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NEW MEASURES OF INFORMATION ENERGY BASED ON KAPUR'S FAMILY

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Abstract: In the present paper we obtained the measure of information energy corresponding to measures of Kapur's [1], [2] family, Havrda-Charvat's [3] measure of entropy, two parametric measures of entropy, generalized measure of entropy, three parametric Bi-measures of entropy respectively and also discussed the particular cases for each information measure of entropy. Entropy, generalized measure of entropy, three parametric Bi-measures of entropy respectively and also discussed the particular cases for each information measure of entropy.

Keywords: Measure of Entropy, Directed Divergence Information Energy, Bi-Measure of Entropy, Generlised Measures of Entropy, Concavity.

1. INTRODUCTION

Information theory is a very effective and efficient tool to set inaccurate and unclear circumstances where precise analysis is makes to impossible. Right from its evaluation the theory established its effectiveness in various studies and practices-biology, computer science logic, artificial intelligence control engineering, expert system, management science, operation research etc.

It was, in 1999 Dongxin Xu [4] provided a brief prospective of various features of energy, entropy and information. It shows interconnectedness between fundamental and general concepts. The energies and information's have many types. If the actual form of energy and the real content of information is taken out the portion remained will be pure quantity [5]. It was in the middle of 20^{th} century, the principal of energy conservation was originated and enhance, however it was in 1940 the real gift of information was studied. On the basis of quantity of energy we can conclude that small portion of U235 has plentiful atomic energy and thus the atomic age commenced. The pure quantity of information enables us to mention that optical cable can transmit more quantity of information than ordinary electrical telephone line. Generally, the rate of information quantity specify the capacity of a communication channel. Although the quantitative measure of information was originated from the study of communication, it is such a fundamental concept and method that it has been widely applied to many areas such as statistics, physics, chemistry, biology, life science, psychology, psychobiology, cognitive science, neuroscience, cybernetics, computer sciences, economics, operation research, linguistics, philosophy [6], [7], [2], [8]. Various information energy measures, their properties and applications are discussed and developed by S.K. Verma and Trivedi [9] in 2004 in his research work

The objective of the present paper is to investigate different measures of entropy having same information energy by developing new information energy measures. In section 2, we discuss preliminaries of the paper, properties and terminologies of information energy measure and investigate the information energy measures corresponding to Shannon's [10] and examine various properties of information energy measures. In section 3, we obtain information energy measures corresponding to Kapur's [1], [2] family of measures of entropy and particular limiting cases of information energy are discussed. The information energy measure related to the new two parametric measures of entropy are developed, the measure of generlised information energy is obtained, and the new three parametric Bi-measures of entropy are introduced along with its measures of information energy and discussed the particular cases of obtained measures of information energy. In section 4 we discussed the conducing remark of the present paper. The reference of the papers is given in section 5.

2. PRELIMINARIES

Let P = $\{p_1, p_2, \ldots, p_n\}$ be a probability distribution

where, $$p_i \geq 0 \tag{2.1}$$

and $$\sum_{i=1}^{n} p_i = 1 \tag{2.2}$$

then the Shannon [10] gave a measure of entropy

$$S(P) = -\sum_{i=1}^{n} p_i \ln p_i \tag{2.3}$$

this measures the uncertainty given by the probability distribution is called entropy of probability distribution P. The measure (2.1) is also be considered as a measure of equality of $p_1, p_2, \ldots, p_n$, among themselves. The basic equation of information theory is,

$$\text{Information Supplied} = \text{Uncertainty Removed} \tag{2.4}$$

We know that when no information is given above P. Except the natural constraint (2.1) and (2.2) then probability distribution is given by uniform distribution U = $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$.

In the present day, the world is information dominated and the information is energy and this information is measured by using (2.4).

Thus no information except (2.1) and (2.2) corresponds to zero level information energy when a information that i^{th} event will happen, there is no uncertainty that is when $p_j = 1$ then $p_i = 0$ for all $i = 1, 2, \ldots, n$, except j and in that case information energy is maximum. Thus information energy should be monotonic decreasing function of measures of uncertainty and since it is energy, it must be greater than or equal to zero. Information energy must increases when uncertainty decreases and since it is energy, it must be non-negative. Thus for a probability distribution P it can be measured by $f(S(P))$, where $f(S(P))$ is a monotonic decreasing S(P). If S(P) is measure of entropy then S(U) – S(P) can be regarded as a measure of information energy. Now we think of information energy associated with any probability distribution P as:

$$E(P) = \text{Constant} - S(P) \tag{2.5}$$

where, E(P) is the information energy and S(P) is the Shannon's [1] measure of entropy

$$E(U) = 0 \tag{2.6}$$

where, E(U) uniform information energy

$$\therefore \quad E\,(\text{degenerate distribution}) = \text{Maximum information energy} \tag{2.7}$$

By (2.5) we can write as

$$\therefore \quad C = S(U) \tag{2.8}$$

$$E(\text{degenerate distribution}) = S(U) - S(P = 0, 0, \ldots, 1, \ldots, 0) \tag{2.9}$$

$$= S(U)$$

So, measure of information energy associated with any probability a distribution P is

$$E(P) = S(U) - S(P) \tag{2.10}$$

where, S(P) be any valid measure of entropy.

E (P) is the measure of information energy for any probability distribution,

P = ($p_1, p_2, \ldots, p_n$) a information energy need to satisfy for the following conditions.

(i) It should be continuous function of $p_1, p_2, \ldots, p_n$ since if $p_1, p_2, \ldots, p_n$ change by small amounts then their maximum value changes by a small amounts.

(ii) It should be permutationally symmetric function of $p_1, p_2, \ldots, p_n$ since, if $p_1, p_2, \ldots, p_n$ are permuted among themselves, then their maximum value does not change .

(iii) It is convex function, function of $p_1, p_2, \ldots, p_n$.

(iv) It should be always non negative and vanishes when, $p_i = q_i$ for all $i = 1, \ldots, n$.

(v) It should be minimum having zero level energy at $p_1 = p_2 = \cdots \ldots p_n = \frac{1}{n}$.

(vi) It should be maximum where $p_1 = 1$ and all others p's are zero.

The Shannon's [10] measure of entropy is given by

$$H(P) = -\sum_{i=1}^{n} p_i \ln p_i$$

where, 0 ln 0 = 0 this measures the degree of equality among the probabilities, that is greater the equality among $p_1, p_2, \ldots, p_n$, greater is the value of H(P) and this measure has its maximum value ln n at equal probabilities that is when

$$p_1 = p_2 = \cdots \ldots\ldots = p_n = \frac{1}{n}$$

$$P = U$$

$$H(U) = -n\frac{1}{n}\ln\frac{1}{n}$$

$$H(U) = \ln n \tag{2.11}$$

The Shannon's entropy (2.3) is also measures the uniformity of P or we can say 'closeness' of P to the uniform distribution U. Since, the information energy corresponding to Shannon's [10] measure of entropy is,

$$E(P) = H(U) - H(P)$$

$$= \ln n + \sum_{i=1}^{n} p_i \ln p_i \tag{2.12}$$

According to Kullback –Leibler [11] measure the directed divergence of P from U is given by

$$D(P:U) = \sum_{i=1}^{n} p_i \ln\frac{p_i}{1/n}$$

$$= \ln n - H(P)$$

$$\therefore \quad E(P) = D(P:U) \tag{2.13}$$

where, D(P:Q) is the directed divergence between two non-degenerate probability distribution $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$ given by Kullback Leibler [11]. Thus greater the value of measure of entropy H(P), the nearest is P to U. This entropy provides a measure of equality or uniformity of probabilities $p_1, p_2, \dots, p_n$ among themselves.

3. MEASURES OF INFORMATION ENERGY

3.1. Measures of Information Energy Corresponding to Kapur's Family of Measures of Entropy

Kapur [1] gave following four measures of entropy with one parameter

$$K_a(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{a}\left[\sum_{i=1}^{n}(1+ap_i)\ln(1+ap_i) - ap_i\right] \; a > 0 \tag{3.1.1}$$

$$K_b(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{b}\left[\sum_{i=1}^{n}(1+bp_i)\ln(1+bp_i) - (1+b)\ln(1+b)\right], \; b > 0 \tag{3.1.2}$$

$$K_c(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{c^2}\left[\sum_{i=1}^{n}(1+cp_i)\ln(1+cp_i) - c\right], \; c > 0 \tag{3.1.3}$$

$$K_k(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{k^2}\left[\sum_{i=1}^{n}(1+kp_i)\ln(1+kp_i) - (1+k)\ln(1+k)\right], \; k > 0 \tag{3.1.4}$$

The Kapur's [1] measures of entropy (3.1.1), (3.1.2), (3.1.3) and (3.1.4) satisfying the property discussed in section two.

The Kapur's measures of information energy for Kapur's [1] measures of entropy (3.1.1) is,

Case (I): $E_a(\mathrm{P}) = K_a(\mathrm{U}) - K_a(\mathrm{P})$

$$= \sum_{i=1}^{n} p_i \ln \frac{p_i}{1/n} + \frac{1}{a}\left[(n+a)\ln\left(\frac{n+a}{n}\right) - \sum_{i=1}^{n}(1+ap_i)\ln(1+ap_i)\right],$$

$$a > 0 \tag{3.1.5}$$

when, $\lim_{a\to 0} E_a(P) = -\sum_{i=1}^{n} p_i \ln \frac{p_i}{1/n} = E_n(P)$

$$\therefore \lim_{a\to 0} E_a(P) = E_n(P) \tag{3.1.6}$$

Case (II): $E_b(P) = K_b(U) - K_b(P)$

The measure of information energy corresponding Kapur's [1] measures of entropy (3.1.2) is

$$E_b(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + \frac{1}{b}\left[(n+b)\ln\left(\frac{n+b}{n}\right) - \sum_{i=1}^{n}(1+bp_i)\ln(1+bp_i)\right],$$

$$b > 0 \tag{3.1.7}$$

when, $b \to 0$ in (3.1.7)

$$\therefore \lim_{b\to 0} E_b(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{1/n}$$

$$= \ln n + \sum_{i=1}^{n} p_i \ln p_i$$

$$\lim_{b\to 0} E_b(P) = E_n(P) \tag{3.1.8}$$

Case (III): $E_c(P) = K_c(U) - K_c(P)$

$$E_b(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + \frac{1}{b}\left[(n+b)\ln\left(\frac{n+b}{n}\right) - \sum_{i=1}^{n}(1+bp_i)\ln(1+bp_i)\right],$$

$$= \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + \frac{1}{c^2}\left[(n+c)\ln\left(\frac{n+c}{n}\right) - \sum_{i=1}^{n}(1+cp_i)\ln(1+cp_i)\right],$$

$$c > 0 \tag{3.1.9}$$

when, $c \to 0$

$$\lim_{c\to 0} E_c(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{1/n} + \frac{1}{2n} - \frac{1}{2}\sum_{i=1}^{n} p_i^2 \tag{3.1.10}$$

Case (IV): $E_k(P) = K_k(U) - K_k(P)$

$$= \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + \frac{1}{k^2}\left[(n+k)\ln\left(\frac{n+k}{n}\right) - \sum_{i=1}^{n}(1+kp_i)\ln(1+kp_i)\right],$$

$$k > 0 \tag{3.1.11}$$

when, $k \to 0$

$$\lim_{k\to 0} E_k(P) = E_n(P) + \frac{1}{2n} - \frac{1}{2}\sum_{i=1}^{n} p_i^2 \qquad (3.1.12)$$

from (3.1.5),(3.1.7), (3.1.9) and (3.1.11) we can write as,

$$\lim_{a\to 0} E_a(P) = \lim_{b\to 0} E_b(P) = E_n(P)$$

and $\lim_{c\to 0} E_c(P) = \lim_{k\to 0} E_k(P)$

$$= E_n(P) + \frac{1}{2n} - \frac{1}{2}\sum_{i=1}^{n} p_i^2 \qquad (3.1.13)$$

3.2. Two Parametric Measure of Information Energy

Case (I): Now we have proposed two parametric measure of entropy as follows,

$$H_{\frac{b}{a}}(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{a}{b}\left[\sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\ln\left(1+\frac{b}{a}p_i\right) - \left(1+\frac{b}{a}\right)\ln\left(1+\frac{b}{a}\right)\right]$$

$$b > 0,\ 0 < a \le 1 \qquad (3.2.1)$$

It can be easily proved that the above expression satisfies all the requisite properties of being a measure of entropy. The Measure of entropy (3.2.1) satisfying the properties of information energy introduced in section one. We have proposed the following measure of information energy corresponding to two parametric measure of entropy,

$$E_{\frac{b}{a}}(P) = H_{\frac{b}{a}}(U) - H_{\frac{b}{a}}(P)$$

$$= \sum_{i=1}^{n} p_i ln \frac{p_i}{\frac{1}{n}} + \frac{a}{b}\left[\left(n+\frac{b}{a}\right)\ln\left(\frac{n+\frac{b}{a}}{n}\right) - \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\ln\left(1+\frac{b}{a}p_i\right)\right]$$

$$b > 0,\ 0 < a \le 1 \qquad (3.2.2)$$

The above mathematical expression is the two parametric information energy measure related to two parametric measure of entropy (3.2.1).

Taking $b \to 0$

$$\lim_{b\to 0} E_{\frac{b}{a}}(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} \qquad (3.2.3)$$

$$\therefore \quad \lim_{b\to 0} E_{\frac{b}{a}}(P) = E_n(P)$$

Hence, $\lim_{b\to 0} E_{\frac{b}{a}}(P) = \lim_{a\to 0} E_a(P) = \lim_{b\to 0} E_b(P) = E_n(P)$ (3.2.4)

Case (II): Again we have introduced a new parametric measures given by the following mathematical expression

$$H^{\propto/\beta}(\mathrm{P}) = \frac{1}{\beta-\alpha}\left(\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} - 1\right), \propto \neq \beta, \beta > 0, \propto > 0 \quad (3.2.5)$$

Hence it can be easily proved that the expression satisfies all the essential properties of being a measure of entropy. Measure of entropy (3.2.5) satisfying all the properties of information energy introduced in section one

$$H^{\propto/\beta}(\mathrm{U}) = \frac{1}{\beta-\alpha}\left(n.\frac{1}{n^{\propto/\beta}} - 1\right), \propto \neq \beta, \beta > 0, \propto > 0 \quad (3.2.6)$$

We have proposed the following measure of information energy corresponding to two parametric measure of entropy,

$$E^{\frac{\alpha}{\beta}}(P) = H^{\frac{\propto}{\beta}}(U) - H^{\propto/\beta}(\mathrm{P})$$

$$= \frac{1}{\beta-\alpha}\left(n^{1-\propto/\beta} - \sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right), \propto \neq \beta, \beta > 0, \propto > 0 \quad (3.2.7)$$

By putting $\beta = 1$ in (3.2.5) we get,

$$H^{\propto}(\mathrm{P}) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right), \propto \neq 1, \propto > 0 \quad (3.2.8)$$

Which is Havarda–charvat's [3] measure of entropy, if we put $\beta = 1$ in (3.2.7) we get measure of information energy corresponding Havarda–charvat's [3] to measure of entropy,

$$E^{\propto}(\mathrm{P}) = \frac{1}{1-\alpha}\left(n^{1-\propto} - \sum_{i=1}^{n} p_i^{\propto}\right), \propto \neq 1, \propto > 0 \quad (3.2.9)$$

3.3. Particular Cases

Case (I): If $a = b$ in (3.1.5) and we get (3.1.7)

$$E_a(P) = E_b(P) \quad (3.3.1)$$

Case (II): If $c = k$ in (3.1.9) then we get (3.1.11)

$$E_c(P) = E_k(P) \quad (3.3.2)$$

this shows that two different valid measure of entropy may have the same measure of information energy.

Case (III): If $a = 1$ in (3.2.2), we get $E_{\frac{b}{a}}(P) = E_b(P)$ (3.3.3)

If $a = 1$ and $b = a$ in (3.2.2) then $E_{\frac{b}{a}}(P) = E_a(P)$ (3.3.4)

Case (IV): Put $b = 1$ in (3.1.2), $k = 1$ in (3.1.4) and $a = b$ in (3.2.1) we get same measure of information energy from these three expression we get,

$$K_1(P) = -\sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n}(1 + p_i)\ln(1 + p_i) - 2\ln 2 \quad (3.3.5)$$

which is a Bose Einstein [2] measure of entropy, similarly, if put $a = 1$ in (3.1.5), $b = 1$ in (3.1.7), $c = 1$ in (3.1.9), $k = 1$ in (3.1.11), $b = a$ in (3.2.2) we get Bose Einstein [2] measure of information energy

$$E_B(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + (n+1) \ln\left(\frac{n+1}{n}\right) - \sum_{i=1}^{n}(1+p_i)\ln(1+p_i) \quad (3.3.6)$$

Case (V): Similar, if we put $b = -1$ in (3.1.2) and put $b = -a$ in (3.2.1) we get

$$K_{-1}(P) = -\sum_{i=1}^{n} p_i ln p_i - \sum_{i=1}^{n}(1-p_i)\ln(1-p_i)$$

Which is Fermi Dirac [2] measure of entropy, if we take $b = -a$ in (3.2.2) by taking $a = -1$ in (3.1.5), $b = -1$ in (3.1.7) and $b = -a$ in (3.2.2) we get, Fermi Dirac [2] measure of information energy,

$$E_F(P) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + (n-1) \ln\left(\frac{n-1}{n}\right) + \sum_{i=1}^{n}(1-p_i)\ln(1-p_i) \quad (3.3.7)$$

3.4 Generalized Measure of Information Energy

Now we obtained a new measures of information energy by developing a new parametric measure of entropy, the new proposed measure is the combination of two generalisations Let,

$$H_{b/a}^{\propto\beta}(\mathrm{P}) = \frac{1}{\beta-\alpha}\left(\sum_{i=1}^{n} p_i^{\propto\beta} - \sum_{i=1}^{n} p_i\right) - \frac{a}{b}\frac{1}{\beta-\alpha}\left[\sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)^{\propto\beta} - \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\right] + \frac{a}{b}\frac{1}{\beta-\alpha}\left[\left(1+\frac{b}{a}\right)^{\propto\beta} - \left(1+\frac{b}{a}\right)\right]$$
$$\propto\neq\beta, \beta>0, \propto>0, b>0, 0<a\leq 1 \quad (3.4.1)$$

$$H_{\frac{b}{a}}^{\propto\beta}(\mathrm{U}) = \frac{1}{\beta-\alpha}\left(n.\frac{1}{n^{\alpha\beta}} - 1\right) - \frac{a}{b}\frac{n}{\beta-\alpha}\left[\left(1+\frac{b}{a}\frac{1}{n}\right)^{\propto\beta} - \left(1+\frac{b}{a}\frac{1}{n}\right)\right] + \frac{a}{b}\frac{1}{\beta-\alpha}\left[\left(1+\frac{b}{a}\right)^{\propto\beta} - \left(1+\frac{b}{a}\right)\right] \propto\neq\beta, \beta>0, \propto>0,$$
$$b>0, 0<a\leq 1 \quad (3.4.2)$$

We have proposed the following measure of information energy corresponding to parametric measure of entropy,

$$E_{\frac{b}{a}}^{\propto\beta}(P) = H_{b/a}^{\propto\beta}(\mathrm{U}) - H_{b/a}^{\propto\beta}(\mathrm{P})$$

$$E_{\frac{b}{a}}^{\propto\beta}(P) = \frac{1}{\beta-\alpha}\left(n^{1-\alpha\beta} - 1\right) - \frac{a}{b}\frac{n}{\beta-\alpha}\left[\left(1+\frac{b}{a}\frac{1}{n}\right)^{\propto\beta} - \left(1+\frac{b}{a}\frac{1}{n}\right) + \left(\sum_{i=1}^{n} p_i^{\propto\beta} - \sum_{i=1}^{n} p_i\right) - \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)^{\propto\beta} + \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\right] \propto\neq\beta, \beta>0, \propto>0,$$
$$b>0, 0<a\leq 1 \quad (3.4.3)$$

Case (I): Taking $\beta = 1$ in (3.4.1) we get,

$$H_{b/a}^{\propto}(\mathrm{P}) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\propto} - \sum_{i=1}^{n} p_i\right) - \frac{a}{b}\frac{1}{1-\alpha}\left[\sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)^{\propto} - \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\right] + \frac{a}{b}\frac{1}{1-\alpha}\left[\left(1+\frac{b}{a}\right)^{\propto} - \left(1+\frac{b}{a}\right)\right] \propto \neq 1, \propto > 0,$$

$$b > 0, 0 < a \leq 1 \tag{3.4.4}$$

if we take $\beta = 1$ in (3.4.3) we get measure of information energy for (3.4.4)

$$E_{\frac{b}{a}}^{\propto}(P) = \frac{1}{1-\alpha}\left(n^{1-\alpha} - 1\right) - \frac{a}{b}\frac{n}{1-\alpha}\left[\left(1+\frac{b}{a}\frac{1}{n}\right)^{\propto} - \left(1+\frac{b}{a}\frac{1}{n}\right) + \left(\sum_{i=1}^{n} p_i^{\propto} - \sum_{i=1}^{n} p_i\right) - \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)^{\propto} + \sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\right] \propto > 0,$$

$$b > 0, 0 < a \leq 1 \tag{3.4.5}$$

Case (II): If we take $a = 1$ and $b = a$ in (3.4.4) it reduces to (3.1.7) which is Kapur's [2] measure of entropy.

By putting $a = 1$ and in (3.4.3) it approaches to measure of information energy (3.1.5) and (3.1.7)

$$E_{b}^{\propto}(P) = \frac{1}{1-\alpha}\left(n^{1-\alpha} - 1\right) + \frac{1}{b}\frac{n}{1-\alpha}\left[\left(1+b\frac{1}{n}\right)^{\propto} - \left(1+b\frac{1}{n}\right) - \left(\sum_{i=1}^{n} p_i^{\propto} - \sum_{i=1}^{n} p_i\right) + \sum_{i=1}^{n}(1+bp_i)^{\propto} - \sum_{i=1}^{n}(1+bp_i)\right] \propto \neq 1, \propto > 0, b > 0 \tag{3.4.6}$$

Case (III): Taking $\beta = 1$ and $\propto \to 1$ in (3.4.1) we get

$$\lim_{\propto \to 1} H_{b/a}^{\propto}(P) = -\sum_{i=1}^{n} p_i \ln p_i + \frac{a}{b}\left[\sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right)\ln\left(1+\frac{b}{a}p_i\right) - \left(1+\frac{b}{a}\right)\ln\left(1+\frac{b}{a}\right)\right] \tag{3.4.7}$$

at $\beta = 1$ and $\propto \to 1$

$$\lim_{\propto \to 1} H_{b/a}^{\propto\beta}(\mathrm{P}) = H_{\frac{b}{a}}(P) \tag{3.4.8}$$

The measure of entropy $H_{\frac{b}{a}}^{1}(P)$ is already studied and the measure of information energy for (3.4.8) is already deduced in section 3.

Therefore, $E_{\frac{b}{a}}^{\propto\beta}(P)$ at $\beta = 1$ and $\propto \to 1$ is obtained as

$$\lim_{\propto \to 1} E_{\frac{b}{a}}^{\propto\beta}(P) = E_{\frac{b}{a}}(P) \tag{3.4.9}$$

Case (IV): Again by taking $a = 1$,$\propto \to 1$ in (3.4.1) we get Kapur's [1] measure of entropy

$$H_{\frac{b}{a}}^{\propto\beta}(\mathrm{P}) = K_b(\mathrm{P}) \tag{3.4.10}$$

which is Kapur's [2] measure of entropy. The measure of information energy related to $H_{\frac{b}{a}}^{\propto\beta}$(P) at , $a = 1$,$\propto\rightarrow 1$ is

$$E_{\frac{b}{a}}^{\propto\beta}(P) = E_b(P) \tag{3.4.11}$$

the information energy measure $E_b(P)$ is already discussed in (3.1.7)

Case (V): When $b \rightarrow 0$ in (3.4.1) we get,

$$\lim_{b\rightarrow 0} H_{b/a}^{\propto\beta}(\text{P}) = H^{\propto/\beta}\ (\text{P}) \tag{3.4.12}$$

by putting $b \rightarrow 0$, $\beta = 1$, $a = 1$ in (3.4.1) , hence

$$\lim_{b\rightarrow 1} H_{b/a}^{\propto\beta}(\text{P}) = H^{\propto}\ (\text{P}) \tag{3.4.13}$$

Which is Havarda-Charvat's [3] measure entropy. The measure $H^{\propto/\beta}$ (P) and $H^{\propto}$ (P) is already discussed in section 3.3. Similarly, the information energy (3.4.3) at $b \rightarrow 0$ gives

$$\lim_{b\rightarrow 0} E_{\frac{b}{a}}^{\propto\beta}(P) = \frac{1}{\beta-\alpha}\left(n^{1-\propto\beta} - \sum_{i=1}^{n} p_i^{\alpha\beta}\right), \propto\neq 1, \propto > 0 \tag{3.4.14}$$

$$\lim_{b\rightarrow 0} E_{\frac{b}{a}}^{\propto\beta}(P) = E^{\propto\beta}\ (\text{P}) \tag{3.4.15}$$

Measure of information energy $E^{\propto\beta}$ (P) is already discussed in section (3.3) Again by taking $b \rightarrow 0$, $\beta = 1$ in (3.4.15) it gives

$$\lim_{b\rightarrow 0} E_{\frac{b}{a}}^{\propto\beta}(P) = \frac{1}{1-\alpha}(n^{1-\propto} - \sum_{i=1}^{n} p_i^{\propto})$$

$$\lim_{b\rightarrow 0} E_{\frac{b}{a}}^{\propto\beta}(P)\ ) = E^{\propto}\ (\text{P}) \tag{3.4.16}$$

Which is a measure of information energy corresponding to Havarda-charvat's [3] measure of entropy.

Case (VI): Taking $\beta = 1$, $a = 1$ and $\propto\rightarrow 1$ in (3.4.1), approaches to

$$\lim_{\propto\rightarrow 1} H_{\frac{b}{a}}^{\propto}(P) = K_b(P)$$

which is Kapur's [1] measure of entropy. Similarly, the information energy measures at $\beta = 1$, $a = 1$ and $\propto\rightarrow 1$

$$\lim_{\propto\rightarrow 1} E_{\frac{b}{a}}^{\propto}(P) = E_b(P) \tag{3.4.17}$$

the information energy measure $E_b(P)$ is already discussed in section (3.4).

Case (VII): Again by taking $\alpha \to 1$, $\beta = 1$, $a = 1$ and $b = 1$ in (3.4.1) it reduces to

$$H_{b/a}^{\propto\beta}(\text{P}) = K_1(P) \quad (3.4.18)$$

Which is a Bose Einstein's [2] measure of entropy $K_1(P)$is already discussed in (3.3.5). When we take $\alpha \to 1$, $\beta = 1$, $a = 1$ and $b = 1$ in (3.4.3)

$$\lim_{\alpha\to 1} E_{\frac{b}{a}}^{\propto\beta}(P) = E_B(P) \quad (3.4.19)$$

the information energy $E_B(P)$ corresponding to Bose Einstein [2] measure of entropy is already discussed in section (3.3).

Case (VIII): Similar, if we take $b = -a$ when $\alpha \to 1$ and $\beta = 1$ in (3.4.1), implies

$$\lim_{\propto\to 1} H_{b/a}^{\propto}(\text{P}) = K_{-1}(P) \quad (3.4.20)$$

Which is Fermi Dirac [2] measure of entropy, it is already discussed in section (3.3). We get measure of information energy corresponding to Fermi Dirac [2] measure of entropy by putting $b = -a$ when $\alpha \to 1$, $\beta = 1$, and $b = -1$ in (3.4.3) it gives

$$\lim_{\alpha\to 1} E_{\frac{b}{a}}^{\propto\beta}(P) = E_F(P) \quad (3.4.21)$$

Which is a measure of information energy corresponding to Fermi Dirac [2] measure of entropy. Hence the measure of entropy (3.4.1) and measure of information energy (3.4.3) approaches to several measures and measures of information energy at various limits.

3.5. New Three Parametric Bi-Measure Information Energy

Consider the new three parametric bi-measure of entropy as

$$H_{a,b,k}(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{b}{a^k}\left[\sum_{i=1}^{n}(1 + ap_i)ln(1 + ap_i) - (1+a)ln(1+a)\right]$$

where, a, b, k are parametric, $a > 0$,$0 < b \leq 1$ (3.5.1)

$$f''(p_i) = -\frac{1}{p_i} + \frac{b}{a^{k-2}}\left[\frac{1}{(1+ap_i)}\right] \quad (3.5.2)$$

$$= -\frac{(1+(a-C)p_i)}{p_i(1+ap_i)} \quad (3.5.3)$$

Hence $H_{a,b,k}(p)$ will be concave, if $1 + (a - C)\, p_i \geq 0$ (3.5.4)

But this is true when $1 \geq C$ or $a \geq C$ i.e. when $a^{k-2} \geq b$ or $a^{k-1} \geq b$ (3.5.5)

It can be easily proved that the expression (3.5.1) satisfies all the required properties of being a measure of entropy. The information energy corresponding to the three parametric measure of entropy $H_{a,b,k}(P)$ is given by following mathematical expression

$$E_{a,b,k}(P) = H_{a,b,k}(U) - H_{a,b,k}(P)$$
$$= \sum_{i=1}^{n} p_i \ln \frac{p_i}{\frac{1}{n}} + \frac{b}{a^k}\left[(n+a)\ln\left(\frac{n+a}{n}\right)\sum_{i=1}^{n}(1+ap_i)\ln(1+ap_i)\right]$$
$$a > 0,\ 0 < b \leq 1 \tag{3.5.6}$$

Case (I): When $a^{k-2} \geq b$ or $a^{k-1} \geq b$, (3.5.7)

$H_{a,b,k}(p)$ represents a three parametric Bi-measure of entropy and $E_{a,b,k}(P)$ represents a three parametric Bi-measure of information energy.

Case (II): When $b = 1$, $a = b$, $k = 1$ in (3.5.1) Hence,

$$H_{a,b,k}(p) = K_b(P)$$

Similarly, the information measure at $b = 1$, $k = 1$, $a = b$, (3.5.6) becomes

$$E_{a,b,k}(P) = E_b(P) \tag{3.5.8}$$

Which is a Kapur's [1] [2] information energy measure discussed in section (3.3).

Case (III): When $b = 1$, $a = k$, $k = 2$ then (3.5.1) becomes $H_{a,b,k}(p) = K_k(p)$, at $b = 1$ and $k = 2$.

Similarly, The information measure at $b = 1$, $k = 2$, $a = b$ then (3.5.6) becomes

$$E_{a,b,k}(P) = E_k(P) \tag{3.5.9}$$

Which is Kapur's [1], [2] measure of information energy

Case (IV): When $b = 1$, $k = 0$, $a = 1$ then (3.5.1) becomes $H_{a,b,k}$ (P) $= K_1(P)$ which is a Bose Einstein [2] measure of entropy which is already discussed in (3.3.5) When we take $k = 0$, $a = 1$ and $b = 1$ in (3.5.1) we get,

$$E_{a,b,k}(P) = E_B(P) \tag{3.5.10}$$

$E_B(P)$ is Bose Einstein [2] measure information energy of is already discussed in section (3.3)

Case (V): When $b = 1$, $k = 0$, $a = -1$ then (3.5.1) becomes

$$H_{a,b,k}(P) = K_{-1}(P) \tag{3.5.11}$$

Which is Fermi Dirac [2] measure of entropy, it is already discussed in section (3.3). We get measure of information energy of Fermi Dirac [2] measure of entropy by putting $b = -1$, $a = -1$, $k = 0$

$$E_{a,b,k}(P) = E_F(P) \tag{3.5.12}$$

which is a Fermi Dirac [2] measure of information energy

Case (VI): When $0 < b \leq 1$ then (3.5.7) will hold if

$$(k-2)\, lna \geq 0 \text{ or } (k-1)\, ln\, a \geq 0 \qquad (3.5.13)$$

But $(k-2)\, n\, a \geq 0$, If $a \geq 1$, $k \geq 2$ or $0 < a < 1$, $k < 2$, and $(k-1)\, ln\, a \geq 0$, if $a \geq 1$, $k \geq 1$ or $0 < a < 1$, $k < 1$.

Case (VII): When $b > 1$, we can find suitable values of 'a', 'k' using (3.5.13). Thus in any case, the family of bi-measures of entropies is much larger than $K_b(P)$ and $K_k(P)$ also the family of bi-measures of information energy also have larger values than $E_b(P)$ and $E_k(P)$. Hence the measure of entropy (3.5.1) and measure of information energy (3.5.6) approaches to various measures and measures of information energy respectively at various limits.

4. CONCLUSION

- In the existing literature of information energy, we found various measures of entropy and information energy; and have examined its different properties and limiting cases their merits demerits and limitations. The development of these measures which can be applied in various disciplines.
- For each measure of entropy, we can develop a corresponding measure of information energy and verify its properties. The information energy measures have a great deal in common and the knowledge of measures entropy can be used to enrich the literature of information energy.
- We have observed some similarities and dissimilarities in the results of four types of new generalized measures of information energy.
- All the four measure of entropy and four measures of information energy are correlated with each other at different limits.
- The measure of entropy with one parameter, two parameters and three parameters have same information energy.
- On different values of 'a' and 'b' we get measures of entropy defined by Kapur [1], [2] which is $K_b(P)$ and $K_k(P)$ and in any case, the obtained family of bi-measure of entropies is much larger than $K_b(P)$ and $K_k(P)$ also the family of bi-measures of information energy have larger values than $E_b(P)$ and $E_k(P)$.

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