm{K}$ endowed with the usual norm

$$\|u\| = \sup_{m \geq \mathrm{M}, k \geq \mathrm{B}} |u(m,k)|, \tag{3}$$

then U is a Banach space.

Definition 2.3.

Let Ω be a subset of Banach space U, Ω is relatively compact if every sequence of function in Ω has a subsequence converging to an element of U. An ε – net for Ω is a set of elements of U such that each u in Ω is within a distance ε of some member of the net. A finite ε – net is an ε – net consisting of a finite number of the elements.

Definition 2.4.

A set Ω of Banach space U is uniformly Cauchy if for every $\varepsilon > 0$ there exist positive reals M_1 and K_1 such that for any $u = \{u(m, k)\}$ in Ω

$$|u(m,k) - u(m',k')| < \varepsilon, \tag{4}$$

whenever $(m,k) \in \mathrm{D}', (m',k') \in \mathrm{D}'$,

where

$$\begin{aligned} \mathrm{D}' &= \mathrm{D}'_1 \cup \mathrm{D}'_2 \cup \mathrm{D}'_3, \\ \mathrm{D}'_1 &= \{(m,k) \mid m > \mathrm{M}_1, k > \mathrm{K}_1\}, \\ \mathrm{D}'_2 &= \{(m,k) \mid \mathrm{M} \leq m \leq \mathrm{M}_1, k > \mathrm{K}_1\}, \\ \mathrm{D}'_3 &= \{(m,k) \mid m > \mathrm{M}_1, \mathrm{K} \leq k \leq \mathrm{K}_1\}. \end{aligned} \tag{5}$$

Lemma 2.5.

[2] A subset Ω of a Banach space U is relatively compact if and only if for each $\varepsilon > 0$, it has a finite ε – net.

3. MAIN RESULTS

Throughout the section, we assume that there exists a continuous function $\mathrm{F} : \mathrm{K}_0 \times \mathrm{K}_0 \times [0,\infty) \to [0,\infty)$ such that $\mathrm{F}(m, k, u)$ is nondecreasing in u and

$$|f(m,k,u)| \leq \mathrm{F}(m,k,|u|),\ (m,k,u) \in \mathrm{K}_0 \times \mathrm{K}_0 \times \mathrm{R}. \tag{6}$$

A solution of (2) is a real double function defined for all $(m,k)\in\{(m,k)\,|\,m\ge min\{\mathrm{M}-n\ell,\mathrm{M}-\tau\ell\},\ k\ge min\{\mathrm{K}-s\ell,\mathrm{K}-\sigma\ell\}\}$ and satisfying (2) for all $(m,k)\in\{(m,k)\,|\,m\ge \mathrm{M},k\ge \mathrm{K}\}$, where $\mathrm{M},\mathrm{K}\in \mathrm{K}_0$.

Lemma 3.1.

Let $m\in\mathbb{N}(1,n-1)$ and $u(k)$ be defined on $\mathbb{N}_\ell(j)$. Then,

1. $\liminf_{k\to\infty} \Delta_\ell^m u(k)>0$ implies $\lim_{k\to\infty} \Delta_\ell^i u(k)=\infty,\ i\in\mathbb{N}(0,m-1)$
2. $\limsup_{k\to\infty} \Delta_\ell^m u(k)<0$ implies $\lim_{k\to\infty} \Delta_\ell^i u(k)=-\infty,\ i\in\mathbb{N}(0,m-1)$

Proof. $\liminf_{k\to\infty} \Delta_\ell^m u(k)>0$ implies that there exists a large $k_1\in\mathbb{N}_\ell(j)$ such that $\Delta_\ell^m u(k)\ge c>0$ for all $k\in\mathbb{N}_\ell(k_1)$ and $k=k_1+n^*\ell,$ for some integer n^*. Since

$$\Delta_\ell^{m-1}u(k) = \Delta_\ell^{m-1}u(k_1)+\sum_{r=0}^{n^*-1}\Delta_\ell^m u(k_1+r\ell)$$

it follows that

$$\Delta_\ell^{m-1}u(k) = \Delta_\ell^{m-1}u(k_1)+c\left((k-k_1)-\left[\frac{k-k_1}{\ell}\right]\ell\right),$$

and hence

$$\liminf_{k\to\infty}\Delta_\ell^{m-1}u(k) = \infty.$$

The rest of the proof is by induction. The case (*ii*) can be treated similarly.

Theorem 3.2.

(Generalized Discrete Kneser's Theorem). Let $u(k)$ be defined on $\mathbb{N}_\ell(j)$, and $u(k)>0$ with $\Delta_\ell^n u(k)$ of constant sign on $\mathbb{N}_\ell(j)$, and not identically zero. Then, there exists an integer $m\in\mathbb{N}(0,n)$ with $n+m$ odd for $\Delta_\ell^n u(k)\le 0$ or $n+m$ even for $\Delta_\ell^n u(k)\ge 0$ and such that $m\le n-1$ implies $(-1)^{m+i}\Delta_\ell^i u(k)>0$ for all large $k\in\mathbb{N}_\ell(j),\ i\in\mathbb{N}(m,n-1),\ m\ge 1$ implies $\Delta_\ell^i u(k)>0$ for all $k\in\mathbb{N}_\ell(j),\ i\in\mathbb{N}(1,m-1)$.

Proof. There are two cases to consider.

Case 1. $\Delta_\ell^n u(k)\le 0$ on $\mathbb{N}_\ell(j)$, First we shall prove that $\Delta_\ell^{n-1}u(k)>0$ on $\mathbb{N}_\ell(j)$, If not, then there exists some $k_1\ge j$ in $\mathbb{N}_\ell(j)$, such that $\Delta_\ell^{n-1}u(k_1)\le 0$. Since $\Delta_\ell^{n-1}u(k)$ is decreasing and not identically constant on $\mathbb{N}_\ell(j)$, there exists $k_2\in\mathbb{N}_\ell(k_1)$ such that $\Delta_\ell^{n-1}u(k)\le\Delta_\ell^{n-1}u(k_2)<\Delta_\ell^{n-1}u(k_1)\le 0$ for all $k\in \mathrm{N}_\ell(k_2)$, But, from Lemma 3.1 we find $\lim_{k\to\infty} u(k)=-\infty$ which is a contradiction to $u(k)>0$. Thus, $\Delta_\ell^{n-1}u(k)>0$ on $\mathbb{N}_\ell(j)$, and there exists a smallest integer $m,\ m\in\mathbb{N}(0,n-1)$ with $n+m$ odd and

$$(-1)^{m+i}\Delta_\ell^i u(k) > 0 \ on\ \mathbb{N}_\ell(j), i\in\mathbb{N}(m,n-1). \tag{7}$$

Next let $m>1$ and

$$\Delta_\ell^{m-1}u(k) < 0 \text{ on } \mathbb{N}_\ell(j), \tag{8}$$

then once again form Lemma 3.1 it follows that

$$\Delta_\ell^{m-2}u(k) > 0 \text{ on } \mathbb{N}_\ell(j). \tag{9}$$

Inequalities (7)-(9) can be unified to

$$(-1)^{(m-2)+i}\Delta_\ell^i u(k) > 0 \text{ on } \mathbb{N}_\ell(j), i\in\mathbb{N}(m-2,n-1)$$

which is a contradiction to the definition of m So, (8) fails and $\Delta_\ell^{m-1}u(k)\ge 0$ on $\mathbb{N}_\ell(j)$, From (6), $\Delta_\ell^{m-1}u(k)$ is nondecreasing and hence $\lim_{k\to\infty}\Delta_\ell^{m-1}u(k)>0$. If $m>2$, we find from Lemma 3.1 that $\lim_{k\to\infty}\Delta_\ell^i u(k)=\infty$, $i\in\mathbb{N}(1,m-2)$. Thus $\Delta_\ell^i u(k)>0$ for all large $k\in\mathbb{N}_\ell(j),\ i\in\mathbb{N}(1,m-1)$.

Case 2. $\Delta_\ell^n u(k) \geq 0$ on $\mathbb{N}_\ell(j)$, Let $k_3 \in \mathbb{N}_\ell(k_2)$ be such that $\Delta_\ell^{n-1} u(k_3) \geq 0$, then since $\Delta_\ell^{n-1} u(k)$ is non decreasing and not identically constant, there exists some $k_4 \in \mathbb{N}_\ell(k_3)$ such that $\Delta_\ell^{n-1} u(k) > 0$ for all $k \in \mathbb{N}_\ell(k_4)$. Thus, $\lim_{k\to\infty} \Delta_\ell^{n-1} u(k) > 0$ and from Lemma 3.1 $\lim_{k\to\infty} \Delta_\ell^{i} u(k) = \infty$, $i \in \mathbb{N}(1, n-2)$ and so $\Delta_\ell^{i} u(k) > 0$ for all large k in $\mathbb{N}_\ell(j), i \in \mathbb{N}(1, n-1)$. This proves the theorem for $m = n$. In case $\Delta_\ell^{n-1} u(k) < 0$ for all $k \in \mathbb{N}_\ell(j)$, we find from Lemma 3.1 that $\Delta_\ell^{n-2} u(k) > 0$ for all $k \in \mathbb{N}_\ell(j)$, The rest of the proof is the same as in Case 1.

Corollary 3.3.

Let $u(k)$ be defined on $\mathbb{N}_\ell(j)$, and $u(k) > 0$ with $\Delta_\ell^n u(k) \leq 0$ on $\mathbb{N}_\ell(j)$, and not identically zero. Then, there exists a large k_1 in $\mathbb{N}_\ell(j)$, such that for all $k \in \mathbb{N}_\ell(k_1)$

$$u(k) \; > \; \frac{1}{(n-m)!}\Delta_\ell^{n-1} u\left(2^{n-m-1}k\right)\frac{k_\ell^{(n-m-1)}}{\ell^{n-m-1}}. \tag{10}$$

Proof. From Theorem 3.2 it follows that $(-1)^{n+i-1}\Delta_\ell^i u(k) > 0$ on $\mathbb{N}_\ell(j)$, $i \in \mathbb{N}(m, n-1)$, and $\Delta_\ell^i u(k) > 0$ for all large k in $\mathbb{N}_\ell(j)$, say, for all $k \geq k_1$ in $\mathbb{N}_\ell(j)$, $i \in \mathbb{N}(1, m-1)$. Using these inequalities, we obtain.

$$\begin{aligned}
-\Delta_\ell^{n-2} u(k) &= -\Delta_\ell^{n-2} u(\infty) + \sum_{r=0}^{\infty}\Delta_\ell^{n-1} u(k+r\ell) \geq \sum_{r=0}^{\left[\frac{k}{\ell}\right]}\Delta_\ell^{n-1} u(k+r\ell) \geq \Delta_\ell^{n-1} u\left(2k\right)\frac{k^{(1)}}{\ell} \\
\Delta_\ell^{n-3} u(k) &= \Delta_\ell^{n-3} u(\infty) - \sum_{r=0}^{\infty}\Delta_\ell^{n-2} u(k+r\ell) \\
&\geq \sum_{r=0}^{\left[\frac{k}{\ell}\right]}\Delta_\ell^{n-1} u(2(k+r\ell))\frac{(k+r\ell)_\ell^{(1)}}{\ell} \\
&\geq \sum_{r=0}^{\left[\frac{k}{\ell}\right]}\Delta_\ell^{n-1} u(2(k+r\ell))\frac{(k+\left(r+k\right)\ell)_\ell^{(1)}}{\ell} \\
&\geq \Delta_\ell^{n-1} u\left(2^2 k\right)\frac{k_\ell^{(2)}}{2!\ell^2} \\
&\quad \cdots \\
\Delta_\ell^{m} u(k) &\geq \Delta_\ell^{n-1} u\left(2^{n-m-1} k\right)\frac{k_\ell^{(n-m-1)}}{(n-m-1)!\ell^{(n-m-1)}}.
\end{aligned}$$

New, we get

$$\begin{aligned}
\Delta_\ell^{m-1} u(k) &= \Delta_\ell^{m-1} u(k_1) + \sum_{r=0}^{n^*-1}\Delta_\ell^{m} u(k_1+r\ell) \\
&\geq \sum_{r=0}^{n^*-1}\Delta_\ell^{n-1} u\left(2^{n-m-1}(k_1+r\ell)\right)\frac{(k_1+r\ell)_\ell^{(n-m-1)}}{(n-m-1)!\ell^{n-m-1}} \\
&\geq \frac{1}{(n-m)!}\Delta_\ell^{n-1} u\left(2^{n-m-1}k\right)\frac{k_\ell^{(n-m-1)}}{\ell^{n-m-1}}.
\end{aligned}$$

Hence, after $(m-1)$ summations, we obtain (10).

Corollary 3.4.

Let $u(k)$ be as in Corollary 3.3 and bounded. Then,

1. $\lim_{k\to\infty} \Delta_\ell^i u(k) = 0, i \in \mathbb{N}(1, m-1)$
2. $(-1)^{i+1}\Delta_\ell^{n-i} u(k) \ge 0$ for all $k \in \mathbb{N}_\ell(j),\ i \in \mathbb{N}(1, m-1)$.

Proof. Part (a) follows from Lemma 3.1. Also, for Part (b) we denote that in the conclusion of Theorem 3.2, m cannot be greater than 1.

Corollary 3.5.

Let $u(k)$ be as in Corollary 3.3. Then, exactly one of the following is true

1. $\lim_{k\to\infty} \Delta_\ell^i u(k) = 0, i \in \mathbb{N}(1, m-1)$
2. There is an odd integer a, $1 \le a \le n-1$ such that $\lim_{k\to\infty} \Delta_\ell^{n-a} u(k) = 0$ for $1 \le i \le a-1$, $\lim_{k\to\infty} \Delta_\ell^{n-a} u(k) \ge 0$ (finite), $\lim_{k\to\infty} \Delta_\ell^{n-a-1} u(k) > 0$ and $\lim_{k\to\infty} \Delta_\ell^i u(k) = \infty,\ 0 \le i \le n-a-2$.

Proof. The proof is contained in Theorem 3.2 and Corollary 3.4.

Theorem 3.6.

(Generalized Discrete Arzela-Ascoli's theorem) A bounded, uniformly Cauchy subset Ω of U is relatively compact.

Proof . By Lemma 2.5, it suffices to construct a finite $\varepsilon-$ net for any $\varepsilon > 0$. We know that for any $\varepsilon > 0$, there are real numbers M_1 and K_1 such that for any $u \in \Omega$

$$|u(m,k) - u(m',k')| < \frac{\varepsilon}{2} \textit{ for } (m,k) \in D', (m',k') \in D'. \tag{11}$$

Let B be a bounded of Ω, that is, $\|u\| \le B,\ u \in \Omega$. Choose a real number L and real numbers $v_{\ell+j} < v_{2\ell+j} < \cdots < v_L$ where $j = k - [k/\ell]\ell$ such that $v_{\ell+j} = -B$, $v(L) = B$ and

$$|v((i+1)\ell + j) - v(i)| < \frac{\varepsilon}{2}, i \in [1, L) \tag{12}$$

We define a double sequence $y = \{y(m,k)\}, m \ge M,\ k \ge K$ as follows: let $y(m,k)$ be one of the values $\{v_{\ell+j}, v_{2\ell+j}, \cdots, v_L\}$ for $M \le m \le M_1$, $K \le k \le K_1$; let $y(m,k) = y(m, K_1)$ for $(m,k) \in D_3'$; let $y(m,k) = y(M_1, K_1)$ for $(m,k) \in D_1'$. Clearly, the double function $y = \{y(m,k)\}\ m \ge M,\ k \ge K$ belongs to U Let V be the set of all double function y defined as above. Note that V includes $L^{[(M_1 - M + \ell)(K_1 - K + \ell)]/\ell}$ such double function.

We claim that V is a finite $\varepsilon-$net for Ω. For any u in Ω we must show that V contains a double function y which differs from u by less than ε at all positive integer pairs (m, k), $m \ge M$, $k \ge K$. For each $M \le m \le M_1$, $K \le k \le K_1$, choose $y(m, k)$ in $\{v_{\ell+j}, v_{2\ell+j}, \cdots, v_L\}$ such that

$$|u(m,k) - y(m,k)| = \min_{1\le p\le L} |u(m,k) - v(p)|. \tag{13}$$

Let

$$\begin{aligned} y(m,k) &= y(m, K_1), (m,k) \in D_2', \\ y(m,k) &= y(M_1, k), (m,k) \in D_3', \\ y(m,k) &= y(M_1, K_1), (m,k) \in D_1', \end{aligned} \tag{14}$$

Hence, $y = \{y(m,k)\},\ m \geq \mathrm{M},\ k \geq \mathrm{K}$ belongs to V In view of (12) and (13), we have

$$|u(m,k) - y(m,k)| < \frac{\varepsilon}{2}, \mathrm{M} \leq m \leq \mathrm{M}_1, \mathrm{K} \leq k \leq \mathrm{K}_1. \tag{15}$$

For $(m,k) \in \mathrm{D}_2'$, (6) and (15) imply that

$$\begin{aligned} |u(m,k) - y(m,k)| &= |u(m,k) - y(m,\mathrm{K}_1)| \\ &\leq |u(m,k) - u(m,\mathrm{K}_1)| + |u(m,\mathrm{K}_1) - y(m,\mathrm{K}_1)| < \varepsilon. \end{aligned} \tag{16}$$

For $(m,k) \in \mathrm{D}_3'$, (6) and (15) imply that

$$\begin{aligned} |u(m,k) - y(m,k)| &= |u(m,k) - y(\mathrm{M}_1,k)| \\ &\leq |u(m,k) - u(\mathrm{M}_1,k)| + |u(\mathrm{M}_1,k) - y(\mathrm{M},k)| < \varepsilon \end{aligned} \tag{17}$$

For $(m,k) \in \mathrm{D}_1'$, (6) and (15) imply that

$$\begin{aligned} |u(m,k) - y(m,k)| &= |u(m,k) - y(\mathrm{M}_1,\mathrm{K}_1)| \\ &\leq |u(m,k) - u(\mathrm{M}_1,\mathrm{K}_1)| + |u(\mathrm{M}_1,\mathrm{K}_1) - y(\mathrm{M}_1,\mathrm{K}_1)| < \varepsilon \end{aligned} \tag{18}$$

Equation (15), (16), (17), and (18) imply that $\|y - u\| < \varepsilon$. The proof is complete.

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