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MODELLING OF NON-NEWTONIAN BLOOD FLOW RESISTANCE AND PRESSURE DROP THROUGH AN ARTERY

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ABSTRACT: A model of blood flow through an artery has been formulated for improved generalized geometry of multiple stenosis located at equispaced points. We have assumed that the stenosis is mild and radially non-symmetric. A set of equations describes the resistance to flow ratio of an artery. Analytic solutions are based on homogenous and irrotational flow through mathematically constructed vessels. Variations in resistance to flow ratio are subjected to alterations in flow behaviour index, structural variations in relation to magnitude of vessel stenosis and multiple abnormal segments. Graphical analysis demonstrates that the pressure drop across the stenosis decreases as the parameter δ/R_0 increases. The formulation of this model is mathematically more general and includes the results of the previous investigators as a special situation.

Mathematics Subject Classification: 92C35.

Keywords: Stenosis, Blood flow, Shape parameter, Non-Newtonian, Rheology.

1. Introduction

The abnormal and unnatural growth in the arterial wall thickness that develops at various locations of the cardiovascular system under diseased conditions is called arteriosclerosis or stenosis. The word stenosis is a medical term which means narrowing of any body passage, tube or orifice. This can cause serious circulatory disorders by reducing or occluding the blood supply. For instance, stenosis in the arteries supplying blood to brain can bring about cerebral strokes; likewise in coronary arteries, it can cause myocardial infarction leading to the heart failure. The actual causes of the stenosis are not well known but it has been suggested that the deposits of the cholesterol on the arterial wall and the proliferation of connective tissues may be responsible. Due to stenosis in the human artery the flow of blood is disturbed and resistance to flow becomes higher than that of normal one. As such fluid mechanical behaviour of an arterial stenosis has drawn considerable attention from various researchers like Young and Tsai [1], Lee and Fung [2], Rodbard [3]. The Rheology of circulation was deeply discussed by Whitmore [4]. Sanyal and Maji [5] investigated the unsteady blood flow through an indented tube in presence of stenosis. Young [6] observed the effect of time-dependent stenosis on flow of a Newtonian fluid through a tube. Chakravarty and Datta [7] performed rheological study on the effect of mild stenoses on the flow behavior of blood in a stenosed arterial segment. The various geometries of stenosis have been suggested by the researchers.

The cosine-shaped geometry was considered and analysed with different parameters by many researchers like Young [2], Kapur [8], Chakravarty [9]. The power-law and casson fluid models with cosine-shaped geometry were discussed by Shukla *et al*., [10]. The radially nonsymmetric stenosis has been analysed by Sanyal and Maji [5], Srivastava and Saxena [11], Srivastava [12]. The effects of shape of stenosis on the resistance to blood flow through an artery has been investigated by Haldar [13]. Due to the presence of a new parameter the formulation of our model is mathematically more general and includes the model of Haldar [13] as a special case.

2. Mathematical Models of Blood Flow

The non-Newtonian properties of blood are appropriate for the use of the Power law, Herschel-Bulkey, Casson, and Bingham models.

The power law or Ostwald de-waele model describes a type of time independent non-Newtonian fluid with shear dependent viscosity. The constitutive equation of the power law model is

$$\tau = m\dot{\gamma}^n. \tag{1}$$

Here, τ is the shear stress, $\dot{\gamma}$ is the shear strain rate, m is the consistency and n is the flow behaviour index. There is no yield stress τo so the equation does not model situations where there is a finite shear stress required to overcome viscosity and start flow. The shear strain rate γ is a function of τ and is proportional to the rate of decrease of axial velocity v along the arterial radius:

$$\dot{\gamma} = f(\tau) = -\frac{dv}{dr}. \tag{2}$$

The three main categories of power law fluids are pseudo plastic. Newtoninan and dilatent, which depend on the flow behaviour index. For pseudo plastic fluids, $n < 1$, the apparent viscosity decreases as the shear strain rate increases. If $n = 1$, the Power law model reduces to its Newtonian case and $m = \mu$ is the viscosity of the fluid. Bio-fluids such as blood that is described by the power law model are pseudo plastic.

2.1 Formulation of the Mathematical Model

We have considered an artery having mild stenosis. The flow of blood is assumed to be steady, laminar and fully-developed. Blood is taken as a power-law fluid.

It is assumed that stenosis is symmetrical about the axis but non-symmetrical with respect to radial co-ordinates. The mathematical expression for geometry can be written as,

$$\frac{R(z)}{R_0}=\begin{cases}1-\lambda\,[L_0^{s-1}(z-kd-(k-1)\,L_0)-(z-kd-(k-1)\,L_0)^s]; & \\ \qquad k(d+L_0)-L_0\le z\le k(d+L_0) & \\ 1; \qquad \text{otherwise.} & \end{cases} \tag{3}$$

where

$$\lambda=\frac{\delta s^{s/s-1}}{R_0L_0^s(s-1)} \tag{4}$$

and δ denotes the maximum height of the stenosis at

$$z=kd+(k-1)\,L_0+L_0/s^{1/s-1} \tag{5}$$

where

R_0 : Radius of normal tube
$R(z)$: Radius of stenotic region
L : The length of the artery
L_0 : The length of the stenosis
d : Distance between equispaced points
δ : Maximum height of stenosis
s : Parameter determining the shape of stenosis ($s\ge 2$)
k : Number of stenoses that appear in arterial lumen

The schemateic diagram of the flow is given by the Fig. 1

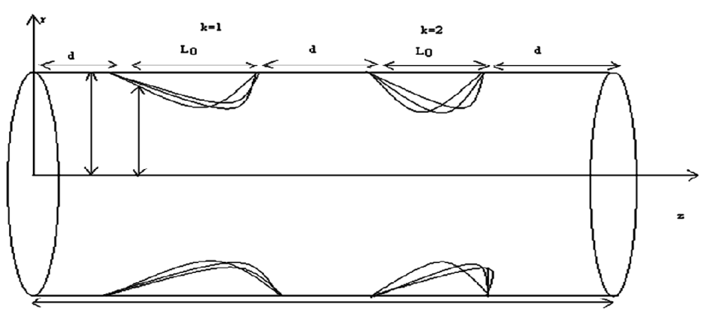

Figure 1

For power-law fluid, we have

$$Q=\int_0^{R(Z)}2\pi r.vdr \tag{6}$$

For variability of circumferential geometry in the transverse section of the artery, the arterial wall radius R(z) is taken as a function of z. Axial velocity is denoted by V. The flow rate through the artery is

$$Q = 2\pi \int_0^{R(z)} rv.dr . \tag{7}$$

The boundary conditions are,

$$V = 0 \text{ at } r = R(z); \qquad k(d + L_0) - L_0 \le z \le k(d + L_0) \tag{8}$$

$$V = 0 \text{ at } r = R_0; \qquad \text{elsewhere.} \tag{9}$$

The equation (7) can be re-expressed as

$$Q = \int_0^{R(z)} rv.dr = \pi \int_0^{R(z)} r^2 \left(-\frac{dv}{dr} \right) dr . \tag{10}$$

Substituting equation (2) into (10) leads to

$$Q = \pi \int_0^{R(z)} r^2 f(\tau)\, dr . \tag{11}$$

The Shear stress

$$\dot{\gamma} = \tau(r) = -\frac{dv}{dr} = -\frac{r}{2}\frac{dp}{dz}, \tag{12}$$

where P is the pressure. Therefore the value of τ at $r = R(z)$ is

$$\tau_R = \tau(R(z)) = -\frac{R(z)}{2}\frac{dp}{dz}. \tag{13}$$

Re-expression of the integration in (10) and using equations (12) and (13) yields

$$Q = \frac{\pi}{\tau_R^3} R^3(z) \int_0^{\tau_R} \tau^2 f(\tau)\, d\tau . \tag{14}$$

Here, the plasma layer is assumed to be negligible and τ_R is the shear stress at the wall. It implies

$$P(z) = \frac{2\mu Q^n}{R(z)^{3n+1}} \left[\frac{3n+1}{n\pi} \right]^n . \tag{15}$$

Now integrating equation (15) along the length of artery and using conditions $p = p_1$ at $Z = 0$ and $p = p_2$ at $z = L$.

We obtain

$$\Delta P = P_1 - P_2 = \left(\frac{3n+1}{n\pi} Q\right)^n \frac{2\mu}{R_0^{3n+1}} \int_0^L \frac{dz}{(R/R_0)^{3n+1}}, \tag{16}$$

where R/R_0 is given by equation (3).

The resistance to flow λ is defined by

$$\lambda = \frac{P_1 - P_2}{Q} = \frac{\Delta P}{Q},$$

which on using equation (16) gives

$$\lambda = \frac{2\mu Q^{(n-1)}}{R_0^{3n+1}} \left(\frac{3n+1}{n\pi}\right)^n \int_0^L \frac{dz}{(R/R_0)^{3n+1}}. \tag{17}$$

For no stenosis (normal artery)

$$\lambda_N = \frac{2\mu L Q^{(n-1)}}{R_0^{3n+1}} \left(\frac{3n+1}{n\pi}\right)^n, \tag{18}$$

since $\int_0^L dz = 1$.

Therefore, division of flow resistance for an abnormal artery with a normal one yields the flow resistance ratio

$$\bar{\lambda} = \frac{\lambda}{\lambda_N} = \frac{1}{L} \int_0^L \frac{dz}{(R/R_0)^{3n+1}}. \tag{19}$$

The flow resistance equation takes into consideration the overlapping wall segments to form a geometrically complex shape. When comparing flow resistance ratios, the arterial length L is arbitrarily set with a unit value of 1 to ease the mathematical formulation. For a normal artery,

$$I_N = \int_0^L dz = 1.$$

Therefore, division of flow resistance for an abnormal artery with a normal one yields the flow resistance ratio.

$$\bar{\lambda} = \frac{I_R}{I_N} = I_R .$$

For normal artery, $R = R_0$ and this gives a flow resistance ratio of one. For a fully occluded artery $R = 0$ and the flow resistance is infinite.

The non-dimensional form of resistance to flow, denoted by $\bar{\lambda}$, is given as

$$\bar{\lambda} = 1 - k_{\max}\frac{L_0}{L} + \frac{1}{L}\sum_{k=1}^{k_{\max}} \int_{k(d+L_0)-L_0}^{k(d+L_0)} \frac{dz}{\left[1 - \frac{\delta}{R_0 L_0^s}\frac{s^{s/s-1}}{s-1}\left[\begin{array}{l} L_0^{s-1}\left(z-kd-(k-1)L_0\right) \\ -\left(z-kd-(k-1)L_0\right)^s \end{array}\right]\right]^{3n+1}} . \quad (20)$$

where n is power law index.

3. RESULTS

Pressure Drop across the Stenosis and across the Whole Length of the Artery

$$P(z) = -\frac{\partial p}{\partial z} = \frac{2\mu Q^n}{R(z)^{3n+1}}\left(\frac{3n+1}{n\pi}\right)^n . \quad (21)$$

So the pressure drop across the length of the stenosis is

$$\Delta P = 2\mu Q^n \left(\frac{3n+1}{n\pi}\right)^n \int_0^L R(z)^{3n+1}\, dz . \quad (22)$$

From equation (3)

$$\Delta P = \frac{2\mu Q^n}{(R_0)^{3n+1}}\left(\frac{3n+1}{n\pi}\right)^n \int_0^L \frac{1}{(1-\lambda[L_0^{s-1}(\beta)-(\beta)^s])^{3n+1}}\, dz \quad (23)$$

where

$$\beta = z - kd - (k-1)L_0$$

$$\lambda = \frac{\delta s^{s/s-1}}{R_0 L_0^s (s-1)} .$$

For symmetric stenosis, $s = 2$.

$$\Delta P = \frac{8\mu Q^n}{R_0^{3n+1}} \left(\frac{3n+1}{n\pi}\right)^n \int_0^L \frac{1}{(1-\lambda[L_0\beta - \beta^2])^{3n+1}}\, dz \tag{24}$$

where

$$\lambda = \frac{4\delta}{R_0 L_0^2}.$$

For non-symmetric stenosis, $S > 2$, for $S = 6$.

$$\Delta P = \frac{8\mu Q^n}{R_0^{3n+1}} \left(\frac{3n+1}{n\pi}\right)^n \int_0^L \frac{1}{(1-\lambda[L_0\beta - \beta^6])^{3n+1}}\, dz$$

where,

$$\lambda = \frac{1.72\delta}{R_0 L_0^2}.$$

If there is no stenosis, then $\delta/R_0 = 1$ and the pressure drop across the stenosis length for symmetric stenosis is

$$(\Delta P)_P = \frac{8\mu Q}{\pi R_0^4} \int_0^L \frac{1}{1 - \frac{4}{L_0^2}(L_0\beta - \beta^2)}\, dz \tag{25}$$

the subscript p denotes Poiseuille flow.

For the whole length of the artery $L = 1$, $k = 1$

$$K_1 = \frac{\Delta P}{(\Delta P)_p}.$$

The value of K_1 is given in the graph. The graph shows that the value of K_1 does not increase significantly as δ/R_0 exceeds 0.1.

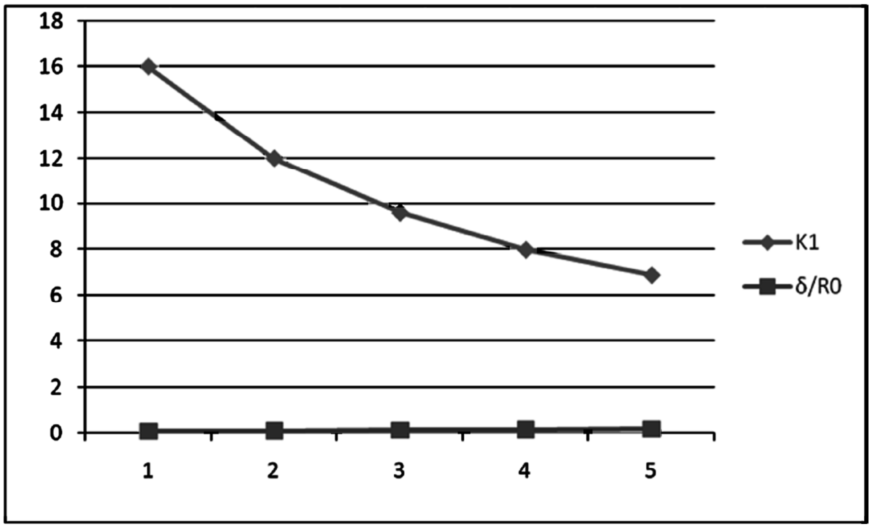

Figure 2

4. Conclusion

The power law model of blood flow through an atherosclerotic artery can solve the flow resistance through the abnormal segment of the artery for varying shape parameters. From graphical representation, our study reveals that for a particular value of length of artery (z), the pressure drop k_1 across the length of the whole artery is performed for the parametric values corresponding to δ/R_0 greater than 0.1.

Clinical data such as pressure drop in atherosclerotic vessels may be useful as reference data for the assessment of flow resistance through variable arterial structures. Future studies to verify our analytical model can be performed using such clinical information.

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