International Journal of Economic Research

ISSN : 0972-9380

available at http: www.serialsjournals.com

Volume 14 • Number 15 • 2017

Adjusting Beliefs via Transformed Fuzzy Returns

Tanarat Rattanadamrongaksorn[1†], Duangthip Sirikanchanarak[1,3], Jirakom Sirisrisakulchai[1,2] and Songsak Sriboonchitta[1,2]

[1] *Faculty of Economics, Chiang Mai University, Chiang Mai 50200, Thailand*
[2] *Centre of Excellence in Econometrics, Chiang Mai University, Chiang Mai 50200, Thailand*
[3] *Bank of Thailand, Bangkok, 10200, Thailand*

Abstract: Change in structural level can cause shifts in the properties of data and, therefore, imposes needs in adjusting belief on the inference. In this study, we consider such a problem in the estimation of financial series. Under Bayesian framework, we propose the idea of combining human approximations and historical observations via transforming fuzzy returns into priors. Fuzzy return has been reintroduced and transformed as an external piece of evidence to the process of Bayesian inference. In addition to the concurrent work [1], this hybrid-prior approach reduces a step of transformation but increases the compatibility with probability theory and, as a result, could be implemented with ease. In our experiment, we selected five samples of financial securities from different markets for examining the proposed methodologies. The problem is multi-dimensional and analytically intractable but conveniently solved by the Markov-chain Monte-Carlo approximation. Both alternatives have been compared and yielded the quite similar results but traded off in the computational efforts. They indicate the importance on the predictive impacts from expert opinions setting baseline on the commonly-used Maximum Likelihood Estimation method.

Keywords: Bayesian inference, data fusion, possibility-probability transformation, fuzzy prior, transformed prior, MCMC.

1. INTRODUCTION

Political policy, administrative style, technological constraint, etc. are the examples of the factors that affect economic conditions and even almost everything. Nowadays, the world is more interconnected so that these dynamics grow bigger in magnitude and higher in velocity and more frequently cause changes in structural level than ever.

Data inference is considered as one of the most fundamental topics in every statistics-intensive branches – Economics is definitely included. The structural change has impacts on the underlying natures of the problem that makes shifts in the parameters. The properties of data can move from the previous location thus the available data are no longer viable and the number of observations have not yet sufficiently existed. As a consequent, the need for incorporating the other kinds of knowledge is inevitable.

Fortunately, even in worst case scenario, there always exists human's *gut feeling* which is a conclusive opinion from expert. The expert opinion is based on the person's education, experience, preference, background, and etc. that can be expressed in linguistic terms. However, human judgement usually contains uncertainties from the lack of complete information leading to imprecise decision. The concept of expert opinion is too conservative because even expert tends to stay on the safe side. Utilizing only either of them may not be a good idea as well. The compromise between the gut feeling and the existing data might be the most proper way to go. While the subjective data can give the approximated insight into the future, the past data still constitute some common characters. Therefore, the best available solution is to combine both thus we propose the data fusion within the framework of Bayesian statistics.

Unlike the Frequentist, the Bayesian is not in favorite probably and partly because the practical implementation confronts the challenging task in choosing a right prior which is the initial belief of the data. We raise an argument that the traditional prior is not comprehensive to user and does not reflect real-life situation. The common selection of prior, for example Beta family or Inverse-Wishart distribution, sounds ridiculous to the person in the field. Instead, we believe the proper prior should be extracted naturally from the qualified personnel and represented verbally by linguistic terms such as "at most", "from-to", "least possible", and so on. This leads to the concept of fuzzy set theory.

Fuzzy set theory is the generalization of set theory to represent the non-random uncertainty. Fuzzy set is characterized by a set of pairs - the values of element and its possibility, $[\theta, \pi(\theta)]$. The element, θ, is a set of parameters in consideration while its possibility is the possibility of occurring for the specific value of the parameters. Fuzzy number is the representation of quantity that cannot be described precisely due to the incomplete knowledge.

In order to combine the different sources of data, the possibility-to-probability transformation comes into play. Fuzzy return is transformed into the transformed return, which is the probability distribution of the return, and input as a prior to the Bayesian statistical process. In our parallel research [1], the transformation is activated twice on fuzzy price and on fuzzy deviation (similar notions to fuzzy return). These fuzzy priors are more comprehensive comparing to the traditional priors and sufficiently intuitive to the ordinary users. However, the computation is quite expensive because of its incompatibility to probability theory. In this work, therefore to avoid such a complex task, we suggest *alternatively* the single transformation exclusively on the fuzzy returns.

2. PROPOSED METHODOLOGY

Traditionally in the context of Bayesian statistics, the commonly-used prior or initial belief of data distribution for mean is beta-family and that for standard deviation is Inverse-Wishart distribution that come out of the thin air. We propose a more solid approach to generate priors through the fuzzy set theory as follows:

Step 1: Translate the linguistic terms from expert opinions to mathematical representation by fuzzy number;

Step 2: Transform the fuzzy returns from the first step into priors for the mean of asset return;

Step 3: Calculate conditional probabilities of standard deviation;

Step 4: Find joint prior probability by the rule of conditional probability;

Step 5: Compute posterior probability by Bayes' theorem.

In the following Sections we shall discuss each step in the necessary concepts in this paper *i.e.* the definitions and conditions of data generating mechanism (Section 2.1), the description and acquisition of fuzzy returns (Section 2.2), the principles and process of possibility-to-probability transformation (Section 2.3), the simplified version of the prior for standard deviation (Section 2.4), Bayes theorem (Section 2.5), the predictive probability for data inference (Section 2.6), and the component-wise Metropolis-Hasting algorithm for approximation of the distribution (Section 2.7).

2.1. Data Generating Mechanism

In this section we describe the condition on data model in our study by considering the data generating mechanism that follows the normal distribution:

$$X \sim N(\mu, \sigma^2), \tag{1}$$

where μ, σ^2, and σ represents mean, variance, and standard deviation respectively. The parameters of mean and standard deviation, $\theta = (\mu, \sigma)$, form the distribution of data, X, and contribute to the building block of data inference in this research.

2.2. Fuzzy Returns

Inferencing financial data is commonly based on the Return On Asset (ROA) or, just, the return. Fuzzy return is an estimate of the ROA incorporated with impression because expert opinion usually contains uncertainty so that the precise estimate for ROA is not possible in most case and can be depicted by fuzzy number. Fuzzy number is defined as a mapping of a value of an element from real number to any number in the unit domain on the scale of possibility which is equivalent to the scale of degree of membership in the original concept, $A : R \rightarrow [0, 1]$. The possibility can be any real number from zero to one meaning the least possible to the most possible, respectively. In the previous research [2], a set of fuzzy returns has been proposed but the two commonly used shapes will be revisited.

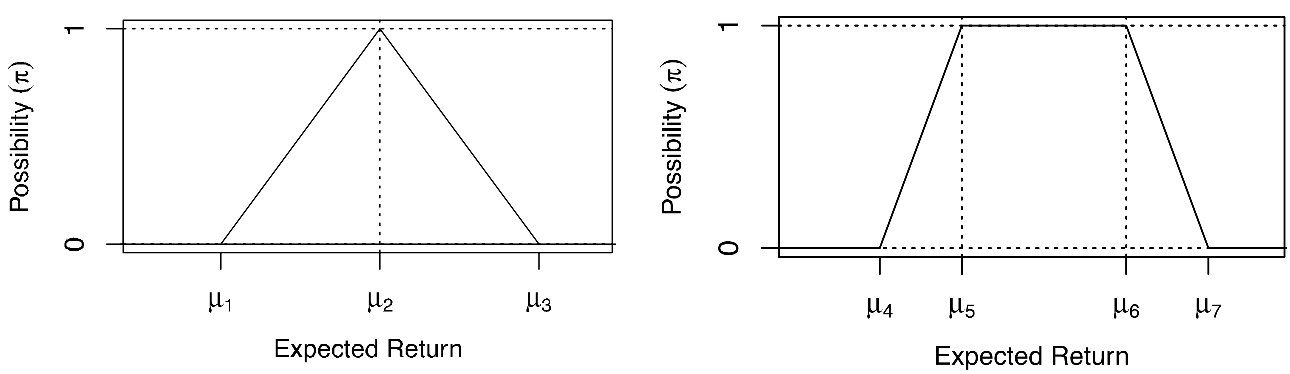

Figure 1: The triangular (left) and trapezoidal (right) fuzzy returns.

The triangular is the three-point estimation of the expected return *i.e.* the most pessimistic (μ_1), the most possible (μ_2) and the most optimistic (μ_3). The trapezoidal shape is the interval-based estimation of the expected return *i.e.* the largest possible ($\mu_4 - \mu_7$) and the most possibility ($\mu_5 - \mu_6$). The representation of the fuzzy return is:

$$N = (a, b, c, d) \tag{2}$$

whose possibilities (0, 1, 1, 0) are omitted from the writing and in the case of triangle b and c are equivalent. The linear approximation for the value in between is common.

Fuzzy return can be used as a prior for a simple reason. We believe that the prior could be acquired more easily and naturally by fuzzy number than by probabilistic distribution. For instance, a person is looking for a stock to invest. He consults one of his friends who has been intensively investing in the stock markets for the 30 years. This trading expert might give the person either 1) the ROA of the stock has its mean at 10 per cent with variance at 15.25 per cent or 2) the ROA of another security is largest possible between 5 to 22.5 per cent and most possible between 10 to 15. Definitely, the second choice is obvious to most human beings. The appropriate prior could be derived by the tailored questions to the expert. For example, the prior for the expectation of ROA may be obtained by interviewing the qualified personnel with the question "What is your opinion for the expected returns of the particular asset?". The answer is the possible mean of ROA and is represented by fuzzy number *i.e.* fuzzy prior.

2.3. Possibility-to-Probability Transformation

In this study, we consider exclusively the transformation from possibility to probability due to the loss of information from the inverse operation [3].

The transformation we employ is based on three principles [3]:

1. Probability-possibility consistency specifying that information is retained or added through the process;
2. Preference preservation that maintains the ordering of possibility and probability; and
3. Principle of indifference stating that the result contains as much uncertainty as possible.

It is noted that several other methods of transformations are also available but based on the different assumptions.

The procedure of transformation starts with selecting the alpha-cut fuzzy possibility, π_a, randomly from the alpha-cut fuzzy number, $A_\alpha = \{\theta \in \Omega | \pi(\theta) \geq \alpha\}$, whose alpha-cut possibility, α, is selected also randomly from the unit uniform distribution, $Unif(0, 1)$. The transformed probability, $p_i(\theta)$, can be calculated by the summation of possibility differences, $\alpha_j(\theta) - \pi_{j+1}(\theta)$, from the considered, $j = i$, to the one before the last order, $T - 1$, and divided by the transforming order, j. Mathematically, the transformed probability is calculated by:

$$P_i(\theta) = \sum_{j=i}^{T-1} \frac{\pi_j(\theta) - \pi_{j+1}(\theta)}{j}, \tag{3}$$

where $\max(\pi_a) \geq \pi_1 \geq \pi_2 \geq \dots\pi_j \dots \geq \pi_{T-1} \geq \pi_T = \min(\pi_a)$. The results of these transformation are a set of probabilities with decelerating decrease in probabilities leveraging on both sides of distribution. For a better comprehension, the process is also illustrated by the pseudo codes in Algorithm 1.

Algorithm 1
Possibility-to-Probability Transformation

For Selection $\alpha = 1$ to T do

Alpha-cut:

$\alpha \sim Unif(0, 1)$

$A_\alpha = \{\theta \in \Omega | \theta \geq \alpha\}$

$\theta_\alpha \sim Unif(\min(A_\alpha), \max(A_\alpha))$

$\pi_a = \{\pi(\theta) | \theta = \theta_\alpha\}$

end for

Transforming order j:

$\max(\pi_\alpha) = \alpha_1 \geq \pi_2 \geq \ldots \pi_j \ldots \geq \pi_{T-1} \geq \pi_T = \min(\pi_a)$

for Transformation $i = 1$ to $T - 1$ do

Transformed probability: $P_i(\theta) = \sum_{j=i}^{T-1} \left(\frac{\pi_j(\theta) - \pi_{j+1}(\theta)}{j} \right)$

end for

2.4. Prior for Standard Deviation

The distribution of standard deviation could be obtained directly by:

$$\sigma | \mu = \sqrt{\frac{\sum (X - \mu)^2}{m}}, \tag{4}$$

where m is the number of observations. The joint probability is computed by:

$$P(\mu, \sigma) = P(\sigma|\mu)P(\mu), \tag{5}$$

where $P(\mu)$ is the probability of mean obtained from the transformation in Section 2.3. This joint probability may be considered as a hybrid prior in a sense that it is not pure external evidence.

2.5 Bayes' Theorem

While the Frequentist is used to find the asymptotically frequency-based properties, the Bayesian is used to describe the degree of belief. Although there are many debates back and forth between the two schools of philosophy, we believe they reflect different things with some parts overlapped but they are not substitutes to each other. It is the user's choice of application by considering the suitability and preferences.

Bayesian approach is attractive in its ability in combining the observed evidences with the external knowledge. The external knowledge is referred to the probability of interested parameters before seeing data or *Prior*, $P(\theta)$. The observed evidence is simply probability of data or *Likelihood*, $P(X|\theta)$. The combination of both information results in the compromise of the two and is called the *posterior*, $P(\theta|X)$ and normalized by the marginal likelihood, $P(X)$. The relationship is expressed by the so-called Bayes' theorem:

$$P(\theta | X) = \frac{P(\theta)P(X | \theta)}{P(X)}. \quad ???$$

The application of Bayesian statistics center around the Bayes' theorem but non-parametric Bayesian is not included, however. In Bayesian scheme, the parameter of interest is considered as a random variable, rather than fixed as in Frequentist's sense.

2.6. Predictive Probability

Often, the posterior is not sufficient nor no the objective of the problem. To be able to compare with the traditional method, we need the predictive probability that could be received iteratively by integrating out all parameters in every chain of simulation (described in the next section):

$$P(X_{t+1}) = \int_{\theta} P(\theta \mid X)\, d\theta, \qquad ???$$

where $P(X_{t+1})$ is the predictive probability or the probability of the predicted data.

2.7. Component-wise Metropolis-Hastings Algorithm

In our experiment, we choose the Metropolis-Hastings algorithm (MH) which is a class of Markov-chain Monte Carlo simulation (MCMC or MC^2) for approximating the distributions. The MH generates, tests, and stays exploring in high probability region the observation by evaluating the present observation based on the previous one. The component-wise MH shown in the Algorithm 2 is our implementation of MCMC that avoids the difficulties in multivariate problem by determining only single parameter in each iteration. The algorithm starts with assigning an initial value from the target distribution, $\theta_0^{(k)} \sim \Gamma(\Theta)$, to each parameter, $\theta^{(k)} \in \Theta$ where (k) represents the order of the parameters and the target distribution, $\Gamma(\Theta)$, is the expected distribution of the posterior. For each parameter, $\theta^{(k)} \in \theta$, a sample called the proposal, θ', is drawn from the desired sampling or the proposal distribution, $\tau^{(k)}\left(\theta_i^{(k)} \mid \theta_{i-1}^{(k)}\right)$. Additionally, the Θ' indicates

Algorithm 2
Component-wise Metropolis-Hastings algorithm

Initialization $\theta_0^{(k)} \sim \Gamma(k)(\theta),\ \forall\theta \in \theta$
for Simulations $i = 1$ to M do
    Component-wise:
    for all Parameters $\theta^{(k)} \in \theta$ do
        Proposal: $\theta' \sim \tau(\theta_i^{(k)} | \theta_{i-1}^{(k)})$
        Acceptance probability: $\Lambda(\theta', \theta_{i-1}^{(k)}) = \min\left(1, \frac{\Gamma(\theta')}{\Gamma(\theta_{i-1}^{(k)})} \frac{\tau(\theta_{i-1}^{(k)} | \theta')}{\tau(\theta' | \theta_{i-1}^{(k)})}\right)$
        Random probability: $u \sim Unif(0, 1)$
        If $\Lambda > u$ then
            Accept: $\theta_i^{(k)} \leftarrow \theta'$
        else
            Reject: $\theta_i^{(k)} \leftarrow \theta_{i-1}^{(k)}$
        end for
    end for
end for

the set of parameters with the considered proposal at parameter k. The test relies on the result of the comparison between the acceptance, $\Lambda(\theta' \mid \theta_{i-1}^{(k-1)})$, and the random probability, u. In case the acceptance probability is greater than the random probability, we accept the proposal for the present observation otherwise reject the proposal and store the previous observation instead. This process iterates for k parameters and M sets of simulations.

3. EXAMPLES

In this section, the proposed methodology were experimented with the financial data from both corresponding expert opinions and historical observations. The following sections contain the discussions on the descriptions and basic properties of the data (Section 3.1), the previously proposed transformation in the concurrent research (Section 3.2). and the graphical and numerical results (Section 3.3).

3.1. Data

The data used in the example are divided into two parts: expert opinions and historical observations. The subjective data of fuzzy returns were extracted from the varieties of sources *i.e.* consultant reports, business newsletters, dissertations, and interviews. The objective data are the corresponding sample assets selected from different markets *i.e.* Forex, Gold, Rice, Foreign Stock, and Domestic Stock ranging from year 2001 to 2016 and freely available from the internet. The basic properties are summarized in Table 1.

Table 1
Basic properties of fuzzy data and statistical data.

Asset	*Most Pessimistic*	*Most Possible*	*Most Positive*	*Centroid*	*Mean*	*Variance*	*Min.*	*Max.*	*C.I.*	*Size*
Forex	1.03	[1.59,1.69]	2.33	1.67	0.00	0.35	-6.04	6.48	[-1.1,1.06]	2294
Gold	-2.61	[2.99,3.66]	7.74	2.83	0.03	1.31	-8.91	9.55	[-2.34,2.32]	4173
Rice	-8.65	[2.69,11.58]	53.18	16.59	0.01	2.87	-24.45	16.25	[-3.25,3.44]	4173
Foreign stock	2.63	[17.29,17.29]	30.07	16.67	0.04	4.06	-11.39	17.00	[-3.98,4.25]	2607
Domestic stock	25.49	[28.79,28.79]	37.18	30.49	0.06	3.75	-8.77	10.98	[-4.04,3.98]	725

Remark: C.I. is the abbreviation for Confidence Interval

3.2. Concurrent Transformations

This work is an alternative development of fuzzy priors under Bayesian framework. The concurrent attempt [1] relies on fuzzy deviation, $\tilde{\sigma}$, that is obtained by the arithmetic operation on fuzzy returns:

$$\tilde{\sigma} = (a - \tilde{\mu}, b - \tilde{\mu}, c - \tilde{\mu}, d - \tilde{\mu}) \tag{8}$$

where $\tilde{x}$ is expectation of fuzzy number obtained by:

$$\tilde{\mu} = \frac{\int \mu\pi(\mu)d\mu}{\int \pi(\mu)\,d\mu}, \tag{9}$$

that is the centroid computed from the area moment against the possibility axis. The fuzzy deviation is the possible deviation from its expectation. Clearly fuzzy deviation is much larger than standard deviation and negative deviation is possible. Even though the interpretation of fuzzy deviation is more useful in that sense, it presents some difficulties in the implementation when applying to Bayesian inference. To be specific, the prior of standard deviation obtained by this method comes with the incompatibility with calculation of likelihood. As a result, the negative side of fuzzy deviation needs another manipulation and complicate the tuning task of MCMC.

We shall refer this previously proposed method as a double transformation and the currently proposed method as a single transformation due to the number of transformation from possibility to probability they utilize. The result of this transformation will also be included for the purpose of comparison.

3.3. Results

Tables 2-5 and Figures 2-6 contains the numerical and graphical results of the same sets of assets. The parameters we report include mean, variance, minimum, maximum, and credible interval (C.R.) but our discussions will pay attention more on the inference by the mean.

The graphical results of the fuzzy (dashed lines) and transformed (solid lines) priors of the expected returns are depicted in the left column of Figure 2. The transformations retain the values of highest and lowest possibilities but the values between them are slightly lowered. Thus the preservation of the distribution shapes are easily conceived. There are, however, the influences of the initial shapes that govern the final result of transformation. The trapezoidal fuzzy returns; *i.e.* forex, gold, and rice; induce less changes in probabilities than those in triangular fuzzy returns; *i.e.* foreign and domestic stocks. There are smaller gaps between the possibilities and probabilities in the trapezoidal because more high possibilities can induced higher probabilities and vice versa.

On the right column of Figure 2, the distributions of standard deviations (solid lines) and those of transformed deviations (dashed lines) are compared. The former are from the single transformations and the latter are from the double transformation. The resultant distributions of both processes look similar with different means. In addition, the newly proposed method yields only the positive outcomes thus is compatible to probability theory and the implementation is much easier. For instance, the minimums of the standard and transformed deviation for the forex are 1.20 and -0.01 respectively. The reader should keep in mind that these priors are different in their interpretations. The first parameter indicates only the variation of the return of the rice but the second also indicates the outcome of the investment, 1.2 per cent of the minimal possible loss for the case investing in the rice, for example.

For the joint distributions, *i.e.* priors (Table 3 and Figure 4), likelihood (Table 4 and Figure 5), and posteriors (Table 5 and Figure 6), it would be less difficult and more comprehensive to have our discussions on marginal probabilities. In Table 2 and Figure 3, the distributions of both expected returns and deviations for the joint priors are less smooth by the effects of the simulations but not much different from the initial shapes. This is because our method does not depend on the parametric distribution of prior.

Interestingly, the distributions of likelihoods in the case of double transformation produce more variance than those in the case of single transformation which is the main reason we propose this specification for acquiring the standard deviation. In the sample of rice in Figure 4 and Table 3, the variance of the standard deviation is 1527.74 for the single transformation but 0.00 of the double transformation — as a

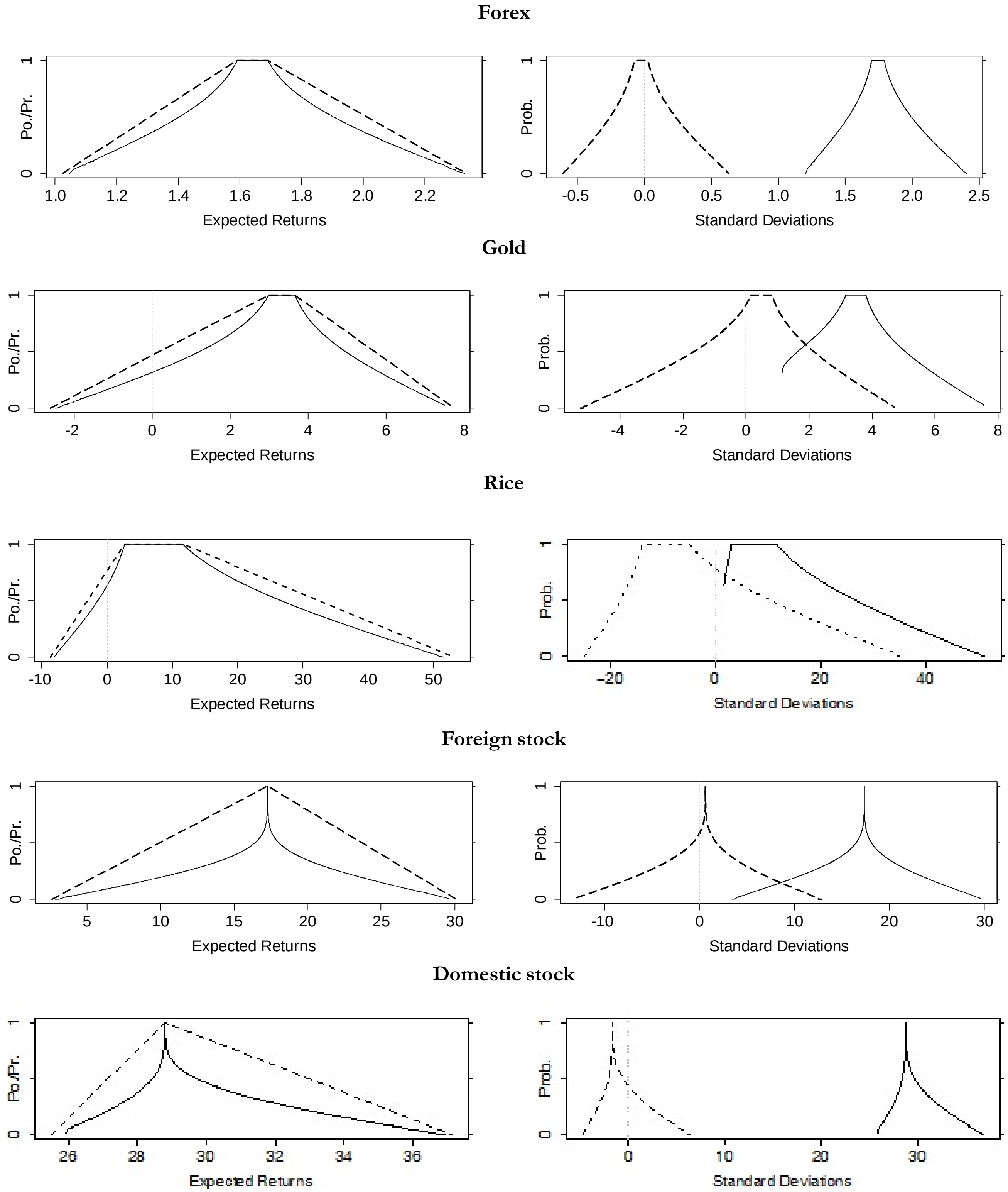

Figure 2: Fuzzy (left, dashed) vs transformed (left, solid) returns and priors for the standard deviations from the single transformation (right, solid) vs Transformed deviation from the double transformations (right, dashed)

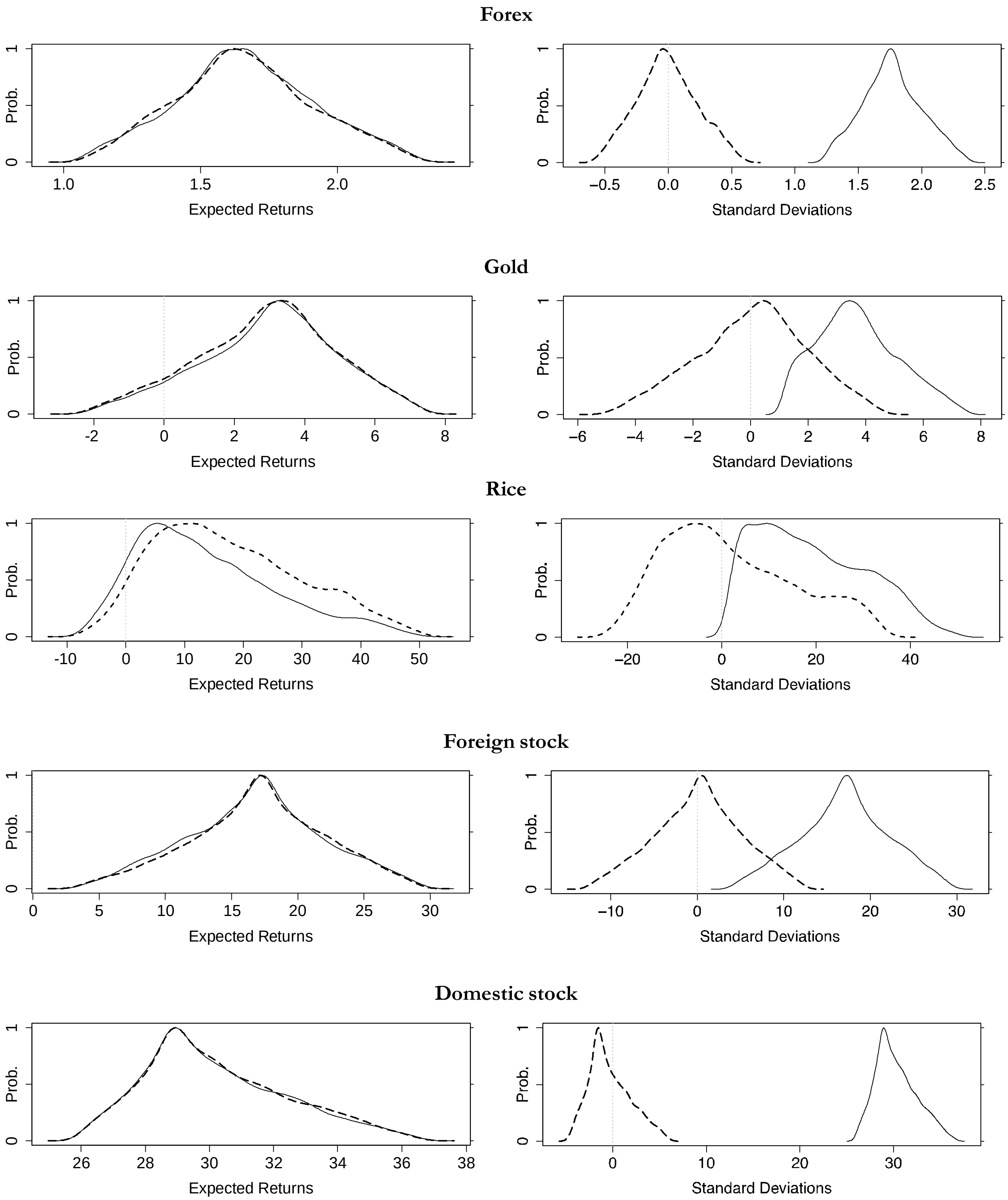

Figure 3: Marginal distribution of joint prior for expected returns by the single (left, solid) vs double (left, dashed) transformations; and Marginal distribution of joint prior for standard deviations by the single (right, solid) vs double (right, dashed) transformations

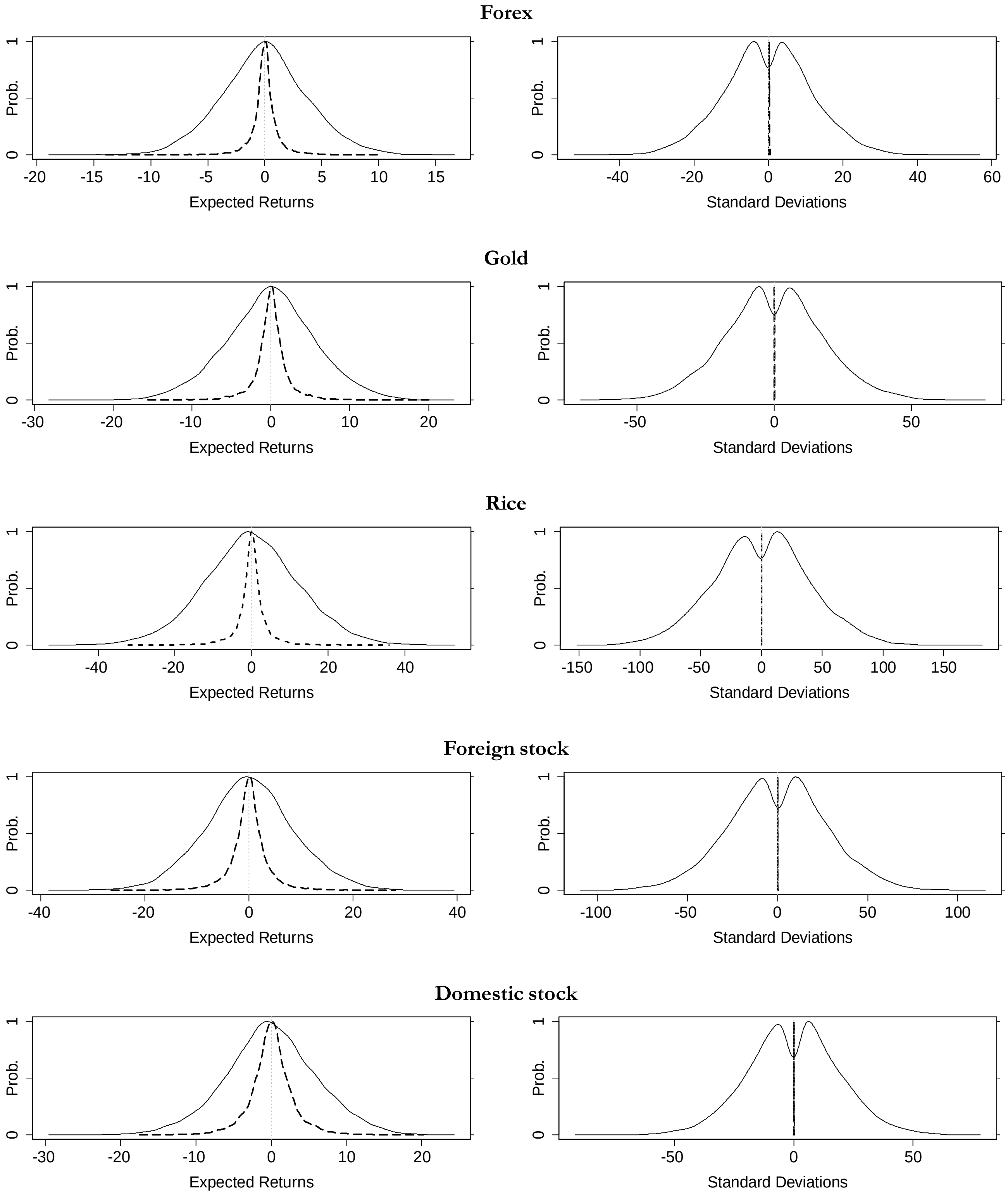

Figure 4: Marginal distribution of likelihoods for expected returns by the single (left, solid) vs double (left, dashed) transformations; and Marginal distribution of likelihoods for standard deviations by the single (right, solid) vs double (right, dashed) transformations

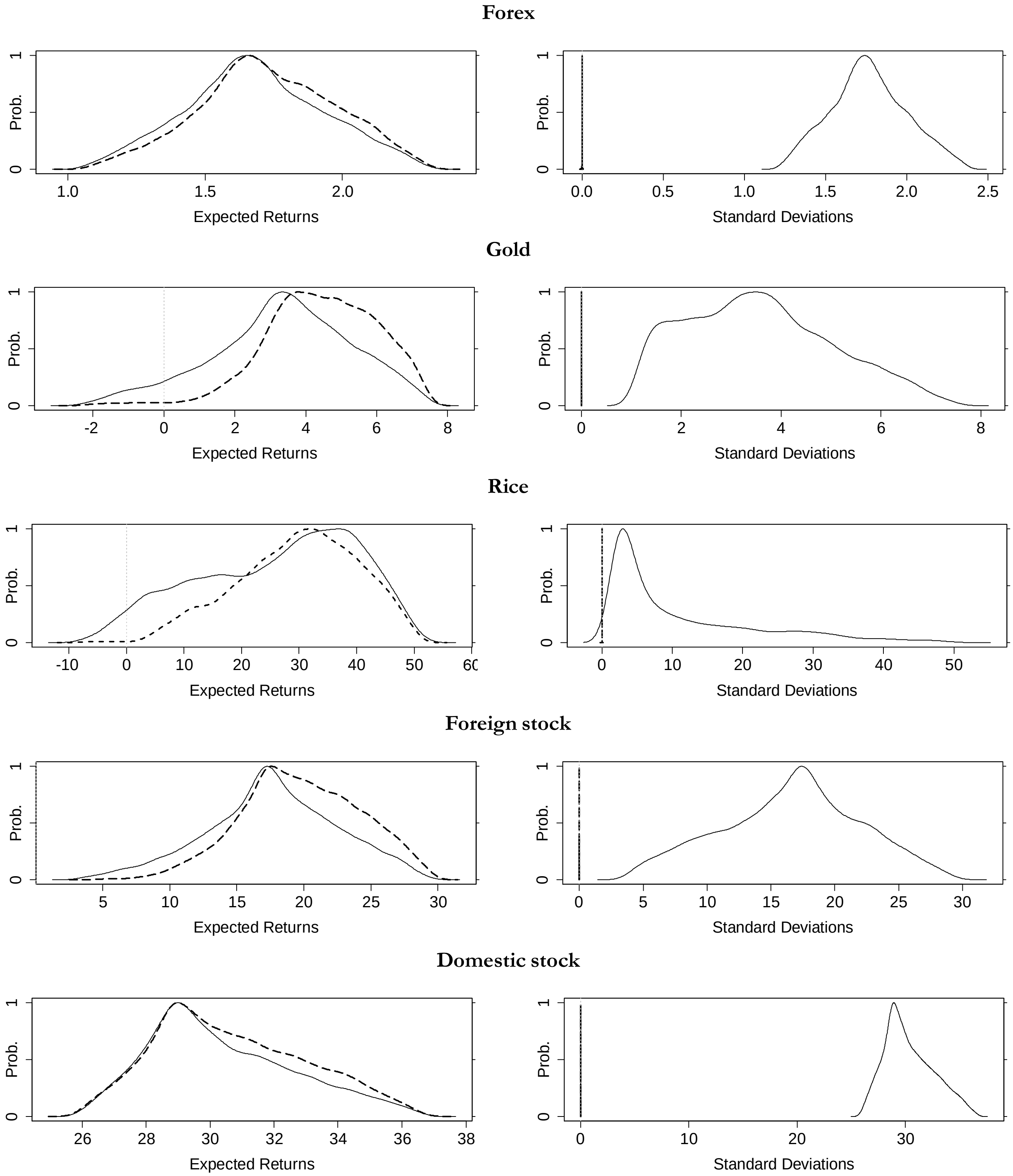

Figure 5: Marginal distribution of posterior for expected returns by the single (left, solid) vs double (left, dashed) transformations; and Marginal distribution of posterior for standard deviations by the single (right, solid) vs double (right, dashed) transformations

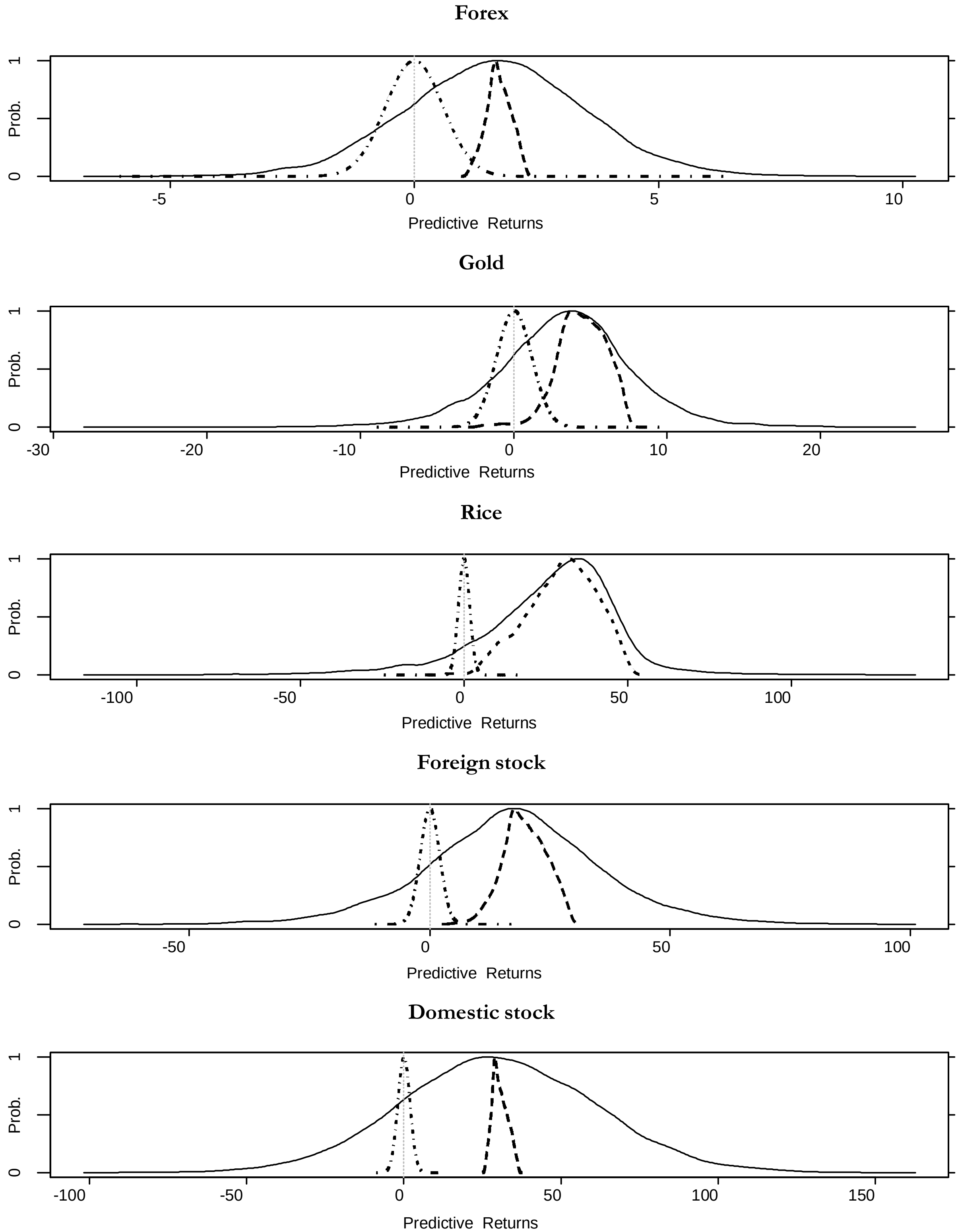

Figure 6: Figure 1: Predictive distributions by the single transformation (solid), double transformation (dashed), and MLE (dot-dashed)

Table 2
Statistics of priors' marginal distributions for expected returns and fuzzy deviations

		Expected Return							*Standard and Transformed Deviation*						
Asset	*Transform. Method*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*
Forex	Single	1.67	0.06	1.05	2.32	1.66	1.62	[1.17,2.18]	1.77	0.06	1.20	2.40	1.76	1.79	[1.32,2.24]
	Double	1.66	0.06	1.06	2.32	1.65	1.72	[1.2,2.17]	–0.01	0.06	–0.60	0.62	–0.01	–0.10	[–0.46,0.47]
Gold	Single	3.00	3.74	–2.41	7.48	3.14	5.46	[–1.07,6.55]	3.71	2.00	1.15	7.54	3.59	3.34	[1.35,6.7]
	Double	2.91	3.80	–2.37	7.47	3.04	3.84	[–1.1,6.5]	0.06	3.64	–5.12	4.65	0.17	0.69	[–3.84,3.64]
Rice	Single	13.70	141.26	–8.06	50.71	11.44	13.70	[–3.82,41.47]	19.04	133.53	1.70	50.51	17.33	2.04	[2.47,42.26]
	Double	17.53	153.24	–7.92	50.61	15.78	–7.83	[–1.93,43.55]	1.76	189.24	–24.69	35.05	–0.73	1.76	[–19.2,30.08]
Foreign Stock	Single	16.79	26.85	3.39	29.58	17.03	16.79	[6.6,26.92]	17.05	26.82	3.84	29.61	17.17	17.03	[6.65,27.04]
	Double	17.01	26.09	3.29	29.61	17.15	17.01	[6.4,26.72]	0.14	25.14	–12.86	12.43	0.31	0.14	[–9.88,9.83]
Domestic Stock	Single	30.21	4.97	25.93	36.66	29.80	28.23	[26.62,35.19]	30.28	4.77	25.94	36.65	29.92	25.98	[26.73,35.08]
	Double	30.26	5.03	25.94	36.66	29.86	28.29	[26.64,35.13]	–0.29	4.98	–4.79	6.34	–0.72	–2.47	[–3.91,4.7]

Remark: C.R. is the abbreviation for Credible Interval.

Table 3
Statistics of likelihoods' marginal distributions for expected returns and standard deviations

		Expected Return							*Standard Deviation*						
Asset	*Transform. Method*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*
Forex	Single	–0.01	14.54	–17.39	15.07	0.00	–0.06	[–7.49,7.72]	–0.08	137.45	–47.34	51.86	–0.39	0.51	[–23.43,22.8]
	Double	0.00	1.69	–13.64	9.59	0.00	0.20	[–2.67,2.81]	0.16	0.00	0.00	0.41	0.16	0.37	[0.05,0.31]
Gold	Single	0.01	32.08	–25.86	20.91	0.05	–0.24	[–11.45,11.36]	0.01	311.20	–63.29	69.62	–0.36	–0.45	[–34.25,35.26]
	Double	0.06	5.14	–15.05	19.45	0.05	–0.11	[–4.86,5.01]	0.11	0.00	0.01	0.32	0.10	0.14	[0.03,0.21]
Rice	Single	0.07	158.27	–47.75	47.77	–0.08	–0.61	[–25.08,25.21]	0.26	1527.74	–135.29	165.22	2.14	2.45	[–76.48,77.48]
	Double	0.01	18.17	–31.24	34.91	0.05	–0.33	[–9.38,8.85]	0.05	0.00	0.00	0.15	0.05	0.10	[0.01,0.09]
Foreign Stock	Single	0.12	74.48	–34.98	35.88	0.02	0.28	[–16.64,17.66]	0.08	735.95	–98.01	103.78	–0.40	1.89	[–53.28,53.39]
	Double	0.03	14.32	–25.48	27.07	0.02	–0.38	[–8.07,8.23]	0.07	0.00	0.00	0.20	0.07	0.15	[0.02,0.13]
Domestic Stock	Single	0.01	37.21	–27.10	21.85	–0.08	0.01	[–12.35,12.17]	–0.05	360.13	–83.57	70.18	–0.35	–1.33	[–37.02,37.34]
	Double	0.05	8.68	–16.58	19.31	0.04	–0.38	[–6.17,6.3]	0.10	0.00	0.00	0.27	0.10	0.22	[0.03,0.2]

Remark: C.R. is the abbreviation for Credible Interval.

Table 4
Statistics of posteriors' marginal distributions for expected returns and standard deviations

		Expected Return							*Standard Deviation*						
Asset	*Transform. Method*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*
Forex	Single	1.68	0.06	1.05	2.32	1.67	1.57	[1.2,2.18]	1.77	0.06	1.21	2.39	1.76	1.72	[1.33,2.25]
	Double	1.73	0.06	1.06	2.32	1.72	1.66	[1.23,2.2]	0.00	0.00	–0.02	0.01	0.00	0.00	[0.00,0.00]
Gold	Single	3.33	3.83	–2.37	7.49	3.43	2.84	[–1,6.79]	3.58	2.14	1.15	7.53	3.47	1.19	[1.27,6.62]
	Double	4.42	2.49	–2.28	7.49	4.46	7.09	[1.13,7.06]	0.00	0.00	0.00	0.00	0.00	0.00	[0.00,0.00]
Rice	Single	26.15	187.26	–7.70	51.38	28.56	51.38	[–0.78,47.29]	11.12	118.67	1.70	50.87	6.10	1.84	[1.79,40.13]
	Double	29.72	110.68	–7.52	51.19	30.59	29.72	[7.92,47.39]	0.00	0.00	–0.26	0.15	0.00	0.00	[0.00,0.00]
Foreign	Single	17.66	24.20	3.26	29.61	17.64	3.32	[7.2,26.99]	16.60	29.59	3.77	29.55	16.99	21.22	[5.75,26.77]
Stock	Double	19.77	19.45	4.38	29.61	19.62	17.20	[11.01,27.9]	0.00	0.00	0.00	0.00	0.00	0.00	[0.00,0.00]
Domestic	Single	30.37	5.22	25.93	36.69	29.93	31.27	[26.77,35.41]	30.35	5.16	25.95	36.60	29.90	28.26	[26.7,35.28]
Stock	Double	30.68	5.49	25.94	36.68	30.36	28.45	[26.75,35.46]	0.00	0.00	0.00	0.00	0.00	0.00	[0.00,0.00]

Remark: C.R. is the abbreviation for Credible Interval.

Table 5
Statistics of predictive posteriors' marginal distributions by Maximum Likelihood Estimation and the proposed methods

Asset	*Transformation Method*	*Mean*	*Variance*	*Min*	*Max*	*Median*	*Mode*	*C.R.*
Forex	Single	1.73	0.06	1.06	2.32	1.72	1.73	[1.23,2.2]
	Diff.	1.73	–0.28	7.10	–4.16	1.72	1.73	[1.25,2.18]
	Double	1.67	3.35	–6.00	9.49	1.67	1.67	[–1.91,5.32]
	Diff.	1.67	3.00	0.04	3.01	1.67	1.67	[–1.89,5.29]
Gold	Single	4.42	2.49	–2.28	7.49	4.46	4.42	[1.13,7.06]
	Diff.	4.39	1.17	6.63	–2.06	4.42	4.39	[1.13,6.99]
	Double	3.33	19.21	–26.34	24.50	3.46	3.33	[–5.75,11.97]
	Diff.	3.30	17.89	–17.43	14.95	3.43	3.30	[–5.75,11.9]
Rice	Single	29.72	110.68	–7.52	51.19	30.59	29.72	[7.92,47.39]
	Diff.	29.71	107.81	16.93	34.94	30.58	29.71	[7.96,47.33]
	Double	26.12	428.31	–108.73	130.44	29.21	26.12	[–23.58,59.78]
	Diff.	26.11	425.44	–84.28	114.19	29.20	26.11	[–23.54,59.72]
Foreign stock	Single	19.77	19.45	4.38	29.61	19.62	19.77	[11.01,27.9]
	Diff.	19.73	15.38	15.77	12.61	19.57	19.73	[11.04,27.78]
	Double	17.27	335.83	–64.80	93.84	17.41	17.27	[–20.4,54.45]
	Diff.	17.23	331.77	–53.41	76.84	17.36	17.23	[–20.37,54.33]
Domestic stock	Single	30.68	5.49	25.94	36.68	30.36	30.68	[26.75,35.46]
	Diff.	30.62	1.75	34.71	25.70	30.30	30.62	[26.83,35.26]
	Double	30.22	952.38	–88.57	149.53	29.63	30.22	[–29.67,90.86]
	Diff.	30.16	948.63	–79.80	138.55	29.57	30.16	[–29.59,90.66]

Remark: Diff. indicates the differences from the results of Maximum Likelihood Estimators.

matter of fact, the values do not appeared or could be considered constant in the second digits in every sample. Even though it is true that the less variances the better in the expected returns as shown in the left column, it seems unreasonable to have very tiny range of variations in the standard deviations as depicted in the right column of the figure.

Such a phenomenon continues in the case of posterior in Figure 5 and Table 4. There are differences in the variations of the standard deviations on the right column. In the same sample (the rice), the variance of rice are 118.67 and 0.00 in the same order as previously. However, the results of the mean of the expected returns seem agreeable in both specifications 26.15 and 29.72 respectively for this sample. It is noticed that the posteriors tends to develop more modes in the some sample like the rice so that care must be taken if one want to make the inference about a posterior mean.

In Figure 6 and Table 5, the Maximum Likelihood Method or MLE (dot-dashed line) was added as a baseline method. The predictive distributions of both single and double processes show the similar results that there are the shifts of the entire distributions from the baselines. The means (19.77 and 19.73 per cent in the same order as previously) of the foreign stock, for example, are moved distantly (17.27 and 17.23 per cent in the same order as previously) from the baseline.

4. CONCLUSIONS

This work offers an alternative insight in making inference by data fusion. The methodology is based on fuzzy returns that could be evaluated by linguistic terms. In normal model, fuzzy return and fuzzy deviation could be plugged in as priors into Bayesian framework by the possibility-to-probability transformation and combined with historical data by updating mechanism.

The conclusions we made is quite satisfactory because the captured behaviors of the results from the proposed method are consistent across the different samples used in the experiment. The outcomes reflect the effects of the structural changes and move the properties of the financial assets when incorporating the opinion from the experts. In addition, the improved method is more realistic in term of the distribution of standard deviation than our previous development [1].

Our contributions include 1) the integration of expert opinion and historical data for statistical inference and resulting in the degree of belief and 2) the newly proposed process to obtain the prior of standard deviation in order to fix the incompatibility in the previous research.

The integration of both subjective and objective data into the same framework has significant differences to the inference that depends solely on data. However, it must also be careful in that 1) the method bases on the subjectivity and cannot be interpreted in Frequentist way, 2) the possibility method is quite flexible and conservative so that applying to the specific application must be considered appropriately. Hopefully, this small step would initiate the awareness of data integration and provide a gateway to the future research in data fusion in social science. Currently, the authors are working in solving the optimization problem with this concept and finding the proper ratio between the number of transformations and observations.

ACKNOWLEDGEMENTS

The first author thanks to Dr. Supanika Leurcharusmee and Asst. Prof. Dr. Napat Harnpornchai for their kind discussions on some ambiguities arising during the problem formulation.

REFERENCES

T. Rattanadamrongaksorn, D. Sirikanchanarak, J. Sirisrisakulchai. and S. Sriboonjitta, Adjusting beliefs via transformed fuzzy prices –- the manuscript submitted to The First International Econometric Conference of Vietnam: ECONVN2018, 2017.

T. Rattanadamrongaksorn, J. Sirisrisakulchai, S. Sriboonchitta, Usages of fuzzy returns on Markowitz's portfolio selection, 2016, In: Proceedings of the Integrated Uncertainty in Knowledge Modelling and Decision Making: 5th International Symposium, IUKM 2016, Da Nang, Vietnam, November 30-December 2, 2016, pp. 124–135. Springer.

D. Dubois, H. Prade, S. Sandri, On possibility/probability transformations, 1993, Springer Netherlands, pp. 103–112.