IJMMS, Vol. 13, No. 2, (July-December 2017) : 327-344

ISSN: 0973-3329

UNIQUENESS OF PRODUCT OF DERIVATIVES AND Q-SHIFT DIFFERENCE OF ENTIRE FUNCTIONS

Renukadevi S Dyavanal* and Rajalaxmi V Desai*

Abstract: *In this paper, we investigate the uniqueness of entire functions of zero order concerning its derivative and q-shift difference. We deduce the results of X M Zheng and H Y Xu[24] as particular case of our results and we extend the results of Y Liu, Y H Cao, X G Qi and H X Yi[16].*

Keywords: *Nevanlinna theory, Meromorphic functions, q-shift difference, Uniqueness, etc.*

Subject Classification: *30D35, 39A13*

1. INTRODUCTION AND RESULTS

In this paper, the term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. We shall use the standard notation in Nevanlinna's value distribution theory of meromorphic functions ([6], [21], [23], [2], [19]), and $S(r, f)$ denotes any quantity that satisfy the condition $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{(1,r]\cap E}\frac{1}{t}dt < \infty$ and also use $S_1(r, f)$ to denote any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, where the logarithmic density of a set F is defined by $\lim_{r\to\infty}\int_{(1,r]\cap F}\frac{1}{t}dt.$.

Moreover, we assume in the whole paper thatm, n are positive integers, q is a non-zero complex constant, $c \in \mathbb{C}$, and $\alpha(z)$ non-zero small function with respect to $f(z)$, that is, $\alpha(z)$ is a non-zero meromorphic function of growth $S(r, f)$.

Recently, many articles have focused on value distribution and uniqueness of difference polynomials of entire or meromorphic functions (see example [1]-[11]).

In this paper, we use following notation.

Let $P_n(z) = a_n(z)z^n + a_{n-1}(z)z^{n-1} + \ldots + a_1(z)z + a_0(z)$ be a non-zero polynomial, where $a_0(z), a_1(z), \ldots, a_n(z)$ ($\neq 0$) are complex constants and t_n is the number of distinct zeros of $P_n(z)$.

* Department of Mathematics, Karnatak University, Dharwad – 580003, India, *E-mail: renukadyavanal@gmail.com and desairajlakshmi@gmail.com*

In 2012, J F Xuand X B Zhang([20]) investigated the zeros of q-shift difference polynomials of meromorphic functions of finite logarithmic order and obtained the following result.

Theorem A. If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$ and q, c are non-zero complex constants, then for $n \geq 2$, $f^n(z)f(qz + c)$ assumes every value $b \in \mathbb{C}$ infinitely often.

In 2014, X M Zheng and H Y Xu([24]) investigated the zeros of differentialq-shiftdifference polynomials of meromorphic functions of finite positive logarithmic order and obtained the following results.

Theorem B. If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$ and q, c are non-zero complex constants, then for $m \geq n + k + 1$, $f^m(z) P_n(f(qz + c)) \prod_{j=1}^{k} f^{(j)}(z)$ assume $\alpha(z)$ infinitely often.

Theorem C. If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$, and q, c are non-zero complex constants, then for $m \geq n + k + 1$, $P_m(f(z)) f^n(qz + c) \prod_{j=1}^{k} f^{(j)}(z)$ assumes $\alpha(z)$ infinitely often.

Theorem D. If $f(z)$ and $g(z)$ be transcendental entire functions of order zero and $m \geq n + 2t_n + 5$. If $f^m(z) P_n(f(qz + c)) f'(z)$ and $g^m(z) P_n(g(qz + c)) g'(z)$ share a non-zero polynomial $p(z)$ CM, then

$$f^m(z) P_n(f(qz + c)) f'(z) = g^m(z) P_n(g(qz + c)) g'(z).$$

Theorem E. If $f(z)$ and $g(z)$ be transcendental entire functions of order zero and $n \geq m + 2t_m + 5$. If $P_m(f(z)) f^n(qz + c) f'(z)$ and $P_m(g(z)) g^n(qz + c) g'(z)$ share a non-zero polynomial $p(z)$ CM, then

$$P_m(f(z)) f^n(qz + c) f'(z) = P_m(g(z)) g^n(qz + c) g'(z).$$

In 2013, Y Liu, Y H Cao, X G Qi and H X Yi([16]) investigated the value sharing results for q-shifts difference polynomials and obtained the following results.

Theorem F. If $f(z)$ and $g(z)$ be two transcendental meromorphic functions with zero order. Suppose that qand c are nonzero complex constants and n is an integer. If $n \geq 14$ and $f^n(z) f(qz + c)$ and $g^n(z) g(qz + c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z) g(z) = t$, where $t^{n+1} = 1$.

Theorem G. Under the conditions of Theorem F, if $n \geq \lambda + 14d + 11$ and $f^n(z) f(qz + c)$ and $g^n(z) g(qz + c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z) g(z) = t$, where $t^{n+1} = 1$.

In 2014, X L Wang, H Y Xu and T S Zhan ([17]) investigated the value distribution of q-shift difference-differential polynomials of meromorphic functions and obtain the following result.

Theorem H. Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order and $F(z)=P(f)\prod_{j=1}^{d} f(q_j z+c_j)^{sj}$. If $k\in\mathbb{N}$ and $n>m(k+1)+2d+1+\lambda$ (*resp.* $n>m(k+1)+d+\lambda$). Then $(F(z))^{(k)}-\alpha(z)$ has infinitely many zeros, where $(F(z))^{(k)}=F(z)$, if $k=0$.

In this paper, we investigate the uniqueness of differential and q-shift-difference polynomials considered in Theorem B and Theorem C for entire functions of zero order, and we prove the uniqueness of q-shift-difference polynomials sharing a value 1CM(IM) considered in Theorem H for transcendental meromorphic functions of zero order which extends Theorem F and Theorem G as follows.

Theorem 1.1. Let $f(z)$ and $g(z)$ be transcendental entire functions of order zero. If $f^m(z)P_n(f(qz+c))\prod_{j=1}^{k} f^{(j)}(z)$ and $g^m(z)P_n(g(qz+c))\prod_{j=1}^{k} g^{(j)}(z)$ share a small function $\alpha(z)$ CM, then

$$f^m(z)P_n\left(f(qz+c)\right)\prod_{j=1}^{k} f^{(j)}(z)=g^m(z)P_n\left(g(qz+c)\right)\prod_{j=1}^{k} g^{(j)}(z)$$

Form $\geq n+2k+2t_n+3$, where t_n is the number of distinct zeros of $P_n(z)$.

Remark 1.1. If $k=1$, in Theorem 1.1 then Theorem 1.1 reduces to Theorem D.

Theorem 1.2. Let $f(z)$ and $g(z)$ be transcendental entire functions of order zero. If $P_m(f(z))f^n(qz+c)\prod_{j=1}^{k} f^{(j)}(z)$ and $P_m\left(g(z)\right)g^n(qz+c)\prod_{j=1}^{k} g^{(j)}(z)$ share a small function $\alpha(z)$ CM, then

$$P_m\left(f(z)\right)f^n(qz+c)\prod_{j=1}^{k} f^{(j)}(z)=P_m(g(z))g^n(qz+c)\prod_{j=1}^{k} g^{(j)}(z)$$

for $n\geq m+2k+2t_m+3$, where t_m is the number of distinct zeros of $P_m(z)$.

Remark 1.2. If $k=1$, in Theorem 1.2, then Theorem 1.2 reduces to Theorem E.

Theorem 1.3: Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with zero order. Suppose that q_j, c_j ($j=1,2,\dots,d$) are nonzero complex constants. n, d, s_j ($j=1,2,\dots,d$) are positive integers, $\lambda=s_1+s_2+\dots+s_d$. If $n\geq\lambda+8d+5$ and $f^n(z)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}$ and $g^n(z)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}$ share 1 CM, then either $f(z)\equiv tg(z)$ or $f(z)\,g(z)=t$, where $t^{n+\lambda}=1$.

Remark 1.3. If $d = 1$, in Theorem 1.3, then Theorem 1.3 reduces to Theorem F.

Theorem 1.4. Under the assumptions of Theorem 1.3, if $n \geq \lambda + 14d + 11$ and $f^n(z)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}$ and $g^n(z)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}$ share 1 IM, then either $f(z) \equiv tg(z)$ or $f(z)\, g(z) = t$, where $t^{n+\lambda} = 1$.

Remark 1.5. If $d = 1$, in Theorem 1.4, then Theorem 1.4 reduces to Theorem G.

2. SOME PRELIMINARY RESULTS

To prove our theorems, we require following lemmas.

Lemma 2.1 ([23]). Let $f(z)$ be a non-constant meromorphic function, then

$$T(r, P_n(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([20]). Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order and q, η be two non-zero complex constants. Then we have

$$T(r, f(qz + \eta)) = T(r, f) + S_1(r, f),$$
$$N(r, f(qz + \eta)) = N(r, f) + S_1(r, f),$$

$$N\left(r, \frac{1}{f(qz+\eta)}\right) = N\left(r, \frac{1}{f}\right) + S_1(r, f).$$

Lemma 2.3([15]). Let $f(z)$ be a non-constant zero-order meromorphic function and q be a non-zero complex number. Then

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.4([23]). Let $f(z)$ be a non-constant meromorphic function in the complex plane and l be a positive integer. Then

$$T\left(r, f^{(l)}\right) \leq T(r, f) + l\overline{N}(r, f) + S(r, f)$$

$$N\left(r, f^{(l)}\right) \leq N(r, f) + l\overline{N}(r, f).$$

Lemma 2.5([22]). Let F and G be two nonconstant meromorphic functions, and let F and G share 1 CM, then one of the following three cases holds:

(i). max $\{T(r,F),\ T(r,G)\} \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G)$,

(ii) $F = G$,

(iii) $FG \equiv 1$,

Where $N_2\left(r,\frac{1}{F}\right)$ denotes the counting function of zero of *F*, such that simple zero are counted once and multiple zeros are counted twice.

Lemma 2.6 ([18]). Let *F* and *G* be two nonconstant meromorphic functions, and let *F* and *G* share 1 IM. Let

$$H=\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1} \tag{2.1}$$

If $H\not\equiv 0$, then

$$\begin{aligned} \mathrm{T(r,F)}+\mathrm{T(r,G)} \le\ & 2[N_2\left(r,\tfrac{1}{F}\right)+N_2\left(r,\tfrac{1}{G}\right)+N_2(r,F)+N_2(r,G)] \\ & +3[\overline{N}\left(r,\tfrac{1}{F}\right)+\overline{N}\left(r,\tfrac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)]+S(r,F) \\ & +S(r,G)]. \end{aligned} \tag{2.2}$$

3. PROOF OF THEOREMS

Proof of Theorem 1.1.

Denote

$$F_1(z)=f^m(z)P_n\big(f(qz+c)\big)\prod_{j=1}^{k}f^{(j)}(z)$$

$$G_1(z)=g^m(z)P_n\big(g(qz+c)\big)\prod_{j=1}^{k}g^{(j)}(z)$$

$\Rightarrow S_1(r, F_1) = S_1(r, f)$ and $S_1(r, G_1) = S_1(r, g)$ (3.1)

Since $f(z)$ is a transcendental entire function of zero order, By Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have

$$T(r,F_1)\le T\big(r,f^m(z)\big)+T\left(r,P_n\big(f(qz+c)\big)\right)+T\left(r,\prod_{j=1}^{k}f^{(j)}(z)\right)$$

$$\le (m+n)T(r,f)+\sum_{j=1}^{k}T(r,f^{(j)}(z))+S_1(r,f)$$

$$\begin{aligned} &\le (m+n)T(r,f)+T(r,f^{(1)}(z))+T(r,f^{(2)}(z))+\cdots+T(r,f^{(k)}(z) \\ &\qquad +S_1(r,f) \\ &\le (m+n)T(r,f)+T(r,f)+\overline{N}(r,f)+T(r,f)+2\overline{N}(r,f)+\cdots \\ &\qquad +T(r,f)+k\overline{N}(r,f)+S_1(r,f) \\ &\therefore T(r,F_1)\le (m+n+k)T(r,f)+S_1(r,f) \end{aligned} \tag{3.2}$$

On the other hand,

$$(m+k)T(r,f) = T\left(r, f^{m+k}(z)\right) + S_1(r,f)$$
$$= T\left(r, \frac{F_1 . f^k}{P_n\big(f(qz+c)\big)\prod_{j=1}^{k} f^{(j)}(z)}\right)$$

$$\leq T(r,F_1) + T\left(r, P_n\big(f(qz+c)\big)\right) + T\left(r, \frac{f^k}{\prod_{j=1}^{k} f^{(j)}(z)}\right)$$
$$+ S_1(r,f)$$
$$\leq T(r,F_1) + nT(r,f) + T\left(r, \frac{\prod_{j=1}^{k} f^{(j)}(z)}{f^k}\right) + S_1(r,f)$$
$$\leq T(r,F_1) + nT(r,f) + m\left(r, \frac{\prod_{j=1}^{k} f^{(j)}(z)}{f^k}\right)$$
$$+N\left(r, \frac{\prod_{j=1}^{k} f^{(j)}(z)}{f^k}\right) + S_1(r,f)$$

$$\leq T(r,F_1) + nT(r,f) + \sum_{j=1}^{k} N\left(r, \frac{f^{(j)}(z)}{f}\right) + S_1(r,f)$$
$$\leq T(r,F_1) + nT(r,f) + \sum_{j=1}^{k} \left[N\left(r, f^{(j)}(z)\right) + N\left(r, \tfrac{1}{f}\right)\right] + S_1(r,f)$$
$$\leq T(r,F_1) + (n+k)T(r,f) + S_1(r,f)$$
$$\therefore (m-n)T(r,f) \leq T(r,F_1) + S_1(r,f) \tag{3.3}$$

From (3.2) and (3.3), we have

$$(m-n)\, T(r,f) + S_1\,(r,f) \leq T(r,F_1) \leq (m+n+k)T(r,f) + S_1\,(r,f) \tag{3.4}$$

Similarly,

$$(m-n)T(r,g) + S_1\,(r,g) \leq T(r,G_1) \leq (m+n+k)\, T(r,g) + S_1\,(r,g) \tag{3.5}$$

Since $f(z)$ and $g(z)$ are entire functions of order zero and F_1 and G_1 share $\alpha(z)$ CM, we have

$$\frac{F_1(z)-\alpha(z)}{G_1(z)-\alpha(z)} = \eta$$

where η is a non-zero constant.

If $\eta \neq 1$, then we have

$$F_1\,(z) - \eta G_1\,(z) = \alpha(z)(1-\eta)$$

Since $P_n(z)$ has t_n distinct zeros, by using the Second main theorem, Lemma 2.2 and Lemma 2.4, we have

$$T(r,F_1) \le \bar{N}(r,F_1) + \bar{N}\left(r,\frac{1}{F_1}\right) + \bar{N}\left(\frac{1}{F_1-\alpha(z)(1-\eta)}\right) + S(r,F_1)$$

$$\le \bar{N}\left(r,\frac{1}{f^m(z)P_n\big(f(qz+c)\big)\prod_{j=1}^{k} f^{(j)}(z)}\right) + \bar{N}\left(r,\frac{1}{G_1}\right) + S_1(r,f)$$

$$\le \sum_{j=1}^{t_n} \bar{N}\left(r,\frac{1}{\big(f(qz+c)-\gamma_j\big)}\right) + \bar{N}\left(r,\frac{1}{\prod_{j=1}^{k} f^{(j)}(z)}\right)$$

$$+\bar{N}\left(r,\frac{1}{f^m(z)}\right) + \bar{N}\left(r,\frac{1}{G_1}\right) + S_1(r,f)$$

$$\le (t_n + 1 + k)\, T(r, f) + (t_n + 1 + k)\, T(r, g) + S_1(r, f) + S_1(r, g), \tag{3.6}$$

Where $\gamma_j, \gamma_j, \ldots, \gamma_{t_n}$ are the distinct zeros of $P_n(z)$. Similarly, we have

$$T(r,G_1) \le (t_n+1+k)T(r,g) + (t_n+1+k)T(r,f) + S_1(r,f) + S_1(r,g). \tag{3.7}$$

From (3.4), (3.5), (3.6) and (3.7), we have

$$(m-n)(T(r,f)+T(r,g)) \le 2(t_n+1+k)(T(r,f)+T(r,g)) + S_1(r,f) + S_1(r,g)$$

$$\therefore (m-n-2t_n-2k-2)(T(r,f)+T(r,g)) \le S_1(r,f) + S_1(r,g)$$

which contradicts for $m \ge n + 2t_n + 2k + 3$.

Therefore $\eta = 1$, then we have $F_1(z) = G_1(z)$

That is,

$$f^m(z)P_n\big(f(qz+c)\big)\prod_{j=1}^{k} f^{(j)}(z) = g^m(z)P_n\big(g(qz+c)\big)\prod_{j=1}^{k} g^{(j)}(z).$$

Proof of Theorem 1.2.

Denote

$F_2(z) = P_m(f(z))\, f^n(qz + c) \prod_{j=1}^{k} f^{(j)}(z)$ and

$$G_2(z) = P_m\big(g(z)\big)g^n(qz+c)\prod_{j=1} g^{(j)}(z)$$

$$\Rightarrow S_1(r, F_2) = S_1(r, f) \text{ and } S_1(r, G_2) = S_1(r, g) \tag{3.8}$$

Since $f(z)$ is a transcendental entire function of zero order, By Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have

$$T(r,F_2) \leq T\left(r,P_m\big(f(z)\big)\right) + T\big(r,f^n(qz+c)\big) + T(r,\prod_{j=1}^{k} f^{(j)}(z))$$
$$\leq (m+n)T(r,f) + \sum_{j=1}^{k} T(r,f^{(j)}(z)) + S_1(r,f)$$
$$T(r,F_2) \leq (m+n+k)T(r,f) + S_1(r,f) \tag{3.9}$$

On the other hand,

$$(n+k)T(r,f) = T\left(r,f^{n+k}(z)\right) \leq T(r,f^n(qz+c)f^k) + S_1(r,f)$$
$$\leq T\left(r,\frac{F_2.f^k}{P_m\big(f(z)\big)\prod_{j=1}^{k} f^{(j)}(z)}\right) + S_1(r,f)$$
$$\leq T(r,F_2) + T\left(r,P_m\big(f(z)\big)\right) + T\left(r,\frac{f^k}{\prod_{j=1}^{k} f^{(j)}(z)}\right) + S_1(r,f)$$
$$\leq T(r,F_2) + mT(r,f) + T\left(r,\frac{\prod_{j=1}^{k} f^{(j)}(z)}{f^k}\right) + S_1(r,f)$$
$$\leq T(r,F_2) + mT(r,f) + \sum_{j=1}^{k} N\left(r,\frac{f^{(j)}(z)}{f^k}\right) + S_1(r,f)$$
$$\leq T(r,F_2) + mT(r,f) + \sum_{j=1}^{k}\left[N\left(r,f^{(j)}(z)\right) + N\left(r,\frac{1}{f}\right)\right] + S_1(r,f)$$
$$\leq T(r,F_2) + (m+k)T(r,f) + S_1(r,f)$$
$$\therefore (n-m)T(r,f) \leq T(r,F_2) + S_1(r,f) \tag{3.10}$$

From (3.9) and (3.10), we have

$$(n-m)T(r,f) + S_1(r,f) \leq T(r,F_2) \leq (m+n+k)T(r,f) + S_1(r,f) \tag{3.11}$$

Similarly,

$$(n-m)T(r,g) + S_1(r,g) \leq T(r,G_2) \leq (m+n+k)\,T(r,g) + S_1(r,g) \tag{3.12}$$

Since $f(z)$ and $g(z)$ are entire functions of order zero and F_2 and G_2 share $\alpha(z)$ CM, we have

$$\frac{F_2(z)-\alpha(z)}{G_2(z)-\alpha(z)} = \eta$$

where η is a non-zero constant.

If $\eta \neq 1$, then we have

$$F_2(z) - \eta G_2(z) = \alpha(z)(1-\eta)$$

Since $P_m(z)$ has t_m distinct zeros, by using the Second main theorem, Lemma 2.2 and Lemma 2.4, we have

$$T(r,F_2) \leq \overline{N}(r,F_2) + \overline{N}\left(r,\frac{1}{F_2}\right) + \overline{N}\left(\frac{1}{F_2-\alpha(z)(1-\eta)}\right) + S(r,F_2)$$

$$\leq \overline{N}\left(r,\frac{1}{P_m(f(z))f^n(qz+c)\prod_{j=1}^{k}f^{(j)}(z)}\right) + \overline{N}\left(r,\frac{1}{G_2}\right) + S_1(r,f)$$

$$\leq \overline{N}\left(r,\frac{1}{P_m(f(z))}\right) + \overline{N}\left(r,\frac{1}{f^n(qz+c)}\right) + \overline{N}\left(r,\frac{1}{\prod_{j=1}^{k}f^{(j)}(z)}\right)$$

$$+\overline{N}\left(r,\frac{1}{G_2}\right) + S_1(r,f)$$

$$\leq \sum_{j=1}^{t_m}\overline{N}\left(r,\frac{1}{(f(z)-\gamma_j)}\right) + T(r,f) + T\left(r,\prod_{j=1}^{k}f^{(j)}(z)\right)$$

$$+\overline{N}\left(r,\frac{1}{G_2}\right) + S_1(r,f)$$

$$\leq (t_m+1+k)\,T(r,f) + (t_m+1+k)\,T(r,g) + S_1(r,f) + S_1(r,g), \qquad (3.13)$$

Where $\gamma_j, \gamma_j, \ldots, \gamma t_m$ are the distinct zeros of $P_m(z)$. Similarly, we have

$$T(r,G_2) \leq (t_m+1+k)T(r,g) + (t_m+1+k)T(r,f) + S_1(r,f) + S_1(r,g). \qquad (3.14)$$

From (3.11), (3.12), (3.13) and (3.14), we have

$$(n-m)(T(r,f)+T(r,g)) \leq 2(t_m+1+k)(T(r,f)+T(r,g)) + S_1(r,f) + S_1(r,g)$$

$$\therefore (n-m-2t_m-2k-2)(T(r,f)+T(r,g)) \leq S_1(r,f) + S_1(r,g)$$

which contradicts for $n \geq m + 2t_m + 2k + 3$.

Therefore $\eta = 1$, then we have $F_2(z) = G_2(z)$

That is,

$$P_m(f(z))f^n(qz+c)\prod_{j=1}^{k}f^{(j)}(z) = P_m(g(z))g^n(qz+c)\prod_{j=1}^{k}g^{(j)}(z)$$

Proof of Theorem 1.3.

Denote

$$F = f^n(z)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j} \text{ and } G = g^n(z)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}$$

Then

$$T(r,F) = T\left(r, f^n(z)\prod_{j=1}^{d} f\left(q_j z + c_j\right)^{s_j}\right)$$

$$\leq T\left(r, f^n(z)\right) + T\left(r, \prod_{j=1}^{d} f\left(q_j z + c_j\right)^{s_j}\right) + S_1(r,f)$$

$$\leq (n+\lambda)T(r,f) + S_1(r,f)$$

On the other hand,

$$nT(r,f) = T\left(r, f^n(z)\right) = T\left(r, \frac{F}{\prod_{j=1}^{d} f\left(q_j z + c_j\right)^{s_j}}\right)$$

$$\leq T(r,F) + \lambda T(r,f) + S_1(r,f)$$

$$\Rightarrow (n-\lambda)T(r,f) + S_1(r,f) \leq T(r,F)$$

$$\therefore (n-\lambda)T(r,f) + S_1(r,f) \leq T(r,F) \leq (n+\lambda)T(r,f) + S_1(r,f) \qquad (3.15)$$

Similarly,

$$(n-\lambda)T(r,g) + S_1(r,g) \leq T(r,G) \leq (n+\lambda)T(r,g) + S_1(r,g) \qquad (3.16)$$

Now

$$N_2\left(r, \frac{1}{f^n(z)}\right) = N_{1)}\left(r, \frac{1}{f^n(z)}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{f^n(z)}\right) = 2\overline{N}_{(2}\left(r, \frac{1}{f^n(z)}\right) = N_{(2}\left(r, \frac{1}{f(z)}\right)$$

$$\leq N\left(r, \frac{1}{f(z)}\right) \leq T\left(r, \frac{1}{f(z)}\right) \leq T(r,f) + S_1(r,f)$$

Therefore,

$$N_2\left(r, \frac{1}{F}\right) = N_2\left(r, \frac{1}{f^n(z)}\right) + N_2\left(r, \frac{1}{\prod_{j=1}^{d} f\left(q_j z + c_j\right)^{s_j}}\right)$$

$$\leq T(r,f) + \sum_{j=1}^{d} N_2\left(r, \frac{1}{\prod_{j=1}^{d} f\left(q_j z + c_j\right)^{s_j}}\right) + S_1(r,f)$$

$$\leq (1+2d)\, T(r,f) + S_1(r,f) \qquad (3.17)$$

Similarly,

$$N_2\left(r,\frac{1}{G}\right) \leq (1+2d)T(r,g) + S_1(r,g) \tag{3.18}$$

$$N_2(r,F) \leq (1+2d)T(r,f) + S_1(r,f) \tag{3.19}$$

$$N_2(r,G) \leq (1+2d)T(r,g) + S_1(r,g) \tag{3.20}$$

Since F and G share 1 CM, let us assume (i) of Lemma 2.5 holds and hence $T(r, F) + T(r, G)$

$$\leq 2[N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G)] + S(r,F) + S(r,G)$$

substituting (3.17)-(3.20), we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2[(1+2d)T(r, f) + (1 + 2d)\, T(r, g) + (1 + 2d)\, T(r, f) \\ &\quad +(1 + 2d)\, T(r, g)] + S_1(r, f) + S_1(r,g) \\ &\leq 2[2(1 + 2d)\, (T(r, f) + T(r, g))] + S_1(r, f) + S_1(r, g) \end{aligned} \tag{3.21}$$

From (3.15), (3.16) and (3.21), we get

$$\begin{aligned} &(n - \lambda)\, (T(r, f) + T(r, g)) \\ &\qquad \leq 4(1+2d)\, (T(r, f) + T(r, g)) + S_1(r, f) + S_1(r,g) \end{aligned}$$

$$\Rightarrow (n - \lambda - 8d - 4)\, (T(r, f) + T(r, g)) \leq S_1(r, f) + S_1(r, g) \tag{3.22}$$

which contradicts for $n \geq \lambda + 8d + 5$.

Thus by Lemma 2.5, we have

either $F \equiv G$ or $F.\, G \equiv 1$ (3.23)

If $F \equiv G$, that is,

$$f^n(z)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j} = g^n(z)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}$$

Let $h(z) = \frac{f(z)}{g(z)}$. Suppose that $h(z)$ is not a constant. Then we have

$$h^n(z)\prod_{j=1}^{d} h(q_j z + c_j)^{s_j} = 1 \tag{3.24}$$

Lemma 2.2 and (3.24) imply that

$$T\big(r,h(z)\big) = T\big(r,h^n(z)\big) = T\left(r,\frac{1}{\prod_{j=1}^{d} f\big(q_j z + c_j\big)^{s_j}}\right) \leq \lambda T(r,h) + S_1(r,h)$$

$$\Rightarrow (n - \lambda)T(r,h) \leq S_1(r,h)$$

Hence $h(z)$ must be a nonzero constant, since $n \geq \lambda + 8d + 5$.

Set $h(z) = t$. By (3.24), we know $t^{n+\lambda} = 1$.

Thus $f(z) = tg(z)$, where $t^{n+\lambda} = 1$.

Which is one of the conclusion of Theorem 1.3.

Again by (3.23), we have

$F.G \equiv 1$, that is,

$$f^n(z)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j} \,.\, g^n(z)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j} = 1$$

Let $l(z) = f(z)\,g(z)$. Suppose that $l(z)$ is not a nonzero constant. Then we have obtain

$$l^n(z)\prod_{j=1}^{d} l(q_jz+c_j)^{s_j} = 1 \tag{3.25}$$

Lemma 2.2 and (3.25), imply that

$$nT\big(r, l(z)\big) = T\big(r, l^n(z)\big) = T\left(r, \frac{1}{\prod_{j=1}^{d} l\left(q_jz+c_j\right)^{s_j}}\right) \leq \lambda T(r,l) + S_1(r,l)$$
$$\Rightarrow (n-\lambda)T(r,l) \leq S_1(r,l)$$

Hence $l(z)$ must be a nonzero constant, since $n \geq \lambda + 8d + 5$.

Set $l(z) = t$. By (3.25), we know $t^{n+\lambda} = 1$.

Thus $f \,.\, g = t$, where $t^{n+\lambda} = 1$.

Proof of Theorem 1.4

Let

$$F = f^n(z)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j} \text{and} G = g^n(z)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}$$

and let H be defined as in Lemma 2.6. Using the similar proof as in the Theorem 1.3, (3.15)-(3.20) holds.

and by Lemma 2.2 , we obtain

$$\overline{N}\big(r, F(z)\big) = \overline{N}\big(r, f^n(z)\big) + \overline{N}(r, \prod_{j=1}^{d} f(q_jz+c_j)^{s_j})$$

$$\leq T(r,f)+\sum_{j=1}^{d}\overline{N}\left(r,f\left(q_jz+c_j\right)^{s_j}\right)+S_1(r,f)$$

$$\leq (d+1)T(r,f)+S_1(r,f) \tag{3.26}$$

Similarly,

$$N\left(r,G(z)\right)\leq (d+1)T(r,g)+S_1(r,g) \tag{3.27}$$

$$\overline{N}\left(r,\frac{1}{F(z)}\right)\leq (d+1)T(r,f)+S_1(r,f) \tag{3.28}$$

$$\overline{N}\left(r,\frac{1}{G(z)}\right)\leq (d+1)T(r,g)+S_1(r,g) \tag{3.29}$$

If $H \not\equiv 0$ and since F and G share 1 IM, then by Lemma 2.6 , we have

$T(r, F) + T(r, G)$

$$\leq 2[N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)]$$

$$+3\left[\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)\right]+S(r,F) +S(r,G)] \tag{3.30}$$

Substituting (3.17)-(3.20) and (3.26)-(3.29), we get

$$\begin{aligned} T(r,F)+T(r,G) &\leq 4(1+2d)\big(T(r,f)+T(r,g)\big) \\ &\quad +3\left[2(d+1)\big(T(r,f)+T(r,g)\big)\right]+S_1(r,f)+S_1(r,g) \\ &\leq [4(1+2d)+6(d+1)]\big(T(r,f)+T(r,g)\big) \\ &\quad +S_1(r,f)+S_1(r,g) \end{aligned} \tag{3.31}$$

From (3.31), (3.15) and (3.16), we get

$$\begin{aligned} &(n-\lambda)\big(T(r,f)+T(r,g)\big) \\ &\qquad \leq (14d+10)\big(T(r,f)+T(r,g)\big)+S_1(r,f)+S_1(r,g) \\ &\Rightarrow (n-\lambda-14d-10)\big(T(r,f)+T(r,g)\big)\leq S_1(r,f)+S_1(r,g) \end{aligned} \tag{3.32}$$

which contradicts for $n \geq \lambda - 14d - 11$.

Hence we have $H \equiv 0$.

Integrating (2.1) twice and using $H \equiv 0$, we have

$$\frac{1}{F-1}=\frac{a}{G-1}+b \tag{3.33}$$

where $a \neq 0$ and b are constants. By (3.33), we have

$$F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)} \text{ and } G = \frac{(b-a)F+(a-b-1)}{bF-(b+1)} \tag{3.34}$$

Next, we consider following cases.

Case (i). $b \neq 0, -1$in (3.34) and for constants a and b.

If $a - b - 1 \neq 0$, by (3.34), we obtain

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right)$$

Using Second fundamental theorem, Lemma 2.2 and (3.16), we obtain

$$(n-\lambda)T(r,g) \leq T(r,G) + S_1(r,g)$$

$$\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) + S(r,G) + S_1(r,g)$$

$$\leq \overline{N}\left(r, \frac{1}{g^n(z)}\right) + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}}\right) + \overline{N}(r, g^n(z))$$

$$+\overline{N}\left(r, \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}\right) + \overline{N}\left(r, \frac{1}{f^n(z)}\right)$$

$$+\overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}\right) + S_1(r,f) + S_1(r,g)$$

$$\leq (2+2d)T(r,g) + (1+d)T(r,f) + S_1(r,f) + S_1(r,g)$$

$$\therefore\ (n-\lambda-2-2d)\, T(r,g) \leq (1+d)\, T(r,f) + S_1(r,f) + S_1(r,g) \tag{3.35}$$

Similarly,

$$(n-\lambda-2-2d)\, T(r,g) \leq (1+d)\, T(r,f) + S_1(r,f) + S_1(r,g) \tag{3.36}$$

From (3.34) and (3.35), we get

$$(n-\lambda-2-2d)\,(T(r,f) + T(r,g)) \leq 2(1+d)\,(T(r,f) + T(r,g)) + S_1(r,f) + S_1(r,g)$$

$$\Rightarrow (n-\lambda-4d-4)\,(T(r,f) + T(r,g)) \leq S_1(r,f) + S_1(r,g)$$

which contradicts with $n \geq \lambda + 14d + 11$.

Hence, we obtain $a - b - 1 = 0$, so

$$F = \frac{(b+1)G}{bG+(a-b)} \text{ and } G = \frac{(b-a)F}{bF-(b+1)}$$

Using the similar method as above, we obtain

$$(n-\lambda)T(r,g) \le T(r,G) + S_1(r,g)$$

$$\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G+\left(\frac{1}{b}\right)}\right) + S(r,G) + S_1(r,g)$$

$$\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}(r,F) + S_1(r,g)$$

$(n-\lambda)\ T(r,g) \le (2+2d)\ T(r,g) + (1+d)\ T(r,f) + S_1(r,g)$

$\Rightarrow (n-\lambda-2-2d)\ T(r,g) \le (1+d)\ T(r,f) + S_1(r,f) + S_1(r,g)$ (3.37)

Similarly,

$(n-\lambda-2-2d)\ T(r,f) \le (1+d)\ T(r,g) + S_1(r,f) + S_1(r,g)$ (3.38)

From (3.37) and (3.38), we get

$\Rightarrow (n-\lambda-4-4d)\ (T(r,f) + T(r,g)) \le S_1(r,f) + S_1(r,g)$

which contradicts withn $\ge \lambda + 14d + 11$.

Case (ii). If $b = -1$ and $a = -1$ in (3.34), then $FG = 1$ follows trivially.

Therefore, we may consider the case $b = -1$ and $a \ne -1$ in (3.34), we have

$$F = \frac{a}{a+1-G} \text{ and } G = \frac{(a+1)F-a}{F}$$

As in Case (i), we get a contradiction.

Set $h(z) = \frac{f(z)}{g(z)}$. Suppose that $h(z)$ is not a constant. Then, we have

$$h^n(z)\prod_{j=1}^{d} h(q_j z + c_j)^{s_j} = 1 \quad (3.39)$$

Lemma 2.2 and (3.39) imply that

$$nT(r,h(z)) = T(r,h^n(z)) = T\left(r,\frac{1}{\prod_{j=1}^{d} h(q_j z+c_j)^{s_j}}\right) \le \lambda T(r,h) + S_1(r,h)$$

$$\Rightarrow (n-\lambda)T(r,h) \le S_1(r,h)$$

Hence $h(z)$ must be a nonzero constant, since $n \ge \lambda\ 14d + 11$.

Set $h(z) = t$. By (3.38), we know $t^{n+\lambda} = 1$.

Thus $f(z) = tg(z)$, where $t^{n+\lambda} = 1$.

Which is one of the conclusion of Theorem 1.2.

Case (iii). If $b = 0$ and $a = 1$ in (3.34), then $F = G$ follows trivially.

Therefore, we may consider the case \$b = 0\$ and $a \neq 1$ in (3.33), we have

$$F = \frac{G + a - 1}{a} \text{ and } G = aF - (a-1)$$

As in Case (i), we get a contradiction.

Let $l(z) = f(z).\, g(z)$. Suppose that $l(z)$ is not a nonzero constant. Then we have obtain

$$l^n(z) \prod_{j=1}^{d} l(q_j z + c_j)^{s_j} = 1 \tag{3.40}$$

Lemma 2.2 and (3.40), imply that

$$nT(r, l(z)) = T(r, l^n(z)) = T\left(r, \frac{1}{\prod_{j=1}^{d} l(q_j z + c_j)^{s_j}}\right) \leq \lambda T(r, l) + S_1(r, l)$$

$$\Rightarrow (n - \lambda) T(r, l) \leq S_1(r, l)$$

Hence $l(z)$ must be a nonzero constant, since $n \geq \lambda + 14d + 11$.

Set $l(z) = t$. By (3.39), we know $t^{n+\lambda} = 1$.

Thus $f \,.\, g = t$, where $t^{n+\lambda} = 1$.

Acknowledgments

The first author is supported by Ref. No. F. 510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016.

References

[1] Bhoosnurmath S. S. and Dyavanal R. S., Uniqueness of meromorphic functions sharing sets, *Bull. Math. Anal. Appl.* Vol. 3, 200-208, (2011).

[2] Chern P T Y, On meromorphic functions with finite logarithmic order, *Trans. Am. Math.* Soc. 358(2), 473-489(2006).

[3] Dyavanal R S, Desai R V, Uniqueness of Difference Polynomials of Entire Functions, *Applied Math. Sci.*, Vol.8, 2014, no.69, 3419-3424.

[4] Dyavanal R. S., Desai R. V., Uniqueness of q- shift difference and differentialpolynomials of Entire functions, *Far East Journal of Applied Mathematics*, Vol. 91, no. 3 (2015), 189-202.

[5] R.S. Dyavanal, A.M. Hattikal, Weighted sharing of uniqueness of difference polynomials of meromorphic functions, *Far East J. Math. Sci.* 98(3) (2015), 293-313.

[6] R.S. Dyavanal and A. M. Hattikal, Weighted sharing of difference-differential polynomials of entire functions, *Mathematical Sciences International Research Journal*, Vol. 4, No. 2, (2015), 276-280.

[7] R.S. Dyavanal, A.M. Hattikal, Unicity theorems on difference polynomials of meromorphic functions sharing one value, *Int. J. Pure Appl. Math. Sci.* 9(2) (2016), 89-97.

[8] R.S. Dyavanal, A.M. Hattikal, On the uniqueness of product of difference polynomials of meromorphic functions, *Konuralp J. Math.* 4(2) (2016), 42-55.

[9] R.S. Dyavanal and A.M. Hattikal, Uniqueness of q-shift Difference Polynomialsof Meromorphic Functions Sharing a Small Function, *Journal of New trends in Mathematical Sciences* (Accepted).

[10] Dyavanal R. S. and Mathai M. M., Uniqueness of Difference-Differential polynomials of meromorphic functions and its applications, *Indian J. Math. Math. Sci.* 12, No. 1, 11-30 (2016).

[11] Dyavanal R. S. and Mathai M. M., Uniqueness of Difference-Differential polynomials of meromorphic functions, Ukrainian Math. J. (Accepted).

[12] Halburd R. G., Korhonen R. J., Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.* 314 (2006), 477-487.

[13] Halburd R. G., Korhonen R. J., Nevanlinna theory for the difference operator, *Ann. Acad. Sci. Fenn.* 31(2006), 463-478.

[14] Hayman W. K., Meromorphic Functions., The Clarendon Press, Oxford (1964).

[15] Liu K., Qi X G., Meromorphic solutions of q-shift difference equations, *Ann. Pol. Math.* 101, 215-225 (2011).

[16] Liu Y, Cao Y H, Qi X G, Yi H X, Value sharing results for q-shifts difference polynomials, Discrete Dynamics in Nature and Society, Vol. 2013, Article ID 152069.

[17] Wang X L, Xu H Y, Zhan T S, Properties of q-shift difference-differential polynomials of meromorphic functions, Advances in Difference equations 2014, 2014: 249.

[18] Xu J, Yi H, Uniqueness of entire functions and differential polynomials, Bulletin of the Korean Mathematical Society, Vol. 44, no. 4, pp. 623-629, 2007.

[19] Xu J F, Zhang X B, The zeros of difference polynomials of meromorphic functions, *Abstr. App. Anal* 2012, Article ID 357203(2012).

[20] Xu J F, Zhang X B, The zeros of q-difference polynomials of meromorphic functions, Adv.Differ. Equ. 2012, Article ID 200 (2012).

[21] Yang L, Value Distribution Theory, Springer, Berlin (1993).

[22] Yang C C, Hua X, Uniqueness and value sharing of meromorphic functions, Annales Academiae Scientiarum Fennicae. *Mathematica*, Vol. 22, no. 2, pp. 395-406, 1997.

[23] Yi H X, Yang C C, Uniqueness Theory of Meromorphic Functions, Kluwer Academic, Dordrecht(2003) Chinese original, Science Press, Beijing, 1995.

[24] Zheng X M, Xu H Y, On value distribution and uniqueness of meromorphic function with finite logarithmic order concerning its derivative and q-shift difference, *Journal of inequalities and applications*, 2014, 2014: 295.