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DISCRETE RAMANUJAN-FOURIER TRANSFORM OF EVEN FUNCTIONS (MOD r)

Pentti Haukkanen *

ABSTRACT

An arithmetical function f or a discrete signal {f(n)} is said to be even (mod r) if f(n) = f((n, r)) for all $n \in \mathbb{Z}$, where (n, r) is the greatest common divisor of n and r. We adopt a linear algebraic approach to show that the Discrete Fourier Transform (DFT) of an even function or an even discrete signal (mod r) can be written in terms of Ramanujan sums, which are integer-valued functions that can be computed recursively. The DFT may thus in this case be referred to as the Discrete Ramanujan-Fourier Transform (DRFT).

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INTRODUCTION

An arithmetical function is usually defined as a complex-valued function defined on the set $\mathbb{Z}^+$ of positive integers. In this paper we define an arithmetical function as a complex-valued function defined on the set $\mathbb{Z}$ of integers. Thus, the concept of an arithmetical function is the same as the concept of a discrete signal. Throughout this paper we use the term "arithmetical function" but we could instead use the term "discrete signal".

For a positive integer r, an arithmetical function f is said to be periodic (mod r) if $f(n+r) = f(n)$ for all $n \in \mathbb{Z}$. Every periodic function f (mod r) can be written uniquely as

$$f(n)=r^{-1}\sum_{k=1}^{r}F_f(k)\epsilon_k(n), \tag{1}$$

where

$$F_f(k)=\sum_{n=1}^{r}f(n)\epsilon_k(-n) \tag{2}$$

* Department of Mathematics, Statistics and Philosophy, University of Tampere, Finland, *mapehau@uta.fi*

and ϵ_k denotes the periodic function (mod r) defined as

$$\epsilon_k(n) = \exp(2\pi ikn/r).$$

The function F_f in (2) is referred to as the Discrete Fourier Transform (DFT) of f, and (1) is the Inverse Discrete Fourier Transform (IDFT).

An arithmetical function f is said to be even (mod r) if

$$f(n) = f((n, r))$$

for all $n \in \mathbb{Z}$, where (n, r) is the greatest common divisor of n and r. It is easy to see that every even function (mod r) is periodic (mod r). Ramanujan's sum $C(n, r)$ is defined as

$$C(n,r)= \sum_{\substack{k(\operatorname{mod} r)\\(k,r)=1}} \exp(2\pi ikn/r)$$

and is an example of an even function (mod r).

In this paper we show that the DFT (2) and IDFT (1) of an even function f (mod r) can be written in a concise form using Ramanujan's sum $C(n, r)$, see Section 3. We also review a proof of (1) and (2) for periodic functions (mod r), see Section 2, and review the Cauchy product of periodic functions (mod r), see Section 4.

The results of this paper may be considered to be known in number theory. They have not been presented exactly in this form and we hope that this paper will provide a clear approach to the elementary theory of even functions (mod r). We also hope that this paper will be useful to the applied mathematics community.

The concept of an even function (mod r) originates from Cohen [2] and was further studied by Cohen in subsequent papers [3, 4, 5]. General accounts of even functions (mod r) can be found in the books by McCarthy [8] and Sivaramakrishnan [13]. For recent papers on even functions (mod r) we refer to [11, 12, 14]. The paper [10] deals with even discrete signals (mod r). Material on periodic functions (mod r) can be found in the book by Apostol [1]. General background material on arithmetical functions can be found in [1, 8, 13] and on discrete signals in [9].

PROOF OF (1) AND (2)

Let P_r denote the set of all periodic arithmetical functions (mod r). It is clear that P_r is a complex vector space under the usual addition and scalar multiplication. In fact, P_r is isomorphic to $\mathbb{C}^r$. Further, P_r is a complex inner product space under the Euclidean inner product given as

$$\langle f,g\rangle = \sum_{n=1}^{r} f(n)\overline{g(n)}. \tag{3}$$

The set $\{r^{-1/2}\epsilon_k : k = 1, 2, \ldots, r\}$ is an orthonormal basis of P_r. Thus, every $f \in P_r$ can be written uniquely as

$$f(n) = \sum_{k=1}^{r} \left\langle f, r^{-1/2}\epsilon_k \right\rangle r^{-1/2}\epsilon_k(n),$$

where

$$\left\langle f, r^{-1/2}\epsilon_k \right\rangle = \sum_{n=1}^{r} f(n)\,\overline{r^{-1/2}\epsilon_k(n)} = r^{-1/2}\sum_{n=1}^{r} f(n)\epsilon_k(-n).$$

This proves (1) and (2).

DFT AND IDFT FOR EVEN FUNCTIONS (MOD r)

Let E_r denote the set of all even functions (mod r). The set E_r forms a complex vector space under the usual addition and scalar multiplication. In fact, E_r is a subspace of P_r. Thus (1) and (2) hold for $f \in E_r$. We can also present (1) and (2) for $f \in E_r$ in terms of Ramanujan's sum as is shown below.

Note that Ramanujan's sum $C(n, r)$ is an integer for all $n \in \mathbb{Z}$ and can be evaluated by addition and subtraction of integers. In fact, $C(n, r)$ can be written as $C(n, r) = \Sigma_{d|(n,r),d>0}\, d\mu(r/d)$, where μ is the Möbius function defined as

$$\mu(r) = \begin{cases} 1 & \text{if } r = 1, \\ (-1)^m & \text{if } r = p_1 p_2 \cdots p_m \ (\text{the } p_i'\text{s being distinct primes}), \\ 0 & \text{if } p^2 \mid r \ \text{ for some prime } p. \end{cases}$$

Ramanujan's sum $C(n, r)$ can also be computed recursively, see [10].

An arithmetical function $f \in E_r$ is completely determined by its values $f(d)$ with $d|r$ ($d > 0$). Thus E_r is isomorphic to $\mathbb{C}^{\tau(r)}$, where $\tau(r)$ is the number of the positive divisors of r. The inner product (3) in P_r can be written in E_r in terms of the positive divisors of r. In fact, we have

$$\sum_{\substack{k=1 \\ (k,r)=d}}^{r} 1 = \sum_{\substack{j=1 \\ (j,r/d)=1}}^{r/d} 1 = \phi(r/d), \tag{4}$$

where ϕ is Euler's totient function ($\phi(r)$ being the number of integers x (mod r) such that $(x, r) = 1$), and thus (3) can be written for $f, g \in E_r$ as

$$\left\langle f, g \right\rangle = \sum_{k=1}^{r} f(k)\overline{g(k)} = \sum_{d|r,d>0} f(d)\overline{g(d)}\phi(r/d).$$

Theorem 1. The set

$$\{(r\phi(d))^{-\frac{1}{2}}C(\cdot,d):d\mid r,d>0\} \tag{5}$$

is an orthonormal basis of the inner product space E_r.

Proof: As the dimension of the inner product space E_r is $\tau(r)$ and the number of elements in the set (5) is $\tau(r)$, it suffices to show the set (5) is an orthonormal subset of E_r. This follows easily from the relation

$$\sum_{e\mid r,e>0} C(r/e,d_1)C(r/e,d_2)\phi(e)=\begin{cases} r\phi(d_1) & \text{if } d_1=d_2,\\ 0 & \text{otherwise},\end{cases}$$

where $d_1 \mid r$ and $d_2 \mid r$ (d_1, $d_2 > 0$), see [8, p. 79].

We now present (1) and (2) for $f \in E_r$.

Theorem 2. Every $f \in E_r$ can be written uniquely as

$$f(n)=r^{-1}\sum_{d\mid r,d>0} R_f(d)C(n,d), \tag{6}$$

where

$$R_f(d)=\phi(d)^{-1}\sum_{n=1}^{r} f(n)C(n,d). \tag{7}$$

Proof: On the basis of Theorem 3.1,

$$f(n)=\sum_{d\mid r,d>0}\left\langle f,(r\phi(d))^{-\frac{1}{2}}C(\cdot,d)\right\rangle (r\phi(d))^{-\frac{1}{2}}C(n,d). \tag{8}$$

Applying (3) to (8) we obtain (6) and (7).

The function R_f in (7) may be referred to as the Discrete Ramanujan-Fourier Transform of f, and (6) may be referred to as the Inverse Discrete Ramanujan-Fourier Transform. Cf. [6].

Another expression of (7) can be obtained easily. Namely, applying (4) to (8) and then applying

$$\phi(e)C(r/e,\ d) = \phi(d)C(r/d,\ e)$$

(see [8, p. 93]) we obtain

$$R_f(d)=\sum_{e\mid r,e>0} f(r/e)C(r/d,e). \tag{9}$$

Note that (6) can also be derived from (1). In fact, if $f \in E_r$, then (2) can be written as

$$\begin{aligned} F_f(k) &= \sum_{n=1}^{r} f(n)\exp(-2\pi ikn/r) \\ &= \sum_{e|r,e>0} \sum_{\substack{n=1 \\ (n,r)=e}}^{r} f(e)\exp(-2\pi ikn/r) \\ &= \sum_{e|r,e>0} f(e) \sum_{\substack{m=1 \\ (m,r/e)=1}}^{r/e} \exp(-2\pi ikm/(r/e)) \\ &= \sum_{e|r,e>0} f(e)C(k,r/e). \end{aligned}$$

A similar argument shows (6) with $R_f(d) = F_f(r/d)$. We omit the details.

THE CAUCHY PRODUCT

The Cauchy product h of periodic functions f and g (mod r) is defined as

$$h(n) = (f \circ g)(n) = \sum_{a+b \equiv n \pmod r} f(a)g(b).$$

It is well known that h is the periodic function (mod r) such that $F_h = F_f F_g$. This follows from the property

$$\sum_{a+b \equiv n \pmod r} \epsilon_k(a)\epsilon_j(b) = \begin{cases} r\epsilon_k(n) & \textit{if } k \equiv j \pmod r, \\ 0 & \textit{otherwise.} \end{cases}$$

Analogously, if h is the Cauchy product of $f \in E_r$ and $g \in E_r$, then $h \in E_r$ with $R_h = R_f R_g$. This follows from the property

$$\sum_{a+b \equiv n \pmod r} C(a,d_1)C(b,d_2) = \begin{cases} rC(a,d_1) & \textit{if } d_1 = d_2, \\ 0 & \textit{otherwise,} \end{cases}$$

where $d_1 \mid r$ and $d_2 \mid r$ ($d_1, d_2 > 0$), see [13, p. 333].

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