

## NEAR-MARTINGALE PROPERTY OF ANTICIPATING STOCHASTIC INTEGRATION

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ABSTRACT. A stochastic process  $X_t$  is called a near-martingale with respect to a filtration  $\{\mathcal{F}_t\}$  if  $E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s]$  for all  $s \leq t$ . It is called a near-submartingale with respect to  $\{\mathcal{F}_t\}$  if  $E[X_t|\mathcal{F}_s] \geq E[X_s|\mathcal{F}_s]$  for all  $s \leq t$ . Near-martingale property is the analogue of martingale property when the Itô integral is extended to non-adapted integrands. We prove that  $X_t$  is a near-martingale (near-submartingale) if and only if  $E[X_t|\mathcal{F}_t]$  is a martingale (near-submartingale, respectively). Doob–Meyer decomposition theorem is extended to near-submartingale. We study stochastic differential equations with anticipating initial conditions and obtain a relationship between such equations and the associated stochastic differential equations of the Itô type.

### 1. Introduction

Let  $B(t)$  be a fixed Brownian motion starting at 0 and  $\{\mathcal{F}_t; a \leq t \leq b\}$  a filtration,  $a \geq 0$ , such that

- (a)  $B(t)$  is  $\{\mathcal{F}_t\}$ -adapted, namely,  $B(t)$  is  $\mathcal{F}_t$ -measurable for each  $t \in [a, b]$ ;
- (b)  $B(t) - B(s)$  and  $\mathcal{F}_s$  are independent for any  $s \leq t$  in  $[a, b]$ .

For example, we can take  $\mathcal{F}_t = \sigma\{B(s); a \leq s \leq t\}$  for  $t \in [a, b]$ .

Consider two continuous stochastic processes  $f(t)$  and  $\varphi(t)$  with  $f(t)$  being  $\{\mathcal{F}_t\}$ -adapted and  $\varphi(t)$  *instantly independent* of  $\{\mathcal{F}_t\}$ , namely,  $\varphi(t)$  and  $\mathcal{F}_t$  are independent for each  $t \in [a, b]$ . In [1, 2] Ayed and Kuo introduced a stochastic integral defined by

$$\int_a^b f(t)\varphi(t) dB(t) = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1})), \quad (1.1)$$

provided that the limit in probability exists. Note that the evaluation points for  $\varphi(t)$  are the right endpoints of subintervals. It is easy to check that the stochastic integral in Equation (1.1) is well-defined.

Next, consider a stochastic process  $\Phi(t)$  of the form

$$\Phi(t) = \sum_{i=1}^m f_i(t)\varphi_i(t), \quad a \leq t \leq b, \quad (1.2)$$

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where for each  $i$ , the stochastic integral  $\int_a^b f_i(t)\varphi_i(t) dB(t)$  exists as defined by Equation (1.1). Then we define the stochastic integral of  $\Phi(t)$  by

$$\int_a^b \Phi(t) dB(t) = \sum_{i=1}^m \int_a^b f_i(t)\varphi_i(t) dB(t). \tag{1.3}$$

By Lemma 2.1 in [4], the stochastic integral  $\int_a^b \Phi(t) dB(t)$  is well-defined.

Now, let  $\Phi(t), a \leq t \leq b$ , be a stochastic process such that there exists a sequence  $\{\Phi_n(t)\}_{n=1}^\infty$  of stochastic processes of the form in Equation (1.2) satisfying the following conditions:

- (a)  $\int_a^b |\Phi(t) - \Phi_n(t)|^2 dt \rightarrow 0$  almost surely as  $n \rightarrow \infty$ ,
- (b)  $\int_a^b \Phi_n(t) dB(t)$  converges in probability as  $n \rightarrow \infty$ ,

where for each  $n \geq 1$ , the stochastic integral  $\int_a^b \Phi_n(t) dB(t)$  is defined by Equation (1.3). Then we define the *stochastic integral* of  $\Phi(t)$  by

$$\int_a^b \Phi(t) dB(t) = \lim_{n \rightarrow \infty} \int_a^b \Phi_n(t) dB(t), \quad \text{in probability.} \tag{1.4}$$

Obviously, this stochastic integral  $\int_a^b \Phi(t) dB(t)$  is well-defined.

It can be easily checked that the stochastic integral  $\int_a^b \Phi(t) dB(t)$  in Equation (1.4) reduces to an Itô integral when  $\Phi(t) = f(t)$  is an adapted stochastic process such that  $\int_a^b |f(t)|^2 dt < \infty$  almost surely. On the other hand, recall that in the Itô theory of stochastic integration, there are two fundamental properties, namely, the martingale property and the Markov property. Hence it is natural to ask the following question:

*“In our extension of the Itô theory, what are the analogues of the martingale property and the Markov property?”*

In this article we will address the analogue of the martingale property for our extension of the Itô integral. In Section 2 we will give a simple example to show how we discover this analogue and prove some theorems to relate it to the martingale property. In the sequel sections we will prove some theorems regarding to this analogue of martingale property and linear stochastic differential equations.

**2. Near-martingale, Near-submartingale, and Near-supermartingale**

We first consider a simple stochastic integral  $\int_0^t B(1) dB(s)$  for  $0 \leq t \leq 1$ . By Equation (1.6) in [1] we have

$$\int_0^t B(1) dB(s) = B(1)B(t) - t, \quad 0 \leq t \leq 1.$$

This equality can also be derived by using Equations (1.1) and (1.3), see Example 2.4 in [4]. Let  $X_t$  be the stochastic process

$$X_t = \int_0^t B(1) dB(s) = B(1)B(t) - t, \quad 0 \leq t \leq 1. \tag{2.1}$$

Note that  $X_t$  is not  $\mathcal{F}_t$ -measurable. Hence  $X_t, 0 \leq t \leq 1$ , is not a martingale with respect to the filtration  $\{\mathcal{F}_t\}$ . However, for  $s \leq t$ , we can easily check the conditional expectation

$$E[X_t|\mathcal{F}_s] = B(s)^2 - s. \quad (2.2)$$

On the other hand, if we put  $t = s$  in Equation (2.2), then we have

$$E[X_s|\mathcal{F}_s] = B(s)^2 - s. \quad (2.3)$$

Now observe that it follows from Equations (2.2) and (2.3) that the stochastic process  $X_t = \int_0^t B(1) dB(s)$  satisfies the following equality:

$$E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s], \quad \forall 0 \leq s \leq t \leq 1. \quad (2.4)$$

The equality in Equation (2.4) is satisfied by many other stochastic processes. For example, from Example 2.3 in [1] we have

$$Y_t = \int_0^t B(1)^2 dB(s) = B(1)^2 B(t) - 2B(1)t, \quad 0 \leq t \leq 1. \quad (2.5)$$

It is easy to check the following conditional expectation for  $s \leq t$ ,

$$E[Y_t|\mathcal{F}_s] = B(s)^3 - 3sB(s) + B(s),$$

which shows that the stochastic process  $Y_t, 0 \leq t \leq 1$  satisfies the equality

$$E[Y_t|\mathcal{F}_s] = E[Y_s|\mathcal{F}_s], \quad \forall 0 \leq s \leq t \leq 1,$$

namely, it satisfies Equation (2.4).

The above discussion leading to Equation (2.4) is the motivation for the concept of near-martingale first introduced in [9].

**Definition 2.1.** A stochastic process  $X_t, a \leq t \leq b$ , with  $E|X_t| < \infty$  for all  $t$  is called a *near-martingale* with respect to a filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$  if for any  $a \leq s \leq t \leq b$  we have the equality

$$E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s], \quad \text{almost surely,}$$

or equivalently

$$E[X_t - X_s|\mathcal{F}_s] = 0, \quad \text{almost surely.} \quad (2.6)$$

*Remark 2.2.* We have learned from Professor A. A. Dorogovtsev that he considered the condition in Equation (2.6) from a different motivation in [3].

Obviously, if a near-martingale  $X_t$  is  $\{\mathcal{F}_t\}$ -adapted, then it is a martingale. We state two theorems from [9].

**Theorem 2.3.** (Theorem 3.5 [9]) *Let  $f(x)$  and  $\varphi(x)$  be continuous functions such that the stochastic integral*

$$X_t = \int_a^t f(B(s))\varphi(B(b) - B(s)) dB(s), \quad a \leq t \leq b,$$

*exists and  $E|X_t| < \infty$  for each  $t \in [a, b]$ . Then  $X_t, a \leq t \leq b$ , is a near-martingale with respect to  $\{\mathcal{F}_t\}$ .*

**Theorem 2.4.** (Theorem 3.6 [9]) *Let  $f(x)$  and  $\varphi(x)$  be continuous functions such that the stochastic integral*

$$Y^{(t)} = \int_t^b f(B(s))\varphi(B(b) - B(s)) dB(s), \quad a \leq t \leq b,$$

*exists and  $E|Y^{(t)}| < \infty$  for each  $t \in [a, b]$ . Then  $Y^{(t)}, a \leq t \leq b$ , is a near-martingale with respect to  $\{\mathcal{F}_t\}$ .*

In view of Theorems 2.3 and 2.4 we see that the concept of near-martingale is the analogue of martingale in the Itô theory for our extension of the Itô integral. In fact, The next theorem gives an intrinsic relationship between the martingale property in the Itô theory and the near-martingale property in our extension of the Itô theory.

**Theorem 2.5.** *Let  $X_t, a \leq t \leq b$ , be a stochastic process with  $E|X_t| < \infty$  for each  $t \in [a, b]$  and let  $Y_t = E[X_t|\mathcal{F}_t]$ . Then  $X_t$  is a near-martingale if and only if  $Y_t$  is a martingale.*

*Proof.* First assume that  $X_t$  is a near-martingale. Then for any  $s \leq t$  we have

$$\begin{aligned} E[Y_t|\mathcal{F}_s] &= E\{E[X_t|\mathcal{F}_t] | \mathcal{F}_s\} \\ &= E[X_t|\mathcal{F}_s] \\ &= E[X_s|\mathcal{F}_s] \\ &= Y_s. \end{aligned}$$

Hence  $Y_t$  is a martingale. Conversely, assume that  $Y_t$  is a martingale. Then for any  $s \leq t$  we have

$$\begin{aligned} E[X_t|\mathcal{F}_s] &= E\{E[X_t|\mathcal{F}_t] | \mathcal{F}_s\} \\ &= E[Y_t|\mathcal{F}_s] \\ &= Y_s \\ &= E[X_s|\mathcal{F}_s]. \end{aligned}$$

Thus  $X_t$  is a near-martingale. □

We give some examples to illustrate Theorem 2.5.

**Example 2.6.** Consider the stochastic process  $X_t = B(1)B(t) - t, 0 \leq t \leq 1$ , from Equation (2.1). We have

$$Y_t = [X_t|\mathcal{F}_t] = E[B(1)B(t) - t|\mathcal{F}_t] = B(t)E[B(1)|\mathcal{F}_t] - t = B(t)^2 - t,$$

which is a well-known martingale. Hence by Theorem 2.5 the stochastic process  $X_t = B(1)B(t) - t, 0 \leq t \leq 1$ , is a near-martingale.

**Example 2.7.** Let  $X_t = B(1)^2B(t) - 2B(1)t, 0 \leq t \leq 1$ , the stochastic process given by Equation (2.5). By direct computation we can derive the conditional expectation  $Y_t = E[X_t|\mathcal{F}_t]$  as given by

$$Y_t = B(t)^3 - 3tB(t) + B(t), \quad 0 \leq t \leq 1.$$

Note that  $B(t)^3 - 3tB(t)$  and  $B(t)$  are martingales in the Itô theory. Hence  $Y_t$  is a martingale. Then by Theorem 2.5  $X_t = B(1)^2B(t) - 2B(1)t$ ,  $0 \leq t \leq 1$ , is a near-martingale.

**Example 2.8.** Consider the following stochastic differential equation with an anticipating initial condition:

$$\begin{cases} dX_t = X_t dB(t), & 0 \leq t \leq 1, \\ X_0 = B(1). \end{cases} \quad (2.7)$$

The solution of this stochastic differential equation is given by

$$X_t = (B(1) - t)e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1.$$

(See Theorem 5.2 in [2] or Equation (3.6) in [6].) Obviously, the conditional expectation  $Y_t = E[X_t | \mathcal{F}_t]$  is given by

$$Y_t = (B(t) - t)e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1.$$

Apply the Itô formula to the function  $\theta(t, x) = (x - t)e^{x - \frac{1}{2}t}$  to get the stochastic differential of  $Y_t$ ,

$$dY_t = \left( Y_t + e^{B(t) - \frac{1}{2}t} \right) dB(t).$$

Hence  $Y_t$  is the solution of the following stochastic differential equation:

$$\begin{cases} dY_t = \left( Y_t + e^{B(t) - \frac{1}{2}t} \right) dB(t), & 0 \leq t \leq 1, \\ Y_0 = 0. \end{cases} \quad (2.8)$$

Note that  $Y_t$  is a martingale and  $X_t$  is a near-martingale. In general, it is an interesting problem to find a relationship between two equations such as Equations (2.7) and (2.8) satisfied by  $X_t$  and  $Y_t$ , respectively. We will address this problem in Section 4.

Next, we consider the analogues of submartingale and supermartingale for our extension of the Itô theory. Obviously, we just modify the equality in Definition 2.1 for a martingale.

**Definition 2.9.** A stochastic process  $X_t$ ,  $a \leq t \leq b$ , with  $E|X_t| < \infty$  for all  $t$  is called a *near-submartingale* with respect to a filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$  if for any  $a \leq s \leq t \leq b$  we have the inequality

$$E[X_t | \mathcal{F}_s] \geq E[X_s | \mathcal{F}_s], \quad \text{almost surely,}$$

or equivalently

$$E[X_t - X_s | \mathcal{F}_s] \geq 0, \quad \text{almost surely.} \quad (2.9)$$

**Definition 2.10.** A stochastic process  $X_t$ ,  $a \leq t \leq b$ , with  $E|X_t| < \infty$  for all  $t$  is called a *near-supermartingale* with respect to a filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$  if for any  $a \leq s \leq t \leq b$  we have the inequality

$$E[X_t | \mathcal{F}_s] \leq E[X_s | \mathcal{F}_s], \quad \text{almost surely,}$$

or equivalently

$$E[X_t - X_s | \mathcal{F}_s] \leq 0, \quad \text{almost surely.} \quad (2.10)$$

By similar arguments as those in the proof of Theorem 2.5 we can easily prove the following theorem.

**Theorem 2.11.** *Let  $X_t, a \leq t \leq b$ , be a stochastic process with  $E|X_t| < \infty$  for each  $t \in [a, b]$  and let  $Y_t = E[X_t|\mathcal{F}_t]$ . Then  $X_t$  is a near-submartingale (near-supermartingale) if and only if  $Y_t$  is a submartingale (supermartingale, respectively).*

Recall a well-known fact: If  $X_t, a \leq t \leq b$ , is a martingale with  $E(|X_t|^2) < \infty$ , then  $X_t^2$  is a submartingale. For the case of near-martingale, this fact does not hold in general. Below is a simple example.

**Example 2.12.** Let  $X_t = B(t)(B(1) - B(t)), 0 \leq t \leq 1$ . Then we have the conditional expectation

$$Y_t = E[X_t|\mathcal{F}_t] = E[B(t)(B(1) - B(t))|\mathcal{F}_t] = B(t)E[B(1) - B(t)|\mathcal{F}_t] = 0.$$

Obviously,  $Y_t$  is a martingale. Hence by Theorem 2.5  $X_t$  is a near-martingale. On the other hand, consider the stochastic process  $X_t^2$

$$X_t^2 = B(t)^2(B(1) - B(t))^2, \quad 0 \leq t \leq 1.$$

We can check the conditional expectation

$$E[X_t^2|\mathcal{F}_t] = E[B(t)^2(B(1) - B(t))^2|\mathcal{F}_t] = B(t)^2(1 - t),$$

which yields the following expectation

$$E\{E[X_t^2|\mathcal{F}_t]\} = t(1 - t), \quad 0 \leq t \leq 1.$$

But the function  $t(1 - t)$  is not an increasing function on  $0 \leq t \leq 1$ . Hence the stochastic process  $E[X_t^2|\mathcal{F}_t]$  is not a submartingale. Thus by Theorem 2.11  $X_t^2$  is not a near-submartingale.

### 3. Doob–Meyer’s Decomposition for Near-submartingales

In [10] we obtained Doob’s decomposition theorem for a near-submartingale sequence  $X_n, n \geq 1$ , with respect to a filtration  $\{\mathcal{F}_n; n \geq 1\}$  (for the case of random sequences, just modify Definitions 2.1 and 2.9 in an obvious way). Here we will prove Doob–Meyer’s decomposition for near-submartingales.

**Theorem 3.1.** *Let  $X_t, a \leq t \leq b$ , be a continuous near-submartingale with respect to a continuous filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$ . Then  $X_t$  has a unique decomposition*

$$X_t = M_t + A_t, \quad a \leq t \leq b, \tag{3.1}$$

where  $M_t$  is a continuous near-martingale with respect to  $\{\mathcal{F}_t; a \leq t \leq b\}$ , and  $A_t$  is a continuous stochastic process satisfying the conditions:

- (1)  $A_a = 0$ ;
- (2)  $A_t$  is increasing in  $t$  almost surely;
- (3)  $A_t$  is adapted to the filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$ .

*Proof.* We first prove the uniqueness of a decomposition. Suppose  $X_t$  has two decompositions as follows:

$$X_t = M_t + A_t = K_t + C_t, \tag{3.2}$$

where  $M_t$  and  $K_t$  are continuous near-martingales, and  $A_t$  and  $C_t$  are continuous stochastic processes satisfying conditions (1)–(3) in the theorem. Note that  $A_t$  and  $C_t$  are  $\mathcal{F}_t$ -measurable for each  $t$ . Hence by taking conditional expectation with respect to  $\mathcal{F}_t$ , we have

$$E[X_t|\mathcal{F}_t] = E[M_t|\mathcal{F}_t] + A_t = E[K_t|\mathcal{F}_t] + C_t.$$

By Theorem 2.5 the stochastic processes  $E[M_t|\mathcal{F}_t]$  and  $E[K_t|\mathcal{F}_t]$  are martingales. Thus by Doob–Meyer’s decomposition for submartingales we see that  $A_t = C_t$ . Then it follows from Equation (3.2) that  $M_t = K_t$ . This shows the uniqueness of the decomposition of  $X_t$ .

Next, we prove the existence of the decomposition. Let  $X_t$  be a continuous near-submartingale with respect to a continuous filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$ . Let

$$Y_t = E[X_t|\mathcal{F}_t], \quad a \leq t \leq b. \quad (3.3)$$

By Theorem 2.11 the stochastic process  $Y_t$  is a submartingale. Then we apply Doob–Meyer’s decomposition to the submartingale  $Y_t$  to get

$$Y_t = N_t + D_t, \quad a \leq t \leq b, \quad (3.4)$$

where  $N_t$  is a continuous martingale with respect to  $\{\mathcal{F}_t\}$  and  $D_t$  is a continuous stochastic process satisfying the conditions (1)–(3) in the theorem. Use  $N_t$  and  $D_t$  in Equation (3.4) to define two stochastic processes  $M_t$  and  $A_t$  as follows:

$$M_t = X_t - E[X_t|\mathcal{F}_t] + N_t, \quad (3.5)$$

$$A_t = D_t. \quad (3.6)$$

Then by Equations (3.3), (3.4), (3.5), and (3.6), we get

$$\begin{aligned} X_t &= M_t + E[X_t|\mathcal{F}_t] - N_t \\ &= M_t + Y_t - N_t \\ &= M_t + D_t \\ &= M_t + A_t. \end{aligned}$$

Hence we have the equality in Equation (3.1). Since  $A_t = D_t$ ,  $A_t$  is a continuous stochastic process satisfying conditions (1)–(3) in the theorem. To complete the proof of the theorem we only need to show that the stochastic process  $M_t$  defined by Equation (3.5) is a near-martingale. Note that

$$E[M_t|\mathcal{F}_t] = E\left\{X_t - E[X_t|\mathcal{F}_t] + N_t \middle| \mathcal{F}_t\right\} = N_t.$$

But  $N_t$  is a martingale. Hence  $E[M_t|\mathcal{F}_t]$  is a martingale. Then by Theorem 2.5 the stochastic process  $M_t$  is a near-martingale.  $\square$

**Example 3.2.** Let  $Q_t, a \leq t \leq b$ , be a continuous submartingale with respect to a filtration  $\{\mathcal{F}_t; a \leq t \leq b\}$ . Suppose a continuous stochastic process  $\varphi(t)$  is instantly independent of  $\{\mathcal{F}_t\}$  and assume that  $E[\varphi(t)] = c$  for all  $t \in [a, b]$  with  $c \geq 0$  being a constant. Consider the stochastic process  $X_t$  defined by

$$X_t = Q_t \varphi(t), \quad a \leq t \leq b. \quad (3.7)$$

Take the conditional expectation of  $X_t$  to get

$$E[X_t|\mathcal{F}_t] = E[Q_t \varphi(t)|\mathcal{F}_t] = Q_t E[\varphi(t)|\mathcal{F}_t] = Q_t E[\varphi(t)] = c Q_t,$$

which shows that  $E[X_t|\mathcal{F}_t]$  is a submartingale. We want to compare Doob–Meyer’s decompositions of the submartingale  $Q_t$  and the near-submartingale  $X_t = Q_t \varphi(t)$ . Suppose  $Q_t$  has the following decomposition

$$Q_t = N_t + D_t, \quad a \leq t \leq b, \tag{3.8}$$

where  $N_t$  is a martingale and  $D_t$  satisfies the conditions (1)–(3). Note that

$$E[X_t|\mathcal{F}_t] = cQ_t = cN_t + cD_t,$$

which can be used together with Equations (3.5) and (3.6) to derive

$$X_t = \left\{ (X_t - cN_t - cD_t) + cN_t \right\} + cD_t = (Q_t\varphi(t) - cD_t) + cD_t, \tag{3.9}$$

which is Doob–Meyer’s decomposition of the near-submartingale  $X_t = Q_t \varphi(t)$ .

**Example 3.3.** Take a special case in Example 3.2 with  $Q_t$  and  $\varphi(t)$  given by

$$Q_t = B(t)^2, \quad \varphi(t) = e^{B(1)-B(t)-\frac{1}{2}(1-t)}, \quad 0 \leq t \leq 1.$$

It is well known that  $B(t)^2$  has the Boob–Meyer decomposition

$$Q_t = B(t)^2 = (B(t)^2 - t) + t,$$

where the first term  $B(t)^2 - t$  is a martingale. On the other hand, by Equation (3.9) we have the decomposition of  $X_t = Q_t\varphi(t)$ ,

$$X_t = Q_t\varphi(t) = (B(t)^2\varphi(t) - ct) + ct,$$

where the first term  $(B(t)^2\varphi(t) - ct)$  is a near-martingale.

#### 4. Linear Stochastic Differential Equations

In this section we will address the problem concerning a relationship between  $X_t$  such as the one in Equation (2.7) and the corresponding  $Y_t$  in Equation (2.8). First we quote a theorem from [6] regarding to linear stochastic differential equations with an anticipating initial condition.

**Theorem 4.1.** (Theorem 4.1 [6]) *Let  $\alpha(t)$  be a deterministic function in  $L^2([a, b])$  and  $\beta(t)$  an adapted stochastic process such that  $E \int_a^b |\beta(t)|^2 dt < \infty$ . Suppose  $P(x)$  is a polynomial. Then the solution of the stochastic differential equation*

$$\begin{cases} dX_t = \alpha(t)X_t dB(t) + \beta(t)X_t dt, & a \leq t \leq b, \\ X_a = P(B(b) - B(a)), \end{cases} \tag{4.1}$$

is given by

$$X_t = P\left(B(b) - B(a) - \int_a^t \alpha(s) ds\right) \exp \left[ \int_a^t \alpha(s) dB(s) + \int_a^t \left(\beta(s) - \frac{1}{2}\alpha(s)^2\right) ds \right]. \tag{4.2}$$

*Remark 4.2.* It is quite easy to show the uniqueness of a solution of the stochastic differential equation in Equation (4.1).



**Example 4.3.** We continue the same idea as given in Example 2.8 to discuss the stochastic differential equation:

$$\begin{cases} dX_t = X_t dB(t), & 0 \leq t \leq 1, \\ X_0 = B(1)^2. \end{cases} \quad (4.3)$$

By Theorem 4.1 the solution is given by

$$X_t = (B(1) - t)^2 e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1.$$

Writing the first factor  $(B(1) - t)^2$  as a sum of three terms

$$(B(1) - t)^2 = (B(1)^2 - 1) - 2tB(1) + (1 + t^2) \quad (4.4)$$

and using the fact that  $B(t)^2 - t$  and  $B(t)$  are martingales, we can easily derive the conditional expectation  $Y_t = E[X_t | \mathcal{F}_t]$  as follows:

$$Y_t = ((B(t) - t)^2 + 1 - t) e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1.$$

Then we apply the Itô formula to the function  $\theta(t, x) = ((x - t)^2 + 1 - t)e^{x - \frac{1}{2}t}$  to derive the stochastic differential of  $Y_t$

$$dY_t = \left( Y_t + 2(B(t) - t) e^{B(t) - \frac{1}{2}t} \right) dB(t).$$

Hence  $Y_t$  is the solution of the following stochastic differential equation:

$$\begin{cases} dY_t = \left( Y_t + 2(B(t) - t) e^{B(t) - \frac{1}{2}t} \right) dB(t), & 0 \leq t \leq 1, \\ Y_0 = 1. \end{cases} \quad (4.5)$$

More general than Examples 2.8 and 4.3, we can consider the same stochastic differential equations with the initial condition  $B(1)^n$  for an integer  $n \geq 3$ . But in view of Equation (4.4) for the case when  $n = 2$ , we need to express  $(B(1) - t)^n$  in terms of Hermite polynomials.

Recall that the Hermite polynomial of degree  $n$  with parameter  $\rho$  is given by

$$H_n(x; \rho) = (-\rho)^n e^{x^2/2\rho} D_x^n e^{-x^2/2\rho}.$$

(See, e.g., page 157 [8].) We list three equalities which will be needed below:

$$D_x H_n(x; \rho) = n H_{n-1}(x; \rho), \quad (4.6)$$

$$\frac{\partial}{\partial \rho} H_n(x; \rho) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x; \rho), \quad (4.7)$$

$$H_n(x + y; \rho) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x; \rho) y^k. \quad (4.8)$$

(See e.g., page 355 [7] and page 159 [8].)

Now, let  $n$  be a fixed natural number and consider the following stochastic differential equation:

$$\begin{cases} dX_t = X_t dB(t), & 0 \leq t \leq 1, \\ X_0 = H_n(B(1); 1). \end{cases} \quad (4.9)$$

By Theorem 4.1 the solution of Equation (4.9) is given by

$$X_t = H_n(B(1) - t; 1) e^{B(t) - \frac{1}{2}t}. \tag{4.10}$$

We need to compute the conditional expectation  $Y_t = E[X_t | \mathcal{F}_t]$ . First note that we have

$$Y_t = E[H_n(B(1) - t; 1) e^{B(t) - \frac{1}{2}t} | \mathcal{F}_t] = e^{B(t) - \frac{1}{2}t} E[H_n(B(1) - t; 1) | \mathcal{F}_t]. \tag{4.11}$$

Use Equation (4.8) with  $x = B(1)$ ,  $y = -t$ , and  $\rho = 1$  to get

$$H_n(B(1) - t; 1) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(B(1); 1) (-t)^k. \tag{4.12}$$

Then put Equation (4.12) into Equation (4.11) and use the fact that  $H_m(B(t); t)$  is a martingale for any natural number  $m$  (see page 59 or 114 [8]) to show that

$$\begin{aligned} Y_t &= e^{B(t) - \frac{1}{2}t} \sum_{k=0}^n \binom{n}{k} (-t)^k E[H_{n-k}(B(1); 1) | \mathcal{F}_t] \\ &= e^{B(t) - \frac{1}{2}t} \sum_{k=0}^n \binom{n}{k} H_{n-k}(B(t); t) (-t)^k. \end{aligned} \tag{4.13}$$

Observe that the summation in Equation (4.13) is exactly the summation in the right-hand side of Equation (4.8) with  $x = B(t)$ ,  $y = -t$ , and  $\rho = t$ . Thus Equations (4.8) and (4.13) yield the equality

$$Y_t = H_n(B(t) - t; t) e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1. \tag{4.14}$$

In order to find the stochastic differential  $dY_t$ , we apply the Itô formula to the function  $\theta(t, x) = H_n(x - t; t) e^{x - \frac{1}{2}t}$  to get

$$dY_t = d\theta(t, B(t)) = \theta_t dt + \theta_x dB(t) + \frac{1}{2} \theta_{xx} dt. \tag{4.15}$$

By Equations (4.6) and (4.7) we have

$$\begin{aligned} \theta_t &= \left\{ -\frac{\partial}{\partial x} H_n + \frac{\partial}{\partial \rho} H_n \right\} e^{B(t) - \frac{1}{2}t} - \frac{1}{2} H_n e^{B(t) - \frac{1}{2}t} \\ &= \left\{ -nH_{n-1} - \frac{1}{2} \frac{\partial^2}{\partial x^2} H_n - \frac{1}{2} H_n \right\} e^{B(t) - \frac{1}{2}t} \\ &= \left\{ -nH_{n-1} - \frac{1}{2} n(n-1)H_{n-2} - \frac{1}{2} H_n \right\} e^{B(t) - \frac{1}{2}t}, \\ \theta_x &= (nH_{n-1} + H_n) e^{B(t) - \frac{1}{2}t}, \\ \theta_{xx} &= (n(n-1)H_{n-2} + 2nH_{n-1} + H_n) e^{B(t) - \frac{1}{2}t}. \end{aligned}$$

Put the above values of  $\theta_t$ ,  $\theta_x$ , and  $\theta_{xx}$  into Equation (4.15) to show that

$$dY_t = (H_n + nH_{n-1}) e^{B(t) - \frac{1}{2}t} dB(t),$$

which, in view of Equation (4.14), can be rewritten as

$$dY_t = \left( Y_t + nH_{n-1}(B(t) - t; t) e^{B(t) - \frac{1}{2}t} \right) dB(t).$$

Finally, observe that  $nH_{n-1} = D_x H_n$  by Equation (4.6). Therefore,

$$dY_t = \left( Y_t + (D_x H_n)(B(t) - t; t) e^{B(t) - \frac{1}{2}t} \right) dB(t).$$

Thus we have proved the next theorem.

**Theorem 4.4.** *Let  $n$  be a fixed natural number and let  $X_t$  be the solution of the stochastic differential equation*

$$\begin{cases} dX_t = X_t dB(t), & 0 \leq t \leq 1, \\ X_0 = H_n(B(1); 1), \end{cases}$$

where  $H_n(x; \rho)$  is the Hermite polynomial of degree  $n$  with parameter  $\rho$ . Then the conditional expectation  $Y_t = E[X_t | \mathcal{F}_t]$  is given by

$$Y_t = H_n(B(t) - t; t) e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1.$$

and satisfies the stochastic differential equation

$$\begin{cases} dY_t = \left( Y_t + (D_x H_n)(B(t) - t; t) e^{B(t) - \frac{1}{2}t} \right) dB(t), & 0 \leq t \leq 1, \\ Y_0 = 0. \end{cases}$$

**Theorem 4.5.** *Let  $n, m$  be natural numbers and  $c$  a constant. Suppose  $X_t$  is the solution of the stochastic differential equation*

$$\begin{cases} dX_t = X_t dB(t), & 0 \leq t \leq 1, \\ X_0 = cH_n(B(1); 1) + H_m(B(1); 1). \end{cases} \quad (4.16)$$

Then  $Y_t = E[X_t | \mathcal{F}_t]$  satisfies the stochastic differential equation

$$\begin{cases} dY_t = \left\{ Y_t + \left( c(D_x H_n)(B(t) - t; t) + (D_x H_m)(B(t) - t; t) \right) e^{B(t) - \frac{1}{2}t} \right\} dB(t), \\ \hspace{15em} 0 \leq t \leq 1, \\ Y_0 = 0. \end{cases} \quad (4.17)$$

*Proof.* We can carry out the same computation as in the above derivation for the case of Theorem 4.4. Alternatively, we can argue as follows. Let  $X_t^{(1)}$  be the solution of the stochastic differential equation

$$\begin{cases} dX_t^{(1)} = X_t^{(1)} dB(t), & 0 \leq t \leq 1, \\ X_0^{(1)} = cH_n(B(1); 1), \end{cases}$$

and let  $X_t^{(2)}$  be the solution of the stochastic differential equation

$$\begin{cases} dX_t^{(2)} = X_t^{(2)} dB(t), & 0 \leq t \leq 1, \\ X_0^{(2)} = H_m(B(1); 1). \end{cases}$$

Then the solution  $X_t$  of Equation (4.16) is given by  $X_t = X_t^{(1)} + X_t^{(2)}$ . Let  $Y_t^{(i)} = E[X_t^{(i)} | \mathcal{F}_t]$ ,  $i = 1, 2$ . Obviously, we have  $Y_t = E[X_t | \mathcal{F}_t] = Y_t^{(1)} + Y_t^{(2)}$ . Then apply Theorem 4.4 to  $X_t^{(i)}$ ,  $i = 1, 2$ , (the constant  $c$  causes no problem for the case  $i = 1$ ) to conclude that  $Y_t$  satisfies Equation (4.17).  $\square$

Next, notice that any polynomial can be expressed as a linear combination of Hermite polynomials. Hence we can apply Theorems 4.4 and 4.5 to find the stochastic differential equation satisfied by  $Y_t = E[X_t|\mathcal{F}_t]$  for the case when the initial condition for  $X_0$  is a polynomial in  $B(1)$ .

**Example 4.6.** Consider the stochastic differential equation

$$\begin{cases} dX_t = X_t dB(t), & 0 \leq t \leq 1, \\ X_0 = B(1)^3. \end{cases} \tag{4.18}$$

By Theorem 4.1 the solution  $X_t$  is given by

$$X_t = (B(1) - t)^3 e^{B(t) - \frac{1}{2}t}, \quad 0 \leq t \leq 1.$$

To find  $Y_t = E[X_t|\mathcal{F}_t]$ , rewrite  $(B(1) - t)^3$  as

$$(B(1) - t)^3 = [B(1)^3 - 3B(1)] - 3t[B(1)^2 - 1] + 3(1 + t^2)B(1) - (t^3 + 3t).$$

Since  $B(t)^3 - 3tB(t)$ ,  $B(t)^2 - t$ , and  $B(t)$  are martingales, we see that

$$\begin{aligned} E[(B(1) - t)^3|\mathcal{F}_t] &= [B(t)^3 - 3tB(t)] - 3t[B(t)^2 - t] + 3(1 + t^2)B(t) - (t^3 + 3t) \\ &= (B(t) - t)^3 + 3(1 - t)(B(t) - t). \end{aligned}$$

Therefore,

$$Y_t = \left\{ (B(t) - t)^3 + 3(1 - t)(B(t) - t) \right\} e^{B(t) - \frac{1}{2}t}.$$

To find the stochastic differential equation satisfied by  $Y_t$ , we express the initial condition  $B(1)^3$  in terms of the Hermite polynomials in  $B(1)$  as follows:

$$B(1)^3 = (B(1)^3 - 3B(1)) + 3B(1) = H_3(B(1); 1) + 3H_1(B(1); 1).$$

Then we can apply Theorem 4.5 with  $c = 3, n = 1$ , and  $m = 3$  to assert that  $Y_t$  satisfies the stochastic differential equation

$$\begin{cases} dY_t = \left\{ Y_t + 3[(B(t) - t)^2 + 1 - t] e^{B(t) - \frac{1}{2}t} \right\} dB(t), & 0 \leq t \leq 1, \\ Y_0 = 0. \end{cases}$$

In the above discussion we have studied special cases with the time interval  $[0, 1]$ , just to keep the ideas simple. We now give a theorem for the general linear stochastic differential equation in Theorem 4.1.

**Theorem 4.7.** Let  $\alpha(t)$  be a deterministic function in  $L^2([a, b])$  and  $\beta(t)$  an adapted stochastic process such that  $E \int_a^b |\beta(t)|^2 dt < \infty$ . Suppose  $n$  is a fixed natural number. Let  $X_t$  be the solution of the stochastic differential equation

$$\begin{cases} dX_t = \alpha(t)X_t dB(t) + \beta(t)X_t dt, & a \leq t \leq b, \\ X_a = H_n(B(b) - B(a); b - a), \end{cases} \tag{4.19}$$

where  $H_n(x; \rho)$  is the Hermite polynomial of degree  $n$  with parameter  $\rho$ . Then the conditional expectation  $Y_t = E[X_t|\mathcal{F}_t]$  is given by

$$Y_t = H_n\left(B(t) - B(a) - \int_a^t \alpha(s) ds; t - a\right) \mathcal{E}_{\alpha, \beta}(t), \quad a \leq t \leq b, \tag{4.20}$$

where  $\mathcal{E}_{\alpha,\beta}(t)$  stands for the exponential process

$$\mathcal{E}_{\alpha,\beta}(t) = \exp \left[ \int_a^t \alpha(s) dB(s) + \int_a^t \left( \beta(s) - \frac{1}{2} \alpha(s)^2 \right) ds \right], \quad a \leq t \leq b.$$

Moreover,  $Y_t$  satisfies the stochastic differential equation

$$\begin{cases} dY_t = \left[ \alpha(t)Y_t + (D_x H_n)(B(t) - B(a) - \int_a^t \alpha(s) ds; t - a) \mathcal{E}_{\alpha,\beta}(t) \right] dB(t) \\ \quad + \beta(t)Y_t dt, \quad a \leq t \leq b, \\ Y_a = 0. \end{cases} \quad (4.21)$$

*Remark 4.8.* It is easy to check the uniqueness of a solution for Equations (4.19) and (4.21). Moreover, the case when  $X_a = P(B(b) - B(a))$  with  $P(\cdot)$  being a polynomial can be handled by the same arguments as those in Example 4.6.

*Proof.* For brevity, we will write  $\mathcal{E}_{\alpha,\beta}(t)$  as  $\mathcal{E}_t$  in the proof. We first derive Equation (4.20). By Theorem 4.1 we have

$$X_t = H_n \left( (B(b) - B(a) - \int_a^t \alpha(s) ds; b - a) \right) \mathcal{E}_t.$$

For convenience, let  $c(t) = \int_a^t \alpha(s) ds$ . Then

$$X_t = H_n \left( (B(b) - B(a) - c(t); b - a) \right) \mathcal{E}_t, \quad a \leq t \leq b. \quad (4.22)$$

Note that  $\mathcal{E}_t$  is adapted with respect to the filtration  $\{\mathcal{F}_t\}$ . Hence

$$Y_t = E[X_t | \mathcal{F}_t] = \mathcal{E}_t E \left[ H_n \left( (B(b) - B(a) - c(t); b - a) \right) \middle| \mathcal{F}_t \right]. \quad (4.23)$$

Apply Equation (4.8) with  $x = B(b) - B(a)$ ,  $y = -c(t)$ , and  $\rho = b - a$  to get

$$\begin{aligned} & H_n(B(b) - B(a) - c(t); b - a) \\ &= \sum_{k=0}^n \binom{n}{k} H_{n-k}(B(b) - B(a); b - a) (-c(t))^k. \end{aligned}$$

Then we use the fact that  $H_m(B(t) - B(a); t - a)$  is a martingale to see that

$$\begin{aligned} & E \left[ H_n \left( (B(b) - B(a) - c(t); b - a) \right) \middle| \mathcal{F}_t \right] \\ &= \sum_{k=0}^n \binom{n}{k} H_{n-k}(B(t) - B(a); t - a) (-c(t))^k, \end{aligned}$$

which is exactly the summation in the right-hand side of Equation (4.8) with  $x = B(t) - B(a)$ ,  $y = -c(t)$ , and  $\rho = t - a$ . Hence by Equation (4.8),

$$E \left[ H_n \left( (B(b) - B(a) - c(t); b - a) \right) \middle| \mathcal{F}_t \right] = H_n \left( (B(t) - B(a) - c(t); t - a) \right). \quad (4.24)$$

Upon putting Equation (4.24) into Equation (4.23) we get

$$Y_t = \mathcal{E}_t H_n \left( (B(t) - B(a) - c(t); t - a) \right),$$

which proves the assertion in Equation (4.20).

To find the stochastic differential of  $Y_t$ , we apply the Itô product rule,

$$dY_t = H_n d\mathcal{E}_t + \mathcal{E}_t dH_n + (d\mathcal{E}_t)(dH_n). \quad (4.25)$$

Using the Itô formula, we can easily derive the following stochastic differentials:

$$dH_n = nH_{n-1} dB(t) - n\alpha(t)H_{n-1} dt, \quad (4.26)$$

$$d\mathcal{E}_t = \alpha(t)\mathcal{E}_t dB(t) + \beta(t)\mathcal{E}_t dt. \quad (4.27)$$

Then we put Equations (4.26) and (4.27) into Equation (4.25) to derive

$$dY_t = \left( \alpha(t)Y_t + nH_{n-1} \mathcal{E}(t) \right) dB(t) + \beta(t)Y_t dt,$$

which is Equation (4.21) since  $D_x H_n = nH_{n-1}$  by Equation (4.6).  $\square$

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