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# CLAW DECOMPOSITION OF PRODUCT GRAPHS

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**Abstract:** A decomposition of a graph $G$ is a family of edge-disjoint subgraphs $\{G_1, G_2, \ldots, G_k\}$ such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. If each $G_i$ is isomorphic to $H$ for some subgraph $H$ of $G$, then the decomposition is called a $H$-decomposition of $G$. A star with three edges is called a claw. In this paper, we give necessary and sufficient condition for the decomposition of cartesian product of standard graphs into claws. Also, we give a sufficient condition for the claw decomposition of lexicographic product of standard graphs.

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***Keywords:*** Decomposition, claw decomposition, cartesian product, lexicographic product

## 1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph without loops or multiple edges. A path on n vertices is denoted by $P_n$, cycle on $n$ vertices is denoted by $C_n$ and complete graph on $n$ vertices is denoted by $K_n$. The *neighbourhood* of a vertex $v$ in $G$ is the set $N(v)$ consisting of all vertices that are adjacent to $v$. $|N(v)|$ is called the degree of $v$ and is denoted by $d(v)$. A complete bipartite graph with partite sets $V_1$ and $V_2$, where $|V_1| = r$ and $|V_2| = s$, is denoted by $K_{r,s}$. The graph $K_{1,r}$ is called a star and is denoted by $S_r$. The vertex of degree $r$ in the star $S_r$ is called the central vertex of the star. Claw is a star with three edges. The complement of a graph $G$ is denoted by $\overline{G}$. $kG$ denotes the union of $k$ copies of $G$. The join $G + H$ of two graphs $G$ and $H$ consists of $G \cup H$ and all edges joining each vertex of $G$ to all the vertices of $H$. Terms not defined here are used in the sense of [5].

A decomposition of a graph $G$ is a family of edge-disjoint subgraphs $\{G_1, G_2, \ldots, G_k\}$ such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. If each $G_i$ is isomorphic to $H$ for some subgraph $H$ of $G$, then the decomposition is called a $H$-decomposition of $G$. If $H$ has at least three edges, then the problem of deciding if a graph $G$ has a $H$-decomposition is $NP$-complete [2]. In 1975, Sumiyasu Yamamoto et al., [6] gave necessary and sufficient condition for the $S_k$-decomposition of complete graphs and complete bipartite graphs. In 1996, C. Lin and T. W. Shyu [4] presented a necessary and sufficient condition for decomposing $K_n$ into stars $S_{k_1}, S_{k_2}, \ldots, S_{k_t}$. In 2004, H. L. Fu et al., [3] decomposed a complete graph into cartesian product of two complete graphs $K_r$ and

$K_c$. In 2012, Darryn E. Bryant et al., [1] gave necessary and sufficient condition for the existence of $k$-star factorizations of any power $K_q^s$ where $q$ is prime and the products $C_{r_1} \times C_{r_2} \times ... \times C_{r_k}$ of $k$ cycles of arbitrary length. In 2013, Tay-Woei Shyu [7] gave necessary and sufficient condition for the decomposition of complete graph into $C_l$'s and $S_k$'s. In this paper, we give necessary and sufficient condition for the decomposition of cartesian product of standard graphs into claws. Also, we give a sufficient condition for the claw decomposition of lexicographic product of standard graphs.

## 2. BUILDING BLOCKS

In this section, we collect certain lemmas and results which are used in the subsequent sections. These are the building blocks in the construction of the main theorems.

**Definition 2.1:** The corona of two graphs $G$ and $H$, is the graph $G \circ H$ formed from one copy of $G$ and $|V(G)|$ copies of $H$ where the $i^{\text{th}}$ vertex of $G$ is adjacent to every vertex in the $i^{\text{th}}$ copy of $H$.

**Definition 2.2:** The Cartesian product of two graphs $G$ and $H$ is a graph, denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent precisely if $g = g'$ and $hh'' \in E(H)$, or $gg'' \in E(G)$ and $h = h'$. Thus,

$$V(G \times H) = \{(g, h)/g \in V(G) \text{ and } h \in V(H)\},$$

$$E(V \times H) = \{(g, h)(g', h')/g = g' \text{ and } hh' \in E(H), \text{ or}$$

$$gg' \in E(G) \text{ and } h = h'\}.$$

**Theorem 2.3:** [6] A complete graph, $K_l$ with $l$ points and $\binom{l}{2}$ lines can be decomposed into a union of line disjoint $\binom{l}{2}\Big/c$ claws, $K_{1,c}$, with $c$ lines each if and only if

(1) $\left(\frac{l}{c}\right)$ is an integral multiple of c, and

(2) $l \geq 2c$.

**Theorem 2.4:** [6] A complete bigraph, $K_{m,n}$, with $m$ and $n$ points and $mn$ lines can be decomposed into union of $mn/c$ line disjoint $\left(\frac{l}{2}\right)\Big/c$ claws, $K_{1,c}$, with $c$ lines each if and only if $m$ and $n$ satisfy one of the following three conditions:

(1) $n \equiv 0 \ (mod\ c)$ when $m < c$

(2) $m \equiv 0\ (mod\ c)$ when $n < c$

(3) $mn \equiv 0\ (mod\ c)$ when $m \geq c$ and $n \geq c$.

**Lemma 2.5:** The graph $C_n \circ \overline{K}_2$ is claw decomposable for all $n$.

**Proof:** Let $V(C_n) = \{v_1, v_2, \ldots v_n\}$ and let $u_i$ and $w_i$ be the pendant vertices at $v_i$. Then $< \{u_i, w_i, v_i, v_{i+1}\} > \cong K_{1,3}$ for all $1 \leq i \leq n - 1$ and $< \{u_n, w_n, v_n, v_1\} > \cong K_{1,3}$.

Thus $$E(C_n \circ \overline{K}_2) = \underbrace{E(K_{1,3}) \cup \ldots \cup E(K_{1,3})}_{n\ times}.$$

Hence $C_n \circ \overline{K}_2$ is claw decomposable. □

**Lemma 2.6:** *If* $n$ is even and $n \equiv 0(mod\ 3)$, then $K_2 \times C_n$ is claw decomposable.

**Proof:** Let $V(K_2) = \{x_1, x_2\}$ and let $V(C_n) = \{y_1, y_2, \ldots, y_n\}$.

Then $V(K_2 \times C_n) = \{(x_i, y_j)/i = 1, 2 \text{ and } 1 \leq j \leq n\}$.

Rename $(x_1, y_j) = v_j$ and $(x_2, y_j) = u_j$ for all $1 \leq j \leq n$.

Now, $< \{v_1, v_2, v_n, u_1\} > \cong K_{1,3}$,

$< \{u_1, u_{n-1}, u_n, v_n\} > \cong K_{1,3}$,

$< \{u_{i+1}, v_i, v_{i+1}, v_{i+2}\} > \cong K_{1,3}$ for all $i \in \{2, 4, \ldots, n-2\}$ and

$< \{u_i, u_{i+1}, u_{i+2}, v_{i+1}\} > \cong K_{1,3}$ for all $i \in \{1, 3, \ldots, n-3\}$.

Thus $$E(K_2 \times C_n) = \underbrace{E(K_{1,3}) \cup \ldots \cup E(K_{1,3})}_{n\ times}.$$

Hence $K_2 \times C_n$ is claw decomposable. □

**Lemma 2.7:** $K_n \circ K_1$ is claw decomposable if and only if $n > 3$ and $n \neq 1(mod\ 3)$.

**Proof:** Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let $u_i$ be the pendant vertex at $v_i$ for all $1 \leq i \leq n$.

Suppose that $n > 3$ and $n \neq 1(mod\ 3)$.

***Case (i):*** $n \equiv 2(\text{mod } 3)$.

Now, $< \{v_5, v_6, \ldots, v_n\} > \cong K_{n-4}$,

$< \{v_3, v_4, v_i, u_i\} > \cong K_{1,3}$ for all $5 \leq i \leq n$,

$< \{v_1, v_2, v_4, u_4\} > - \{v_1v_2\} \cong K_{1,3}$,

$< \{v_1, v_3, v_4, u_3\} > - \{v_1v_4\} \cong K_{1,3}$,

$< \{u_1, v_1, v_2, v_5, v_6, \dots, v_n\} > - E\,(< \{v_5, v_6, \dots, v_n\} >) \cong K_{1,n-2}$ and

$< \{u_2, v_2, v_3, v_5, v_6, \dots, v_n\} > - E(< \{v_5, v_6, \dots, v_n\} >) \cong K_{1,n-2}$.

Thus $E(K_2 \text{ o } C_1) = E(K_{n-4}) \cup \underbrace{E(K_{1,3}) \cup \dots \cup E(K_{1,3})}_{(n-2)\ times} \cup E(K_{1,n-2}) \cup E(K_{1,n-2})$.

Since $n \equiv 2(mod\ 3)$, $n - 4 \equiv 1(mod\ 3)$. Hence by Theorem 2.3, $K_{n-4}$ is claw decomposable. Also, $K_{1,n-2}$ is claw decomposable.

Hence $K_n$ o $K_1$ is claw decomposable.

***Case (ii):*** $n \equiv 0(mod\ 3)$.

Then $< \{v_1, v_2, \dots, v_{n-1}, u_1, u_2, \dots, u_{n-1}\} > \cong K_{n-1}$ o $K_1$ and

$< \{v_1, v_2, \dots, v_n\} > - E(< \{v_1, v_2, \dots, v_{n-1}\} >) + \{u_nv_n\} \cong K_{1,n}$.

Thus $E(K_n \text{ o } K_1) = E(K_{n-1} \text{ o } K_1) \equiv E(K_{1,n})$.

Since $n \equiv 0(mod\ 3)$, $n - 1 \equiv 2(mod\ 3)$. Hence by Case (*i*), $K_{n-1}$ o $K_1$ is claw decomposable. Also, $K_{1,n}$ is claw decomposable.

Hence $K_n$ o $K_1$ is claw decomposable.

Conversely, suppose that $K_n$ o $K_1$ is claw decomposable.

Then $|E(K_n \text{ o } K_1)| \equiv 0\ (mod\ 3)$. That is, $\frac{n(n-1)}{2} + n \equiv 0\ (mod\ 3)$ which implies $\frac{n(n+1)}{2} \equiv 0\ (mod\ 3)$ and thus $n \equiv 0(mod\ 3)$ or $n \equiv 2\ (mod\ 3)$. Hence $n \neq 1(mod\ 3)$. Also, $K_3$ o $K_1$ is not claw decomposable. Thus $n > 3$.

Hence $n > 3$ and $n \neq 1(mod\ 3)$. □

**Lemma 2.8:** The graph $K_2 \times K_n$ is claw decomposable if and only if $n > 3$ and $n \equiv 0\ (mod\ 3)$.

**Proof:** Let $V(K_2) = \{x_1, x_2\}$ and let $V(C_n) = \{y_1, y_2, \dots, y_n\}$.

Then $V(K_2 \times C_n) = \{(x_i, y_j)/i = 1, 2 \text{ and } 1 \leq j \leq n\}$.

Rename $(x_1, y_j) = v_j$ and $(x_2, y_j) = u_j$ for all $1 \leq j \leq n$.

Now, $< \{v_1, v_2,..., v_n, u_1, u_2,..., u_n\} > - E(< \{u_1, u_2,..., u_n\} >) \cong K_n \text{ o } K_1$

and $< \{u_1, u_2,..., u_n\} > \cong K_n$.

Thus $E(G) = E(K_n \text{ o } K_1) \cup E(K_n)$.

Suppose that $n > 3$ and $n \equiv 0\ (mod\ 3)$.

Then by Lemma 2.7, $K_n$ o $K_1$ is claw decomposable. Also, by Theorem 2.3, $K_n$ is claw decomposable.

Hence $K_2$ o $K_n$ is claw decomposable.

Conversely, suppose that $K_2 \times K_n$ is claw decomposable.

Then $|E(K_2 \times K_n)| \equiv 0\ (mod\ 3)$. That is, $2.\frac{n(n-1)}{2} + 1.n \equiv 0\ (mod\ 3)$ which implies $n^2 \equiv 0\ (mod\ 3)$ and hence $n \equiv 0\ (mod\ 3)$. Also, $K_2 \times K_3$ is not claw decomposable. Thus $n > 3$. Hence $n > 3$ and $n \equiv 0\ (mod\ 3)$.

**Lemma 2.9:** The graph $K_2 \times K_n$ together with a pendant vertex attached to each vertex of one copy of $K_n$ is claw decomposable if and only if $n \neq 1(mod\ 3)$.

**Proof:** Let $G$ be the graph $K_2 \times K_n$ together with a pendant vertex attached to the each vertex of one copy of $K_n$.

Let $V(K_2) = \{x_1, x_2\}$ and let $V(K_n) = \{y_1, y_2,..., y_n\}$.

Then $V(K_2 \times K_n) = \{(x_i, y_j)/i = 1, 2 \text{ and } 1 \leq j \leq n\}$.

Rename $(x_1, y_j) = v_j$ and $(x_2, y_j) = u_j$ for all $1 \leq j \leq n$.

Let $w_j$ be the pendant vertex at $v_j$ in $G$ for all $1 \leq j \leq n$.

Now, $< \{u_1, u_2,..., u_n, v_1, v_2,..., v_n\} > - E(< \{v_1, v_2,..., v_n\} >) \cong K_n \text{ o } K_1$

and $< \{v_1, v_2,..., v_n, w_1, w_2,..., w_n\} > \cong K_n \text{ o } K_1$.

Thus $E(G) = E(K_n \text{ o } K_1) \cup E(K_n \text{ o } K_1)$.

Suppose that $n \neq 1\ (mod\ 3)$.

Then by Lemma 2.7, $K_n$ o $K_1$ is claw decomposable.

Hence $G$ is claw decomposable.

Conversely, suppose that $G$ is claw decomposable.

Then $|E(G)| \equiv 0\ (mod\ 3)$. That is, $2.\frac{n(n-1)}{2} + 1.n + n \equiv 0\ (mod\ 3)$ which implies $n(n + 1) \equiv 0\ (mod\ 3)$ and thus $n \equiv 0\ (mod\ 3)$ or $n \equiv 2(mod\ 3)$.

Hence $n \neq 1(mod\ 3)$. □

## 3. CLAW DECOMPOSITION OF CARTESIAN PRODUCT OF GRAPHS

In this section, we give necessary and sufficient condition for the decomposition of cartesian product of some standard graphs into claws.

**Theorem 3.1:** If $G_1$ and $G_2$ are $H$-decomposable, then $G_1 \times G_2$ is $H$-decomposable.

**Proof:** Let $V(G_1) = \{v_1, v_2, \ldots, v_k\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_n\}$.

Then $V(G_1 \times G_2) = \{(v_i, u_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(v_i, u_j) = v_{ij}$; $1 \leq i \leq k$, $1 \leq j \leq n$.

Now, $< \{v_{1j}, v_{2j}, \ldots, v_{kj}\} > \cong G_1$ for all $1 \leq j \leq n$ and

$< \{u_{i1}, u_{i2}, \ldots, u_{in}\} > \cong G_2$ for all $1 \leq i \leq k$.

Thus, $E(G_1 \times G_2) = \underbrace{E(G_1) \cup \ldots \cup E(G_1)}_{n\ times} \cup \underbrace{E(G_2) \cup \ldots \cup E(G_2)}_{k\ times}$.

Since $G_1$ and $G_2$ are $H$-decomposable, $G_1 \times G_2$ is $H$-decomposable. □

**Corollary 3.2:** If $m, n \equiv 0 \ (mod\ 3)$, then $K_{1,m} \times K_{1,n}$ is claw decomposable.

**Corollary 3.3:** If $m \equiv 0 \ (mod\ 3)$ and $n \neq 2 \ (mod\ 3)$ then $K_{1,m} \times K_n$ is claw decomposable.

***Proof:*** It follows from Theorems 2.3 and 3.1.

**Corollary 3.4:** *If* $rs \equiv 0 \ (mod\ 3)$ and $n \equiv 2 \ (mod\ 3)$, then $K_{r,s} \times K_n$ *is* claw decomposable.

**Proof:** It follows from Theorems 2.3, 2.4 and 3.1.

**Corollary 3.5:** If $rs \equiv 0 \ (mod\ 3)$ and $n \equiv 0 \ (mod\ 3)$, then $K_{r,s} \times K_{1,n}$ is $K_{1,3}$ -decomposable.

**Proof:** It follows from Theorems 2.4 and 3.1. □

**Remark 3.6:** $P_n \circ K_1$ and $C_n \circ K_1$ are not claw decomposable for any values of $n$.

**Remark 3.7:** If $G = P_m \circ K_1$, then $G \circ C_n$ is not claw decomposable.

**Theorem 3.8:** Let $G_1 = P_m \circ K_1$. If $G_2$ and $G_2 \circ K_1$ are claw decomposable, then $G_1 \times G_2$ is claw decomposable.

**Proof:** Let $V(G_1) = \{u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_m\}$ where $w_i$ is the pendant edge at $u_i$ for all $1 \leq i \leq m$, $u_1 u_2 \ldots u_m$ is the $m$-path in $G$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$.

Then $V(G_1 \times G_2) = \{(u_i, v_j), (w_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$.

Rename $(u_i, v_j) = u_{ji}$ and $(w_i, v_j) = w_{ji}$ for all $1 \le i \le m,$ and $1 \le j \le n.$

Now, $< \{u_{1j}, u_{2j},..., u_{nj}, w_{1j}, w_{2j},...,w_{nj}\} > - E(< \{u_{1j}, u_{2j},..., u_{nj}\} >)$

$\cong G_2 \text{ o } K_1$ for all $1 \le j \le m$,

$< \{u_{1j}, u_{2j},..., u_{nj}, u_1(j+1), u_2(j+1),..., u_n(j+1)\}$

$> - E(< \{u_{1(j+1)}, u_{2(j+1)},..., u_{n(j+1)}\} >) \cong G_2 \text{ o } K_1$ for all $1 \le j \le m-1$

and $< \{u_{1m}, u_{2m},..., u_{nm}\} > \cong G_2.$

Thus $E(G_1 \times G_2) = \underbrace{E(G_2 o K_1 \cup ... \cup E(G_2 o K_1)}_{(2m-1)\ times} \cup E(G_2)$.

By assumption, $G_2$ and $G_2$ o $K_1$ are claw decomposable.

Hence $G_1 \times G_2$ is claw decomposable. □

**Corollary 3.9:** If $G = P_m \text{ o } K_1$ and $n \equiv 0$ (*mod* 3), then $G \times K_n$ is claw decomposable.

**Proof:** Since $n \equiv 0$ (*mod* 3), by Theorem 2.3, $K_n$ is claw decomposable. Also, by Lemma 2.7, $K_n$ o $K_1$ is claw decomposable. Hence the result follows from above theorem. □

**Remark 3.10:** If $G = P_m \text{ o } K_1$, then $G \times K_{1,n}$ is not claw decomposable.

**Proof:** Suppose not. Then let $S = \{S_1, S_2,..., S_k\}$ be a claw decomposition of $G \times K_{1,n}$. Let $V(G) = \{u_1, u_2,..., u_m, w_1, w_2,...,w_m\}$ where $w_i$ is the pendant edge at $u_i$ for all $1 \le i \le m$ and $u_1u_2....u_m$ is the $m$-path in $G$.

Let $V(K_{1,n}) = \{v_0, v_1,...,v_n\}$ where $d(v_0) = n$.

Then $V(G \times K_{1,n}) = \{(u_i, v_j), (w_i, v_j)/1 \le i \le m, 0 \le j \le n\}$.

Rename $(u_i, v_j) = u_{ji}$ and $(w_i, v_j) = w_{ji}$ for all $1 \le i \le m, 0 \le j \le n$.

Now, $w_{11}u_{11} \in E(G \times K_{1,n})$ and hence must be in some member of $S$, say $S_1$. Since $d(u_{11}) = 3$ and $d(w_{11}) = 2$, $u_{11}u_{12} \in S_1$. Similarly, $w_{1i}\,u_{1i}$ and $u_{1i}\,u_{1(i+1)}$ will be in the same member of $S$, say $S_i$ for all $1 \le i \le m-1$.

Then in $G \times K_{1,n} - U_{i=1}^{n} E(S_i)$, $d(u_{1n}) = 2$ and $d(w_{1n}) = 2$. Thus $w_{1n}\,u_{1n} \notin S$, a contradiction.

Hence $G \times K_{1,n}$ is not claw decomposable. □

**Theorem 3.11:** If $n \equiv 0$ (*mod* 3), then $P_k \times K_n$ is claw decomposable for all values of $k$.

**Proof:** Let $V(K_n) = \{v_1, v_2,..., v_n\}$ and $V(P_k) = \{u_1, u_2,..., u_k\}$ where $P_k = u_1u_2...u_k$.

Then $V(P_k \times K_n) = \{(u_i, v_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $1 \leq i \leq k$, $1 \leq j \leq n$.

Assume that $n \equiv 0$ (*mod* 3).

Now, $< \{v_{1j}, v_{2j},..., v_{nj}, v_{1(j+1)}, v_{2(j+1)},...., v_{n(j+1)}\}$

$> - E(< \{v_{1(j+1)}, v_{2(j+1)},...., v_{n(j+1)}\} >) \cong K_n \text{ o } K_1$ for all $1 \leq j \leq k-1$

and $< \{v_{1k}, v_{2k},..., v_{nk}\} > \cong K_n$.

Thus $E(G) = \underbrace{E(K_n o K_1 \cup ... \cup E(K_n o K_1)}_{(k-1)\, times} \cup E(K_n)$.

Since $n \equiv 0$ (*mod* 3), by Lemma 2.7, $K_n$ o $K_1$ is claw decomposable. Also, by Theorem 2.3, $K_n$ is claw decomposable.

Hence $P_k \times K_n$ is claw decomposable. □

**Conjecture 3.12:** The graph $P_k \times K_n$ is claw decomposable if and only if $n \equiv 0$ (*mod* 3).

**Theorem 3.13:** If $n \neq 1$(*mod* 3), then $C_k \times K_n$ is claw decomposable.

**Proof:** Let $V(K_n) = \{v_1, v_2,..., v_n\}$ and $V(C_k) = \{u_1, u_2,..., u_k\}$.

Then $V(C_k \times K_n) = \{(u_i, v_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $1 \leq i \leq k$, $1 \leq j \leq n$.

Assume that $n \neq 1$(*mod* 3).

Now, $< \{v_{1i}, v_{2i},...,v_{ni}, v_{1(i+1)}, v_{2(i+1)},...., v_{n(i+1)}\}$

$> -E(< \{v_{1(i+1)}, v_{2(i+1)},..., v_n(i+1)\} >) \cong K_n \text{ o } K_1$ for all $1 \leq i \leq k-1$

and $< \{v_{1k}, v_{2k},..., v_{nk}, v_{11}, v_{21},..., v_{n1}\} > - E(< \{v_{11}, v_{21},..., v_{n1}\} >) \cong K_n \text{ o } K_1$.

Thus $E(G) = \underbrace{E(K_n o K_1 \cup ... \cup E(K_n o K_1)}_{k\, times}$

Since $n \neq 1$(*mod* 3), by Lemma 2.7, $K_n$ o $K_1$ is claw decomposable.

Hence $C_k \times K_n$ is claw decomposable. □

**Conjecture 3.14:** The graph $C_k \times K_n$ is claw decomposable if and only if $n \neq 1$(*mod* 3).

**Theorem 3.15:** The graph $K_{1,m} \times K_{1,n}$ is claw decomposable if and only if $2mn + m + n \equiv 0(mod\ 3)$.

**Proof:** Let $V(K_{1,m}) = \{u_0, u_1,...,u_m\}$ and $V(K_{1,n}) = \{v_0, v_1,...,v_n\}$ where $d(u_0) = m$ and $d(v_0) = n$.

Then $V(K_{1,m} \times K_{1,n}) = \{(u_i, v_j)/0 \le i \le m, 0 \le j \le n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $0 \le i \le m$ and $0 \le j \le n$.

Suppose that $2mn + m + n \equiv 0\ (mod\ 3)$.

***Case (i):*** $m \equiv 0\ (mod\ 3)$.

Since $2mn + m + n \equiv 0\ (mod\ 3)$, $n \equiv 0\ (mod\ 3)$. Thus both $K_{1,m}$ and $K_{1,n}$ are claw decomposable. Hence by Theorem 3.1, $K_{1,m} \times K_{1,n}$ is claw decomposable.

***Case (ii):*** $m \equiv 1\ (mod\ 3)$.

Then $2mn + m + n \equiv 1(mod\ 3)$ for all values of n, a contradiction.

Hence this case does not arise.

***Case (iii):*** $m \equiv 2\ (mod\ 3)$.

If $n \equiv 0\ (mod\ 3)$, then $2mn + m + n \equiv 2(\ mod\ 3)$, a contradiction.

If $n \equiv 1\ (mod\ 3)$, then $2mn + m + n \equiv 1(mod\ 3)$, a contradiction.

Thus $n \equiv 2\ (mod\ 3)$.

Now, $< \{v_{0j}, v_{1j},...,v_{nj}\} > + \{v_{0j}\, v_{00}\} \cong K_{1,n+1}$ for all $1 \le j \le m$,

$< \{v_{i0}, v_{i1},...,v_{i(m-2)}\} > \cong K_{1,(m-2)}$ for all $1 \le i \le n$ and

$< \{v_{00}, v_{10},...,v_{n0}, v_{1(m-1)}, v_{2(m-1)},...,v_{n(m-1)}, v_{1m}, v_{2m},..., v_{nm}\} > \cong G'$

where $G'$ is the graph obtained by identifying one pendant vertex of each copy of $K_{1,3}$ in $nK_{1,3}$.

$$\text{Thus}\quad E(K_{1,m} \times K_{1,n}) = \underbrace{E(K_{1,(n+1)}) \cup ... \cup E(K_{1,(n+1)})}_{m\ times} \cup$$

$$\underbrace{E(K_{1,(m-2)}) \cup ... \cup E(K_{1,(m-2)})}_{m\ times} \cup E(G').$$

Since $n, m \equiv 2(mod\ 3)$, $K_{1,(n+1)}$ and $K_{1,(m-2)}$ areclaw decomposable.

Also, $G'$ is claw decomposable.

Hence $K_{1,m} - K_{1,n}$ is claw decomposable.

Conversely, suppose that $K_{1,m} \times K_{1,n}$ is claw decomposable.

Then $|E(K_{1,m} \times K_{1,n})| \equiv 0 \ (mod\ 3)$.

That is, $(m + 1)\, n + (n + 1)\, m \equiv 0 \ (mod\ 3)$.

That is, $2mn + m + n \equiv 0 \ (mod\ 3)$. □

**Remark 3.16.** $K_2 \times C_5$ is not claw decomposable.

**Theorem 3.17:** Let $n$ be even and $n \equiv 0 \ (mod\ 3)$ and $m \equiv 1 (mod\ 3)$. Then $K_{1,m} \times C_n$ is claw decomposable.

**Proof:** Let $V(K_{1,m}) = \{u_0, u_1, \ldots, u_m\}$ where $d(u_0) = m$ and

$$V(C_n) = \{v_1, v_2, \ldots, v_n\}.$$

Then $V(K_{1,m} \times C_n) = \{(u_i, v_j)/0 \le i \le m, 1 \le j \le n\}$.

Rename $(u_i, v_j) = v_{ji}$; $0 \le i \le m$, $1 \le j \le n$.

Assume that $n$ is even, $n \equiv 0 \ (mod\ 3)$ and $m \equiv 1 (mod\ 3)$.

**Claim:** $G_2 = K_{1,3} \times C_n - E(C_n)$ where $E(C_n)$ denotes the edges of the cycle $C_n$ corresponding to the central vertex is claw decomposable if $n$ is even and $n \equiv 0 (mod\ 3)$.

Then $G' = K_{1,3} \times C_n - \{v_{i0}v_{(i+1)0}, v_{10}v_{n0}/1 \le i \le n \le 1\}$.

Now, $< \{v_{ni}, v_{1i}, v_{2i}, v_{10}\} > \cong K_{1,3}$ for all $1 \le i \le 3$,

$< \{v_{i0}, v_{i1}, v_{i2}, v_{i3}\} >$ <"= $K_{1,3}$; $i \in \{2, 4, \ldots, n\}$,

$< \{v_{ij}, v_{(i+1)j}, v_{(i+2)j}, v_{(i+1)0}\} > \cong K_{1,3}$ for all $1 \le j \le 3$ and $i \in \{2, 4, \ldots, n - 2\}$.

Thus $E(G') = E(K_{1,3}) \cup E(K_{1,3}) \cup E(K_{1,3}) \cup \underbrace{E(K_{1,3}) \cup \ldots \cup E(K_{1,3})}_{\left(\frac{n}{2}\right) times} \cup$

$$\underbrace{E(K_{1,3}) \cup \ldots \cup E(K_{1,3})}_{3\left(\frac{n-2}{2}\right) times}.$$

Hence $G'$ is claw decomposable if $n$ is even and $n \equiv 0 \ (mod\ 3)$.

Since $m \equiv 1 (mod\ 3)$, $m = 3t + 1$; $t \in Z$.

Thus $E(K_{1,m} \times C_n) = E(K_2 \times C_n) \cup \underbrace{E(G') \cup \ldots \cup E(G')}_{t\ times}$.

By the Claim and Lemma 2.6, $G'^2$ and $K_2 \times C_n$ are claw decomposable.

Hence $K_{1,m} \times C_n$ is claw decomposable. □

**Theorem 3.18:** $K_1, \times K_n$ is claw decomposable if and only if $n \equiv 0$ (*mod* 3) or $mn + m + n \equiv 1$(*mod* 3).

**Proof:** Let $V(K_{1,\mathrm{m}}) = \{u_0, u_1, ..., u_m\}$ where $d(u_0) = m$ and $V(K_n) = \{v_1, v_2, ..., v_n\}$.

Then $V(K_1, m \times K_n) = \{(u_i, v_j)/0 \le i \le m, 1 \le j \le n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $0 \le i \le m$, $1 \le j \le n$.

Suppose that $n \equiv 0$ (*mod* 3) or $mn + m + n \equiv 1$(*mod* 3).

***Case (i):*** $n \equiv 0$ (*mod* 3)

**Subcase 1:** $m \equiv 0$ (*mod* 3)

Then $K_{1,m}$ is claw decomposable. Also, since $n \equiv 0$ (*mod* 3), by Theorem 2.3, $K_n$ is claw decomposable.

Hence by Theorem 3.1, $K_{1,m} \times K_n$ is claw decomposable.

**Subcase 2:** $m \equiv 1$(*mod* 3)

Now, < {v1j , v2j , . . . , vnj} > <“= Kn for all 0 d” j d” m “ 1,

< {vi0, vi1, . . . , vi(m”1)} > <“= K1,m”1 for all 1 d” i d” n and

$< \{v_{10}, v_{20}, ..., v_{n0}, v_{1m}, v_{2m}, ..., v_{nm}\} > - E(< \{v_{10}, v_{20}, ..., v_{n0}\} >) \cong K_n \circ K_1$.

Thus $E(K_{1,m} \times K_n) = \underbrace{E(K_n) \cup ... \cup E(K_n)}_{n\ times} \cup$

$$\underbrace{E(K_{1,m-1}) \cup ... \cup E(K_{1,m-1})}_{n\ times} \cup E(K_n o K_1).$$

By Lemma 2.7, $K_n$ o $K_1$ is claw decomposable. Since $n \equiv 0$ (*mod* 3), by Theorem 2.3, $K_n$ is claw decomposable. Since $m \equiv 1$(*mod* 3), $K_{1,m-1}$ is claw decomposable.

Hence $K_{1,m} \times K_n$ is claw decomposable.

**Subcase 3:** $m \equiv 2$(*mod* 3)

Now, $< \{v_{1j}, v_{2j}, ..., v_{nj}\} > \cong K_n$ for all $0 \le j \le m - 2$,

$< \{v_{i0}, v_{i1}, ..., v_{i(m-2)}\} > \cong K_{1}, _{m-2}$ for all $1 \le i \le n$,

$< \{v_{10}, v_{20},..., v_{n0}, v_{1(m-1)}, v_{2(m-1)},..., v_{n(m-1)}\}$

$> - E(< \{v_{10}, v_{20},..., v_{n0}\} >) \cong K_n \text{ o } K_1$ and

$< \{v_{10}, v_{20},..., v_{n0}, v_{1m}, v_{2m},..., v_{nm}\} > - E(< \{v_{10}, v_{20},...,v_{n0}\} >) \cong K_n \text{ o } K_1$.

Thus $E(K_{1,m} \times K_n) = \underbrace{E(K_n) \cup ... \cup E(K_n)}_{(m-1)\ times} \cup$

$$\underbrace{E(K_{1,m-2}) \cup ... \cup E(K_{1,m-2})}_{(m-1)\ times} \cup E(K_n o K_1) \cup E(K_n o K_1).$$

Since $n \equiv 0\ (mod\ 3)$, by Theorem 2.3, $K_n$ is claw decomposable. Since $m \equiv 2(mod\ 3)$, $K_{1,m-2}$ is claw decomposable. Also by Lemma 2.7, $K_n$ o $K_1$ is claw decomposable.

Hence $K_{1,m} \times K_n$ is claw decomposable.

***Case (ii):*** $mn + m + n \equiv 1(mod\ 3)$

**Subcase 1:** $m \equiv 0\ (mod\ 3)$

Since $mn + m + n \equiv 1(mod\ 3)$, $n \equiv 1(mod\ 3)$. Thus by Theorem 2.3, $K_n$ is claw decomposable. Also, $K_{1,m}$ is claw decomposable. Hence by Theorem 3.1, $K_{1,m} \times K_n$ is claw decomposable.

**Subcase 2:** $m \equiv 1(mod\ 3)$

Since $mn + m + n \equiv 1(mod\ 3)$, $n \equiv 0\ (mod\ 3)$. This case is already dealt in Subcase 2 of Case *(i)*.

**Subcase 3:** $m \equiv 2(mod\ 3)$

If $m \equiv 2(mod\ 3)$, then $mn + m + n \equiv 2(mod\ 3)$ for all values of $n$, a contradiction. Hence this case does not arise.

Hence in all the cases, $K_{1,m} \times K_n$ is claw decomposable.

Conversely, suppose that $K_{1,m} \times K_n$ is claw decomposable.

Then $|E(K_{1,m} \times K_n)| \equiv 0\ (mod\ 3)$. Thus, $(m+1)\dfrac{n(n-1)}{2} + mn \equiv 0\ (mod\ 3)$. which implies $\dfrac{n}{2}[mn + m + n - 1] \equiv 0\ (mod\ 3)$ and hence $n \equiv 0\ (mod\ 3)$ or $mn + m + n \equiv 1(mod\ 3)$.

## 4. CLAW DECOMPOSITION OF LEXICOGRAPHIC PRODUCT OF GRAPHS

In this section, we give sufficient condition for the lexicographic product of any graph $G$ with $\overline{K}_n$, $K_n$, $K_{m,n}$ and $K_2 \times K_n$ to be claw decomposable.

**Definition 4.1:** The lexicographic product of two graphs $G$ and $H$ is a graph, denoted by $G * H$, whose vertex set is $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent precisely if $gg' \in E(G)$, or $g = g'$ and $hh' \in E(H)$.

The other way of viewing $G * H$ is by replacing each vertex in $G$ by a copy of $H$ and two vertices in $G$ are adjacent if and only if there exists a complete bipartite subgraph with the corresponding vertices of $H$ as partite sets in $G * H$.

**Theorem 4.2:** Let $G$ be any non trivial graph. If $n \equiv 0$ $(mod\ 3)$, then $G * \overline{K}_n$ is claw decomposable.

**Proof:** Assume that n a" 0(mod 3).

Let $V(G) = \{v_1, v_2,..., v_k\}$ and $V(\overline{K}_n) = \{u_1, u_2,..., u_n\}$.

Then $V(G * \overline{K}_n) = \{(v_i, u_j)/1 \leq i \leq k$ and $1 \leq j \leq n\}$.

Rename $(v_i, u_j) = v_{ji}$; $1 \leq i \leq k$ and $1 \leq j \leq n$.

Now, for each $v_iv_j \in E(G)$, $< \{v_{1i}, v_{2i},..., v_{ni}, v_{1j}, v_{2j},...,v_{nj}\} > \cong K_{n,n}$.

Thus, $$E(G * \overline{K}_n) = \underbrace{E(K_{n,n}) \cup ... \cup E(K_{n,n})}_{|E(G)|\ times}.$$

Since $n \equiv 0$ $(mod\ 3)$, by Theorem 2.4, $K_{n,n}$ is claw decomposable.

Hence $G * \overline{K}_n$ is claw decomposable. □

**Theorem 4.3:** Let $G$ be any non trivial graph. If $n > 3$ and $n \equiv 0$ $(mod\ 3)$, then $G * K_n$ is claw decomposable.

**Proof:** Assume that $n > 3$ and $n \equiv 0$ $(mod\ 3)$.

Let $V(G) = \{v_1, v_2,..., v_k\}$ and $V(K_n) = \{u_1, u_2,..., u_n\}$.

Then $V(G * K_n) = \{(v_i, u_j)/1 \leq i \leq k$ and $1 \leq j \leq n\}$.

Rename $(v_i, u_j) = v_{ji}$; $1 \leq i \leq k$ and $1 \leq j \leq n$.

Now, $< \{v_{1i}, v_{2i},..., v_{ni}\} > \cong K_n$ for all $1 \leq i \leq k$.

Also, for each $v_iv_j \in E(G)$,

$$< \{v_{1i}, v_{2i},...,v_{ni}, v_{1j}, v_{2j},...,v_{nj}\} > -E(< \{v_{1i}, v_{2i},..., v_{ni}\} >)$$
$$- E(< \{v_{1j}, v_{2j},..., v_{nj}\} >) \cong K_{n,n}.$$

Thus, $E(G * K_n) = \underbrace{E(K_n) \cup ... \cup E(K_n)}_{k\ times} \cup \underbrace{E(K_{n,n}) \cup ... \cup E(K_{n,n})}_{|E(G)|\ times}.$

Since $n \equiv 0(mod\ 3)$, by Theorem 2.3 and 2.4, $K_n$ and $K_{n,n}$ are claw decomposable.

Hence $G * K_n$ is claw decomposable. □

**Theorem 4.4:** Let $G$ be any non trivial graph. If $m \equiv 0\ (mod\ 3)$ and $n \equiv 0\ (mod\ 3)$, then $G * K_{m,n}$ is claw decomposable.

**Proof:** Assume that $m \equiv 0\ (mod\ 3)$ and $n \equiv 0\ (mod\ 3)$.

Let $V(G) = \{v_1, v_2,...,v_k\}$ and $V(K_{m,n}) = \{u_1, u_2,..., u_m, w_1, w_2,...,w_n\}$ where $d(u_i) = n$ for all $1 \le i \le m$ and $d(w_j) = m$ for all $1 \le j \le n$.

Then $V(G \times K_{m,n}) = \{(v_i, u_j), (v_i, w_l)/1 \le i \le k, 1 \le j \le m, 1 \le 1 \le n\}$.

Rename $(v_i, u_j) = u_{ji}$ and $(v_i,w_l) = w_{li}$ for all $1 \le i \le k$, $1 \le j \le m$, $1 \le 1 \le n$.

Now for each $v_iv_j \in E(G)$,

$$< \{u_{1i}, u_{2i},..., u_{mi}, w_{1i}, w_{2i},...,w_{ni}, u_{1j}, u_{2j},..., u_{mj}, w_{1j}, w_{2j},...,w_{nj}\}$$
$$> - E(< \{u_{1i}, u_{2i},..., u_{mi}, w_{1i}, w_{2i},...,w_{ni}\} >)$$
$$- E(< \{u_{1j}, u_{2j},..., u_{mj}, w_{1j}, w_{2j},...,w_{nj}\} >) \cong Km + n, m + n \text{ and}$$
$$< \{u_{1i}, u_{2i},..., u_{mi}, w_{1i}, w_{2i},...,w_{ni}\} > \cong K_{m,n} \text{ for all } 1 \le i \le k.$$

Thus, $E(G * K_{m,n}) = \underbrace{E(K_{m,n}) \cup ... \cup E(K_{m,n})}_{k\ times} \cup \underbrace{E(K_{m+n,m+n}) \cup ... \cup E(K_{m+n,m+n})}_{|E(G)|\ times}$

Since $m \equiv 0(mod\ 3)$ and $n \equiv 0\ (mod\ 3)$, by Theorem 2.4, $K_{m,n}$ and $K_{m+n,m+n}$ are claw decomposable.

Hence $G * K_{m,n}$ is claw decomposable.

**Theorem 4.5:** Let $G$ be any non trivial graph. If $n > 3$ and $n \equiv 0\ (mod\ 3)$, then $G * [K_2 \times K_n]$ is claw decomposable.

**Proof:** Assume that $n > 3$ and $n \equiv 0\ (mod\ 3)$.

Let $V(G) = \{w_1,w_2,...,w_k\}$, $V(K_2 \times K_n) = \{v_1, v_2,..., v_n, u_1, u_2,...,u_n\}$ and $E(K_2 \times K_n) = \{v_iv_j, u_iu_j, u_iv_i/1 \le i, j \le n, i\ 6 = j\}$.

Then $V(G * [K_2 \times K_n]) = \{(w_i, v_j), (w_i, u_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(w_i, v_j) = v_{ji}$ and $(w_i, u_j) = u_{ji}$ for all $1 \leq i$ d" $k$, $1 \leq j \leq n$.

Now, $< \{v_{1i}, v_{2i},..., v_{ni}, u_{1i}, u_{2i},..., u_{ni}\} > \cong K_2 \times K_n$ for all $1 \leq i \leq k$.

Also, for each $w_i w_j \in E(G)$,

$$< \{v_{1i}, v_{2i},..., v_{ni}, u_{1i}, u_{2i},..., u_{ni}, v_{1j}, v_{2j},..., v_{nj}, u_{1j}, u_{2j},..., u_{nj}\} > - E(< \{v_{1i}, v_{2i},...., v_{ni}, u_{1i}, u_{2i},...., u_{ni}\} >) - E(< \{v_{1j}, v_{2j},...,v_{nj}, u_{1j}, u_{2j},..., u_{nj}\} >) \cong K_{2n,2n}.$$

Thus $E(G * [K_2 \times K_n]) = \underbrace{E(K_2 \times K_n) \cup ... \cup E(K_2 \times K_n)}_{k\ times} \cup \underbrace{E(K_{2n,2n}) \cup ... \cup E(K_{2n,2n})}_{|E(G)|\ times}$.

Since $n > 3$ and $n \equiv 0$ (*mod* 3), by Lemma 2.8, $K_2 \times K_n$ is claw decomposable. Also, by Theorem 2.4, $K_{2n,2n}$ is claw decomposable.

Hence $G * [K_2 \times K_n]$ is claw decomposable. □

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