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THE DISTRIBUTION OF THE NUMBER OF CLUSTERS IN THE ARRATIA FLOW

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Abstract. In this paper we study the distribution of the number of elements in the image of a closed interval under the action of the Arratia flow. We present two possible approaches to finding this distribution. The first is based on the well-known Karlin–McGregor formula and the second on the Pfaffian formulae for systems of coalescing and annihilating Brownian motions recently obtained in [15]. We also show that the corresponding mean value can be found using the notion of the total free time of particles in the Arratia flow introduced in [3].

1. Introduction

The Arratia flow is a one-dimensional stochastic flow of Brownian particles in which any two particles move independently until they meet and after that coalesce and move together. It was introduced in [1] and later studied by many authors (see, e. g., [2, 3, 4, 5, 7] etc.). For the reader's convenience, here we recall the formal definition of the Arratia flow.

Definition 1.1. The *Arratia flow* is a random field $\{x(u,t),\ u \in \mathbb{R},\ t \geqslant 0\}$ satisfying the following conditions:

1) for any $u \in \mathbb{R}$ the stochastic process $\{x(u,t),\ t \geqslant 0\}$ is a Brownian motion with respect to the common filtration
$$\mathcal{F}_t := \sigma\{x(u,s),\ u \in \mathbb{R},\ 0 \leqslant s \leqslant t\}, \quad t \geqslant 0,$$
starting from the point $u \in \mathbb{R}$;
2) for any $u, v \in \mathbb{R}$
$$\text{if } u \leqslant v, \text{ then } x(u,t) \leqslant x(v,t) \text{ for all } t \geqslant 0;$$
3) for any $u, v \in \mathbb{R}$ the joint quadratic variation of $\{x(u,t),\ t \geqslant 0\}$ and $\{x(v,t),\ t \geqslant 0\}$ is given by
$$\langle x(u,\cdot), x(v,\cdot)\rangle_t = \int_0^t 1\!\!\mathrm{I}\{x(u,s) = x(v,s)\}\, ds, \quad t \geqslant 0,$$
where $1\!\!\mathrm{I}\{\}$ stands for the indicator function of the corresponding set.

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In 1984 T. E. Harris [7] proved that the image of every compact set $K \subset \mathbb{R}$ under the action of the Arratia flow consists of finitely many elements almost surely for every positive time. In particular, it means that

$$\forall\, t > 0 \quad \forall\, u > 0: \quad \mathbf{P}\left\{\nu_t([0;u]) < +\infty\right\} = 1,$$

where $\nu_t([0;u])$ is the number of elements of the set $x([0;u],t)$, i. e.

$$\nu_t([0;u]) := |x([0;u],t)|\,.$$

Thus, there arises the natural problem of finding the distribution of $\nu_t([0;u])$. However, to the best of our knowledge, this problem has not been solved yet. Among the results related to it we should mention the following.

In paper [12] the asymptotic behaviour of the uniform distance on the unit interval between the mappings generated by the Arratia flow (more generally, an arbitrary Harris flow satisfying some non-restrictive conditions) and the identity mapping was studied. In particular, it was proved that

$$\lim_{t\to 0+}\left(\frac{1}{\sqrt{t\ln\frac{1}{t}}}\sup_{0\leqslant u\leqslant 1}|x(u,t)-u|\right) = 1 \quad \text{a. s.} \tag{1.1}$$

From this relation we can easily obtain that

$$\varliminf_{t\to 0+}\left(2\sqrt{t\ln\frac{1}{t}}\cdot\nu_t([0;1])\right) \geqslant 1 \quad \text{a. s.} \tag{1.2}$$

Indeed, if $\nu_t([0;1]) = k$, then at least two of the $k+1$ Brownian motions

$$x\left(0,\cdot\right),\; x\left(\frac{1}{k},\cdot\right),\; x\left(\frac{2}{k},\cdot\right),\; \ldots,\; x\left(\frac{k-1}{k},\cdot\right),\; x\left(1,\cdot\right)$$

coalesce by time t (otherwise we would have $\nu_t([0;1]) \geqslant k+1$). Therefore, for some $i_0 \in \{0,1,2,\ldots,k-1\}$

$$x\left(\frac{i_0}{k},t\right) = x\left(\frac{i_0+1}{k},t\right),$$

and so

$$\max\left\{\left|x\left(\frac{i_0}{k},t\right)-\frac{i_0}{k}\right|,\left|x\left(\frac{i_0+1}{k},t\right)-\frac{i_0+1}{k}\right|\right\} \geqslant \frac{1}{2k}.$$

Thus, we have

$$\sup_{0\leqslant u\leqslant 1}|x(u,t)-u| \geqslant \frac{1}{2\nu_t([0;1])}, \tag{1.3}$$

and (1.2) follows from (1.1) and (1.3).

In paper [6] a representation (in the form of a sum over binary forests) for the action of the semigroup of the n-point motions of the Arratia flow on the functions from the core of its generator was found. Although this allows to obtain a formula for the probabilities $\mathbf{P}\left(A^{(n)}_{i_1 i_2\ldots i_{k-1}}\right)$ in formula (1.4) below, its complexity makes it hardly possible to find their asymptotic behaviour as $n\to\infty$, which is necessary for finding the limit in (1.4).

In paper [15][1] a remarkable connection between linearly ordered systems of coalescing (or annihilating) particles and Pfaffians was discovered. In particular, the authors prove that the clusters of the Arratia flow (there called the system of coalescing Brownian motions under the maximal entrance law) form a Pfaffian point process and find its kernel (for details see [15]; cf. [14]). However, the possibility to find the distribution of the number of surviving particles using (1.4) seems to be left unnoticed.

In this paper we present two possible approaches to finding the distribution of $\nu_t([0;u])$. Both of them are based on the equality

$$\mathbf{P}\left\{\nu_t([0;u]) = k\right\} = \lim_{n\to\infty} \sum \mathbf{P}\left(A^{(n)}_{i_1 i_2 \ldots i_{k-1}}\right), \tag{1.4}$$

where

$$A^{(n)}_{i_1 i_2 \ldots i_{k-1}} := \Bigg\{x(0,t) = x\left(\frac{i_1 u}{2^n}, t\right) \neq x\left(\frac{(i_1+1)u}{2^n}, t\right) = x\left(\frac{i_2 u}{2^n}, t\right) \neq$$
$$\neq x\left(\frac{(i_2+1)u}{2^n}, t\right) = \ldots = x\left(\frac{i_{k-1} u}{2^n}, t\right) \neq x\left(\frac{(i_{k-1}+1)u}{2^n}, t\right) = x(u,t)\Bigg\},$$

and the sum is taken over all indices $i_1, i_2, \ldots, i_{k-1}$ such that

$$1 \leqslant i_1 < i_1 + 1 < i_2 < i_2 + 1 < \ldots < i_{k-1} < i_{k-1} + 1 \leqslant 2^n - 1. \tag{1.5}$$

To prove it one should notice that equality (1.4) with the sum taken over all indices $i_1, i_2, \ldots, i_{k-1}$ satisfying

$$0 \leqslant i_1 < i_2 < \ldots < i_{k-1} \leqslant 2^n - 1$$

instead of (1.5) follows immediately from the (easily verified) corresponding equality with indicators of the events instead of their probabilities and that the difference between the two sums can be estimated by

$$\sum_{i=0}^{2^n-2} \mathbf{P}\left\{x\left(\frac{iu}{2^n}, t\right) \neq x\left(\frac{(i+1)u}{2^n}, t\right) \neq x\left(\frac{(i+2)u}{2^n}, t\right)\right\}. \tag{1.6}$$

Using the Karlin–McGregor formula (see Theorem 2.1 below), one can easily show that (1.6) tends to zero as $n \to \infty$.

The first approach, which is described in Section 2, consists in representing each probability $\mathbf{P}\left(A^{(n)}_{i_1 i_2 \ldots i_{k-1}}\right)$ on the right-hand side of (1.4) as an algebraic sum of the probabilities of the events

$$\left\{x\left(\frac{j_1 u}{2^n}, t\right) \neq x\left(\frac{j_2 u}{2^n}, t\right) \neq \ldots \neq x\left(\frac{j_l u}{2^n}, t\right)\right\}, \quad j_1 < j_2 < \ldots < j_l,$$

and then computing these probabilities using the Karlin–McGregor formula. Although it is easy to show that this can be done for all $k \geqslant 2$, this representation is, generally speaking, not unique and because of its quickly growing complexity (for instance, in the case when $k = 4$ one of the possible representations for the probability $\mathbf{P}\left(A^{(n)}_{i_1 i_2 \ldots i_{k-1}}\right)$ consists of sixteen terms which have the form of integrals of determinants of varying order from 4×4 to 8×8), it seems difficult to

[1]The author is grateful to Prof. G. V. Riabov for drawing his attention to this paper.

find a usable form of writing it allowing to find the limit in (1.4) in the general case. Therefore, here we consider only the case when $k = 2$.

The second approach, which is described in Section 3, consists in representing each probability $\mathbf{P}\left(A^{(n)}_{i_1 i_2 \ldots i_{k-1}}\right)$ as a Pfaffian of some antisymmetric matrix. The corresponding representation was obtained in paper [15] mentioned above.

In Section 4 we compute the mean value of $\nu_t([0;u])$. This can be easily done by integrating the one-point density of the Pfaffian point process formed by the clusters of the Arratia flow (see Remark 4.4). However, this mean value can also be computed using the total free time of particles of the Arratia flow. This notion was first introduced in [3] as a part of the construction of the stochastic integral with respect to the Arratia flow and its elementary properties in a more general setting were studied in [9]. Although our result presented in this section (see Corollary 4.3 below) follows immediately from the formula for the mean value of the total free time of particles contained in [3], for the reader's convenience here we recall the corresponding definition and repeat the proof of this formula.

Remark 1.2. Unless explicitly stated otherwise, we will always assume u and t to be some fixed strictly positive real numbers.

2. The Distribution of $\nu_t([0;u])$ and the Karlin–McGregor Formula

We begin this section by recalling the Karlin–McGregor formula giving the determinantal representation for the probability density function of non-intersecting Brownian motions. (For two non-empty sets $A, B \subset \mathbb{R}$ the inequality $A < B$ means that $z_1 < z_2$ for all $z_1 \in A$, $z_2 \in B$.)

Theorem 2.1. [8] *Let $w_1, \ldots, w_n$ be independent Brownian motions starting from some points $u_1 < \ldots < u_n$. Then the probability $P_t(u_1, \ldots, u_n; A_1, \ldots, A_n)$ that at time $t > 0$ these Brownian motions are found in some non-empty Borel sets $A_1 < \ldots < A_n$ without any two of them having ever been coincident in the intervening time is given by*

$$P_t(u_1, \ldots, u_n; A_1, \ldots, A_n)$$
$$= \int\limits_{\Delta_n} \begin{vmatrix} p_t(v_1 - u_1) & \cdots & p_t(v_1 - u_n) \\ \vdots & \ddots & \vdots \\ p_t(v_n - u_1) & \cdots & p_t(v_n - u_n) \end{vmatrix} 1\!\!\mathrm{I}\left\{v_1 \in A_1, \ldots, v_n \in A_n\right\} dv_1 \ldots dv_n,$$

where

$$\Delta_n := \{(v_1, \ldots, v_n) \in \mathbb{R} \mid v_1 \leqslant \ldots \leqslant v_n\}$$

and

$$p_t(v) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{v^2}{2t}}, \quad v \in \mathbb{R}.$$

We will also use the following well-known result.

Lemma 2.2. *For any $u_1, u_2 \in \mathbb{R}$, $u_1 < u_2$,*

$$\mathbf{P}\left\{x(u_1, t) \neq x(u_2, t)\right\} = \int\limits_{-(u_2-u_1)/\sqrt{2}}^{(u_2-u_1)/\sqrt{2}} p_t(v)\, dv.$$

Proof. One can derive this equality directly from the Karlin–McGregor formula or by noting that

$$\mathbf{P}\left\{x(u_1,t)\neq x(u_2,t)\right\}=\mathbf{P}\left\{\tau_w\left(-\frac{u_2-u_1}{\sqrt{2}}\right)>t\right\},$$

where $\tau_w(c)$ is the first time when the standard Brownian motion w starting from zero reaches some level $c\in\mathbb{R}$. □

Theorem 2.3. *We have*

$$\mathbf{P}\left\{\nu_t([0;u])=2\right\}=\frac{u}{\sqrt{\pi t}}-\int_0^u\int_{\Delta_3}\begin{vmatrix}p_t(v_1) & p_t'(v_1-r) & p_t(v_1-r)\\ p_t(v_2) & p_t'(v_2-r) & p_t(v_2-r)\\ p_t(v_3) & p_t'(v_3-r) & p_t(v_3-r)\end{vmatrix}\begin{matrix}dv_1\\ dv_2\\ dv_3\end{matrix}\,dr$$
$$-\int_0^u\int_{\Delta_3}\begin{vmatrix}p_t'(v_1-r) & p_t(v_1-r) & p_t(v_1-u)\\ p_t'(v_2-r) & p_t(v_2-r) & p_t(v_2-u)\\ p_t'(v_3-r) & p_t(v_3-r) & p_t(v_3-u)\end{vmatrix}\begin{matrix}dv_1\\ dv_2\\ dv_3\end{matrix}\,dr$$
$$+\int_0^u\int_{\Delta_4}\begin{vmatrix}p_t(v_1) & p_t'(v_1-r) & p_t(v_1-r) & p_t(v_1-u)\\ p_t(v_2) & p_t'(v_2-r) & p_t(v_2-r) & p_t(v_2-u)\\ p_t(v_3) & p_t'(v_3-r) & p_t(v_3-r) & p_t(v_3-u)\\ p_t(v_4) & p_t'(v_4-r) & p_t(v_4-r) & p_t(v_4-u)\end{vmatrix}\begin{matrix}dv_1\\ dv_2\\ dv_3\\ dv_4\end{matrix}\,dr.$$

Proof. Equality (1.4) now has the form

$$\begin{aligned}&\mathbf{P}\left\{\nu_t([0;u])=2\right\}\\ &=\lim_{n\to\infty}\sum_{i=1}^{2^n-2}\mathbf{P}\left\{x(0,t)=x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)=x(u,t)\right\}.\end{aligned}$$

Let us note that

$$\begin{aligned}&\mathbf{P}\left\{x(0,t)=x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)=x(u,t)\right\}\\ &=\mathbf{P}\left\{x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)\right\}\\ &\quad-\mathbf{P}\left\{x(0,t)\neq x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)\right\}\\ &\quad-\mathbf{P}\left\{x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)\neq x(u,t)\right\}\\ &\quad+\mathbf{P}\left\{x(0,t)\neq x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)\neq x(u,t)\right\}.\end{aligned}$$

Therefore,

$$\mathbf{P}\left\{\nu_t([0;u])=2\right\}=I_1-I_2-I_3+I_4,$$

where

$$I_1 := \lim_{n\to\infty} \sum_{i=1}^{2^n-2} \mathbf{P}\left\{x\left(\frac{iu}{2^n},t\right) \neq x\left(\frac{(i+1)u}{2^n},t\right)\right\},$$

$$I_2 := \lim_{n\to\infty} \sum_{i=1}^{2^n-2} \mathbf{P}\left\{x(0,t) \neq x\left(\frac{iu}{2^n},t\right) \neq x\left(\frac{(i+1)u}{2^n},t\right)\right\},$$

$$I_3 := \lim_{n\to\infty} \sum_{i=1}^{2^n-2} \mathbf{P}\left\{x\left(\frac{iu}{2^n},t\right) \neq x\left(\frac{(i+1)u}{2^n},t\right) \neq x(u,t)\right\},$$

$$I_4 := \lim_{n\to\infty} \sum_{i=1}^{2^n-2} \mathbf{P}\left\{x(0,t) \neq x\left(\frac{iu}{2^n},t\right) \neq x\left(\frac{(i+1)u}{2^n},t\right) \neq x(u,t)\right\}.$$

For the first term by Lemma 2.2 we obtain

$$I_1 = \lim_{n\to\infty} \sum_{i=1}^{2^n-2} \int\limits_{-u/(2^n\sqrt{2})}^{u/(2^n\sqrt{2})} p_t(v)\,dv = u\sqrt{2}\cdot \lim_{n\to\infty}\left(\frac{2^n}{u\sqrt{2}} \int\limits_{-u/(2^n\sqrt{2})}^{u/(2^n\sqrt{2})} p_t(v)\,dv\right) = \frac{u}{\sqrt{\pi t}}.$$

To compute the second term we note that by the Karlin–McGregor formula

$$\begin{aligned}
&\mathbf{P}\left\{x(0,t) \neq x\left(\frac{iu}{2^n},t\right) \neq x\left(\frac{(i+1)u}{2^n},t\right)\right\} \\
&= \int\limits_{\Delta_3} \begin{vmatrix} p_t(v_1) & p_t\left(v_1 - \frac{iu}{2^n}\right) & p_t\left(v_1 - \frac{(i+1)u}{2^n}\right) \\ p_t(v_2) & p_t\left(v_2 - \frac{iu}{2^n}\right) & p_t\left(v_2 - \frac{(i+1)u}{2^n}\right) \\ p_t(v_3) & p_t\left(v_3 - \frac{iu}{2^n}\right) & p_t\left(v_3 - \frac{(i+1)u}{2^n}\right) \end{vmatrix} \begin{matrix} dv_1 \\ dv_2 \\ dv_3 \end{matrix} \\
&= \frac{u}{2^n} \int\limits_{\Delta_3} \begin{vmatrix} p_t(v_1) & \frac{2^n}{u}\cdot \Delta p_t\left(v_1 - \frac{iu}{2^n}\right) & p_t\left(v_1 - \frac{iu}{2^n}\right) \\ p_t(v_2) & \frac{2^n}{u}\cdot \Delta p_t\left(v_2 - \frac{iu}{2^n}\right) & p_t\left(v_2 - \frac{iu}{2^n}\right) \\ p_t(v_3) & \frac{2^n}{u}\cdot \Delta p_t\left(v_3 - \frac{iu}{2^n}\right) & p_t\left(v_3 - \frac{iu}{2^n}\right) \end{vmatrix} \begin{matrix} dv_1 \\ dv_2 \\ dv_3 \end{matrix} \\
&= \frac{u}{2^n} \int\limits_{\Delta_3} \begin{vmatrix} p_t(v_1) & \frac{2^n}{u}\cdot \Delta p_t\left(v_1 - \frac{iu}{2^n}\right) - p_t'\left(v_1 - \frac{iu}{2^n}\right) & p_t\left(v_1 - \frac{iu}{2^n}\right) \\ p_t(v_2) & \frac{2^n}{u}\cdot \Delta p_t\left(v_2 - \frac{iu}{2^n}\right) - p_t'\left(v_2 - \frac{iu}{2^n}\right) & p_t\left(v_2 - \frac{iu}{2^n}\right) \\ p_t(v_3) & \frac{2^n}{u}\cdot \Delta p_t\left(v_3 - \frac{iu}{2^n}\right) - p_t'\left(v_3 - \frac{iu}{2^n}\right) & p_t\left(v_3 - \frac{iu}{2^n}\right) \end{vmatrix} \begin{matrix} dv_1 \\ dv_2 \\ dv_3 \end{matrix} \\
&\quad + \frac{u}{2^n} \int\limits_{\Delta_3} \begin{vmatrix} p_t(v_1) & p_t'\left(v_1 - \frac{iu}{2^n}\right) & p_t\left(v_1 - \frac{iu}{2^n}\right) \\ p_t(v_2) & p_t'\left(v_2 - \frac{iu}{2^n}\right) & p_t\left(v_2 - \frac{iu}{2^n}\right) \\ p_t(v_3) & p_t'\left(v_3 - \frac{iu}{2^n}\right) & p_t\left(v_3 - \frac{iu}{2^n}\right) \end{vmatrix} \begin{matrix} dv_1 \\ dv_2, \\ dv_3 \end{matrix}
\end{aligned}$$

where we set

$$\Delta p_t\left(v_j - \frac{iu}{2^n}\right) := p_t\left(v_j - \frac{iu}{2^n}\right) - p_t\left(v_j - \frac{(i+1)u}{2^n}\right), \quad j = 1,2,3.$$

However, using the Taylor expansion one can easily show that

$$\lim_{n\to\infty}\sum_{i=1}^{2^n-2}\frac{u}{2^n}\int\limits_{\Delta_3}\begin{vmatrix} p_t(v_1) & \frac{2^n}{u}\cdot\Delta p_t\left(v_1-\frac{iu}{2^n}\right)-p_t'\left(v_1-\frac{iu}{2^n}\right) & p_t\left(v_1-\frac{iu}{2^n}\right)\\ p_t(v_2) & \frac{2^n}{u}\cdot\Delta p_t\left(v_2-\frac{iu}{2^n}\right)-p_t'\left(v_2-\frac{iu}{2^n}\right) & p_t\left(v_2-\frac{iu}{2^n}\right)\\ p_t(v_3) & \frac{2^n}{u}\cdot\Delta p_t\left(v_3-\frac{iu}{2^n}\right)-p_t'\left(v_3-\frac{iu}{2^n}\right) & p_t\left(v_3-\frac{iu}{2^n}\right)\end{vmatrix}\begin{matrix}dv_1\\dv_2\\dv_3\end{matrix}=0.$$

Thus, we obtain that

$$\begin{aligned} I_2 &= \lim_{n\to\infty}\sum_{i=1}^{2^n-2}\frac{u}{2^n}\int\limits_{\Delta_3}\begin{vmatrix} p_t(v_1) & p_t'\left(v_1-\frac{iu}{2^n}\right) & p_t\left(v_1-\frac{iu}{2^n}\right)\\ p_t(v_2) & p_t'\left(v_2-\frac{iu}{2^n}\right) & p_t\left(v_2-\frac{iu}{2^n}\right)\\ p_t(v_3) & p_t'\left(v_3-\frac{iu}{2^n}\right) & p_t\left(v_3-\frac{iu}{2^n}\right)\end{vmatrix}\begin{matrix}dv_1\\dv_2\\dv_3\end{matrix}\\ &= \int\limits_0^u\int\limits_{\Delta_3}\begin{vmatrix} p_t(v_1) & p_t'(v_1-r) & p_t(v_1-r)\\ p_t(v_2) & p_t'(v_2-r) & p_t(v_2-r)\\ p_t(v_3) & p_t'(v_3-r) & p_t(v_3-r)\end{vmatrix}\begin{matrix}dv_1\\dv_2\\dv_3\end{matrix}\,dr. \end{aligned}$$

The remaining terms I_3 and I_4 can be computed in a similar way. The theorem is proved. □

On the set $\{\nu_t([0;u])=2\}$ we can define θ to be the unique point of discontinuity of the mapping $x(\cdot,t)\colon\mathbb{R}\to\mathbb{R}$ on the interval $[0;u]$ and ξ_1 and ξ_2 to be the lower and the upper clusters in the image $x([0;u],t)$ respectively.

Theorem 2.4. *For all Borel sets $A\subset[0;u]$ and $B_1,B_2\subset\mathbb{R}$ with $B_1<B_2$ we have*

$$\begin{aligned} &\mathbf{P}\left\{\nu_t([0;u])=2,\ \theta\in A,\ \xi_1\in B_1,\ \xi_2\in B_2\right\}\\ &=\int\limits_A\int\limits_{\Delta_2}\begin{vmatrix} p_t'(v_1-r) & p_t(v_1-r)\\ p_t'(v_2-r) & p_t(v_2-r)\end{vmatrix}\,\mathbb{1}\left\{\begin{matrix}v_1\in B_1,\\ v_2\in B_2\end{matrix}\right\}\begin{matrix}dv_1\\dv_2\end{matrix}\,dr\\ &\quad-\int\limits_A\int\limits_{\Delta_3}\begin{vmatrix} p_t(v_1) & p_t'(v_1-r) & p_t(v_1-r)\\ p_t(v_2) & p_t'(v_2-r) & p_t(v_2-r)\\ p_t(v_3) & p_t'(v_3-r) & p_t(v_3-r)\end{vmatrix}\,\mathbb{1}\left\{\begin{matrix}v_2\in B_1,\\ v_3\in B_2\end{matrix}\right\}\begin{matrix}dv_1\\dv_2\\dv_3\end{matrix}\,dr\\ &\quad-\int\limits_A\int\limits_{\Delta_3}\begin{vmatrix} p_t'(v_1-r) & p_t(v_1-r) & p_t(v_1-u)\\ p_t'(v_2-r) & p_t(v_2-r) & p_t(v_2-u)\\ p_t'(v_3-r) & p_t(v_3-r) & p_t(v_3-u)\end{vmatrix}\,\mathbb{1}\left\{\begin{matrix}v_1\in B_1,\\ v_2\in B_2\end{matrix}\right\}\begin{matrix}dv_1\\dv_2\\dv_3\end{matrix}\,dr\\ &\quad+\int\limits_A\int\limits_{\Delta_4}\begin{vmatrix} p_t(v_1) & p_t'(v_1-r) & p_t(v_1-r) & p_t(v_1-u)\\ p_t(v_2) & p_t'(v_2-r) & p_t(v_2-r) & p_t(v_2-u)\\ p_t(v_3) & p_t'(v_3-r) & p_t(v_3-r) & p_t(v_3-u)\\ p_t(v_4) & p_t'(v_4-r) & p_t(v_4-r) & p_t(v_4-u)\end{vmatrix}\,\mathbb{1}\left\{\begin{matrix}v_2\in B_1,\\ v_3\in B_2\end{matrix}\right\}\begin{matrix}dv_1\\dv_2\\dv_3\\dv_4\end{matrix}\,dr. \end{aligned}$$

Proof. Obviously, it is enough to consider the case when A is an interval. In this case we have

$$\begin{aligned} &\mathbf{P}\left\{\nu_t([0;u])=2,\ \theta\in A,\ \xi_1\in B_1,\ \xi_2\in B_2\right\}\\ &=\lim_{n\to\infty}\sum\mathbf{P}\left\{\begin{matrix}x(0,t)=x\left(\frac{iu}{2^n},t\right)\neq x\left(\frac{(i+1)u}{2^n},t\right)=x(u,t),\\ x\left(\frac{iu}{2^n},t\right)\in B_1,\ x\left(\frac{(i+1)u}{2^n},t\right)\in B_2\end{matrix}\right\}, \end{aligned}$$

where the sum is taken over all indices $i \in \{1, \dots, 2^n - 2\}$ such that

$$\left[\frac{iu}{2^n}; \frac{(i+1)u}{2^n}\right] \subset A.$$

It remains to note that

$$\mathbf{P}\left\{\begin{matrix} x(0,t) = x\left(\frac{iu}{2^n}, t\right) \neq x\left(\frac{(i+1)u}{2^n}, t\right) = x(u,t), \\ x\left(\frac{iu}{2^n}, t\right) \in B_1,\ x\left(\frac{(i+1)u}{2^n}, t\right) \in B_2 \end{matrix}\right\}$$

$$= \mathbf{P}\left\{\begin{matrix} x\left(\frac{iu}{2^n}, t\right) \neq x\left(\frac{(i+1)u}{2^n}, t\right), \\ x\left(\frac{iu}{2^n}, t\right) \in B_1,\ x\left(\frac{(i+1)u}{2^n}, t\right) \in B_2 \end{matrix}\right\}$$

$$- \mathbf{P}\left\{\begin{matrix} x(0,t) \neq x\left(\frac{iu}{2^n}, t\right) \neq x\left(\frac{(i+1)u}{2^n}, t\right), \\ x\left(\frac{iu}{2^n}, t\right) \in B_1,\ x\left(\frac{(i+1)u}{2^n}, t\right) \in B_2 \end{matrix}\right\}$$

$$- \mathbf{P}\left\{\begin{matrix} x\left(\frac{iu}{2^n}, t\right) \neq x\left(\frac{(i+1)u}{2^n}, t\right) \neq x(u,t), \\ x\left(\frac{iu}{2^n}, t\right) \in B_1,\ x\left(\frac{(i+1)u}{2^n}, t\right) \in B_2 \end{matrix}\right\}$$

$$+ \mathbf{P}\left\{\begin{matrix} x(0,t) \neq x\left(\frac{iu}{2^n}, t\right) \neq x\left(\frac{(i+1)u}{2^n}, t\right) \neq x(u,t), \\ x\left(\frac{iu}{2^n}, t\right) \in B_1,\ x\left(\frac{(i+1)u}{2^n}, t\right) \in B_2 \end{matrix}\right\},$$

and to proceed further as in the proof of Theorem 2.3 (using the Karlin–McGregor formula for the first term as well, instead of Lemma 2.2). □

Remark 2.5. In the case when $k \geqslant 3$ this approach allows to find only the distribution of the $k-1$ points of discontinuity (without the k points of the set $x([0;u],t)$).

3. The Distribution of $\nu_t([0;u])$ and Pfaffians

We begin this section by recalling the definition of the Pfaffian and its simple properties that we are going to use (for more information on Pfaffians see [13], [10]).

Definition 3.1. The *Pfaffian* $\mathrm{Pf}\,(A)$ of an antisymmetric matrix $A = (a_{ij})_{i,j=1}^{2n}$ of order $2n$ is defined as

$$\mathrm{Pf}\,(A) = \sum_{\sigma} \mathrm{sign}\,(\sigma) \cdot a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n}, \tag{3.1}$$

where the sum is taken over all permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n \end{pmatrix}$$

such that $i_1 < i_2 < \dots < i_n$ and $i_k < j_k$ for all $k \in \{1, 2, \dots, n\}$.

According to the above definition we have

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a,$$

$$\mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

Remark 3.2. In the sequel we will omit the entries of antisymmetric matrices below the diagonal.

The fact that the Pfaffian is defined for antisymmetric matrices of only even order is justified by the first assertion of Theorem 3.3 below and the fact that the determinant of an antisymmetric matrix of odd order is equal to zero. The second assertion of Theorem 3.3 will be used in the proof of Theorem 3.5 (and Theorem 3.6) below. For the proof of Theorem 3.3 see, e. g., [13].

Theorem 3.3. *(i) If A is an antisymmetric matrix of even order, then*

$$\det A = (\mathrm{Pf}\,(A))^2.$$

(ii) If A is an antisymmetric matrix of even order and B is a square matrix of the same order, then the matrix B^TAB is antisymmetric and

$$\mathrm{Pf}\left(B^TAB\right) = \det B \cdot \mathrm{Pf}\,(A)\,. \tag{3.2}$$

Now let us define several auxiliary matrices. First of all, let $\mathbf{O}_{2n}$, $\widehat{\mathbf{I}}_{2n}$, and $\mathbf{I}_{2n}(\lambda)$ with $\lambda \in \mathbb{R}$ be matrices of order $2n$ with the entries given by

$$(\mathbf{O}_{2n})_{ij} := \begin{cases} +1, & \text{if } i = 2, 4, \dots, 2n-2 \text{ and } j = i+1, \\ -1, & \text{if } i = 3, 5, \dots, 2n-1 \text{ and } j = i-1, \\ 0, & \text{otherwise;} \end{cases}$$

$$(\widehat{\mathbf{I}}_{2n})_{ij} := \begin{cases} +1, & \text{if } i = j, \\ -1, & \text{if } i = 2, 4, \dots, 2n-2 \text{ and } j = i+1, \\ 0, & \text{otherwise;} \end{cases}$$

$$(\mathbf{I}_{2n}(\lambda))_{ij} := \begin{cases} 1, & \text{if } i = j \text{ and } i \neq 3, 5, \dots, 2n-1, \\ \lambda, & \text{if } i = j \text{ and } i = 3, 5, \dots, 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\det(\widehat{\mathbf{I}}_{2n}) = 1, \tag{3.3}$$

$$\det(\mathbf{I}_{2n}(\lambda)) = \lambda^{n-1}. \tag{3.4}$$

Also for $u_1 \leqslant \ldots \leqslant u_{2n}$ let $\mathbf{F}_t \equiv \mathbf{F}_t(u_1, \dots, u_{2n})$ be an antisymmetric matrix of order $2n$ with the entries above the diagonal given by

$$(\mathbf{F}_t)_{ij} := F_t(u_j - u_i), \quad 1 \leqslant i < j \leqslant 2n,$$

where

$$F_t(z) := 2 \int\limits_{z/\sqrt{2}}^{+\infty} p_t(v)\, dv, \quad z \in \mathbb{R}.$$

Finally, for $r_0 \leqslant r_1 \leqslant \ldots \leqslant r_k \leqslant r_{k+1}$ let $\widehat{\mathbf{F}}_t \equiv \widehat{\mathbf{F}}_t(r_0, r_1, \ldots, r_k, r_{k+1})$ be an antisymmetric matrix of order $2k+2$ with the entries above the diagonal given by

$$(\widehat{\mathbf{F}}_t)_{ij} := \begin{cases} F_t(r_{[j/2]} - r_{[i/2]}), & \text{if } i \text{ is even or } i = 1, \text{ and } j \text{ is even}, \\ F_t'(r_{[j/2]} - r_{[i/2]}), & \text{if } i \text{ is even or } i = 1, \text{ and } j \text{ is odd}, \\ -F_t'(r_{[j/2]} - r_{[i/2]}), & \text{if } i \text{ is odd, } i \neq 1, \text{ and } j \text{ is even}, \\ -F_t''(r_{[j/2]} - r_{[i/2]}), & \text{if } i \text{ is odd, } i \neq 1, \text{ and } j \text{ is odd}. \end{cases}$$

Lemma 3.4. [15] *Let $N_t([a;b])$ denote the number of particles of the Arratia flow which are found in the interval $[a;b]$ at time t. Then for all $n \geqslant 1$ and $u_1 < \ldots < u_{2n}$ we have*

$$\mathbf{P}\left\{ \begin{matrix} N_t([u_i; u_{i+1}]) = 0 & \text{for all } i = 1, 3, \ldots, 2n-1, \\ N_t([u_i; u_{i+1}]) > 0 & \text{for all } i = 2, 4, \ldots, 2n-2 \end{matrix} \right\}$$
$$= \operatorname{Pf}\left(\mathbf{F}_t(u_1, \ldots, u_{2n}) - \mathbf{O}_{2n}\right).$$

Now we are ready to prove the main results of this section.

Theorem 3.5. *For all $k \geqslant 1$ we have*

$$\mathbf{P}\left\{\nu_t([0;u]) = k+1\right\} = \int_{\Delta_k(u)} \!\!\cdots\! \int \operatorname{Pf}\left(\widehat{\mathbf{F}}_t(0, r_1, \ldots, r_k, u)\right) dr_1 \ldots dr_k,$$

where

$$\Delta_k(u) := \{(r_1, \ldots, r_k) \in \mathbb{R}^k \mid 0 \leqslant r_1 \leqslant \ldots \leqslant r_k \leqslant u\}.$$

Proof. The duality formulae (see Section 2.2 in [15] and the references therein) imply that the probability of the event $A^{(n)}_{i_1 i_2 \ldots i_k}$ coincides with that of the event

$$\left\{ \begin{matrix} N_t\left(\left[\dfrac{(i_l+1)u}{2^n}; \dfrac{i_{l+1}u}{2^n}\right]\right) = 0 & \text{for all } l = 0, 1, 2, \ldots, k, \\ N_t\left(\left[\dfrac{i_l u}{2^n}; \dfrac{(i_l+1)u}{2^n}\right]\right) > 0 & \text{for all } l = 1, 2, \ldots, k \end{matrix} \right\},$$

where we set for convenience $i_0 := -1$ and $i_{k+1} := 2^n$. Computing the latter with the help of Lemma 3.4, we obtain that

$$\mathbf{P}(A^{(n)}_{i_1 i_2 \ldots i_k}) = \operatorname{Pf}\left(\mathbf{F}_t - \mathbf{O}_{2k+2}\right),$$

where

$$\mathbf{F}_t = \mathbf{F}_t\left(0, \frac{i_1 u}{2^n}, \frac{(i_1+1)u}{2^n}, \frac{i_2 u}{2^n}, \frac{(i_2+1)u}{2^n}, \ldots, \frac{i_k u}{2^n}, \frac{(i_k+1)u}{2^n}, u\right).$$

Therefore, equalities (3.2), (3.3) and (3.4) imply that

$$\mathbf{P}\left\{\nu_t([0;u]) = k+1\right\} = \lim_{n\to\infty} \sum \operatorname{Pf}\left(\mathbf{F}_t - \mathbf{O}_{2k+2}\right)$$
$$= \lim_{n\to\infty} \sum \left(\frac{u}{2^n}\right)^k \operatorname{Pf}\left(\mathbf{I}^T_{2k+2}(2^n/u) \cdot \widehat{\mathbf{I}}^T_{2k+2} \cdot [\mathbf{F}_t - \mathbf{O}_{2k+2}] \cdot \widehat{\mathbf{I}}_{2k+2} \cdot \mathbf{I}_{2k+2}(2^n/u)\right).$$

Finally, we note that the matrix

$$\mathbf{I}^T_{2k+2}(2^n/u) \cdot \widehat{\mathbf{I}}^T_{2k+2} \cdot [\mathbf{F}_t - \mathbf{O}_{2k+2}] \cdot \widehat{\mathbf{I}}_{2k+2} \cdot \mathbf{I}_{2k+2}(2^n/u)$$

has the first and second differences of the function F_t in the same places as the matrix $\widehat{\mathbf{F}}_t$ has the first and second derivatives (and with the same signs). Thus, expanding the Pfaffians using (3.1), noting that the limit of each sum is an integral, and rewriting the result as an integral of a Pfaffian, we arrive at the required formula. □

Thus, for instance, we have

$$\mathbf{P}\left\{\nu_t([0;u])=2\right\}=\int_0^u \operatorname{Pf}\begin{pmatrix} 0 & F_t(r) & F_t'(r) & F_t(u) \\ & 0 & F_t'(0) & F_t(u-r) \\ & & 0 & -F_t'(u-r) \\ & & & 0 \end{pmatrix} dr$$

and

$$\mathbf{P}\left\{\nu_t([0;u])=3\right\}$$
$$=\iint_{\Delta_2(u)} \operatorname{Pf}\begin{pmatrix} 0 & F_t(r_1) & F_t'(r_1) & F_t(r_2) & F_t'(r_2) & F_t(u) \\ & 0 & F_t'(0) & F_t(r_2-r_1) & F_t'(r_2-r_1) & F_t(u-r_1) \\ & & 0 & -F_t'(r_2-r_1) & -F_t''(r_2-r_1) & -F_t'(u-r_1) \\ & & & 0 & F_t'(0) & F_t(u-r_2) \\ & & & & 0 & -F_t'(u-r_2) \\ & & & & & 0 \end{pmatrix} \begin{matrix} dr_1 \\ dr_2 \end{matrix}.$$

Now on the set $\{\nu_t([0;u])=k+1\}$ let us define $\theta_1,\ldots,\theta_k\in[0;u]$, $\theta_1<\ldots<\theta_k$, as the points of discontinuity of the mapping $x(\cdot,t)\colon\mathbb{R}\to\mathbb{R}$ on the interval $[0;u]$.

Theorem 3.6. *For all $k\geqslant 1$ and non-empty Borel sets $A_1,\ldots,A_k\subset[0;u]$ with $A_1<\ldots<A_k$ we have*

$$\mathbf{P}\left\{\nu_t([0;u])=k+1,\ \theta_1\in A_1,\ \ldots,\ \theta_k\in A_k\right\}$$
$$=\int_{A_1\times\ldots\times A_k}\!\!\cdots\int \operatorname{Pf}\left(\widehat{\mathbf{F}}_t(0,r_1,\ldots,r_k,u)\right) dr_1\ldots dr_k.$$

Proof. Obviously, it is enough to consider the case when the sets $A_1,\ldots,A_k$ are (disjoint) intervals. In this case we note that the probability of the event

$$\{\nu_t([0;u])=k+1,\ \theta_1\in A_1,\ \ldots,\ \theta_k\in A_k\}$$

is equal to the limit on the right-hand side of (1.4) (with $k+1$ instead of k) with the sum taken over all indices $i_1,i_2,\ldots,i_k$ satisfying (1.5) and such that

$$\left[\frac{i_l u}{2^n};\frac{(i_l+1)u}{2^n}\right]\subset A_l \quad \text{for all } l=1,2,\ldots,k.$$

Proceeding further as in the proof of Theorem 3.5 we obtain the desired result. □

4. The Total Free Time of Particles and the Mean Value

For $n \geqslant 2$ distinct, but not necessarily ordered, points $u_1, u_2, \ldots, u_n \in [0; u]$ let us set

$$\tau_1 := t,$$
$$\tau_k := t \wedge \inf\left\{ s \geqslant 0 \;\middle|\; \prod_{j=1}^{k-1} (x(u_k, s) - x(u_j, s)) = 0 \right\}, \quad 2 \leqslant k \leqslant n, \tag{4.1}$$

where the infimum of an empty set is taken to be equal to $+\infty$. Each τ_k is equal to the time that the particle with index k spends up to time t before coalescing with one of the particles with a smaller index. Therefore, it is natural to call the sum

$$S_t(\{u_1, u_2, \ldots, u_n\}) := \sum_{k=1}^{n} \tau_k$$

the total free time of the particles starting from $u_1, u_2, \ldots, u_n$ (up to time t).

It can be checked that $S_t(\{u_1, u_2, \ldots, u_n\})$ does not depend on the order of the initial points (which justifies the use of the set rather than the ordered n-tuple in the notation) and so

$$S_t(\{u_1, u_2, \ldots, u_n\}) = \sum_{k=1}^{n} \sigma_k,$$

where σ_k, $1 \leqslant k \leqslant n$, are defined as in (4.1), but for the ordering $u_{(1)} < u_{(2)} < \ldots < u_{(n)}$ of the points $u_1, u_2, \ldots, u_n$.

Moreover,

$$S_t(\{u_1, u_2, \ldots, u_n\}) = \int_0^t \nu_s(\{u_1, u_2, \ldots, u_n\})\, ds, \tag{4.2}$$

where

$$\nu_s(\{u_1, u_2, \ldots, u_n\}) := |x(\{u_1, u_2, \ldots, u_n\}, s)|, \quad 0 < s \leqslant t.$$

Theorem 4.1. [3] *For any dense countable subset $U = \{u_1, u_2, \ldots, u_n, \ldots\} \subset [0; u]$ there exists a. s. the limit*

$$\lim_{n \to \infty} S_t(\{u_1, u_2, \ldots, u_n\}). \tag{4.3}$$

Moreover, this limit does not depend on the choice of the subset U.

The random variable $S_t([0; u])$ defined by the limit (4.3) is called the *total free time* of the particles of the interval $[0; u]$ up to time t.

Using the monotone convergence theorem, from equality (4.2) we obtain that

$$S_t([0; u]) = \int_0^t \nu_s([0; u])\, ds. \tag{4.4}$$

Theorem 4.2. [3] *The mean value of $S_t([0; u])$ is given by*

$$\mathbf{E} S_t([0; u]) = t + \frac{2u\sqrt{t}}{\sqrt{\pi}}.$$

Proof. Let $\{u_1, u_2, \dots, u_n, \dots\}$ be an arbitrary dense countable subset of the interval $[0; u]$. Then

$$\mathbf{E}S_t([0;u]) = \lim_{n\to\infty} \mathbf{E}S_t(\{u_1, u_2, \dots, u_n\}) = t + \lim_{n\to\infty} \sum_{k=2}^{n} \mathbf{E}\sigma_k.$$

Now for $v_1 < v_2$ let us set

$$\tau := \inf\{s \geqslant 0 \mid x(v_1, s) = x(v_2, s)\}.$$

Then by Lemma 2.2

$$\mathbf{P}\{\tau \geqslant s\} = \mathbf{P}\{\tau > s\} = \mathbf{P}\{x(v_1,s) \neq x(v_2,s)\} = \int\limits_{-(v_2-v_1)/\sqrt{2}}^{(v_2-v_1)/\sqrt{2}} p_s(v)\,dv,$$

and so

$$\mathbf{E}(\tau \wedge t) = \int\limits_0^{+\infty} \mathbf{P}\{\tau\wedge t \geqslant s\}\,ds = \int\limits_0^t \mathbf{P}\{\tau \geqslant s\}\,ds = \int\limits_0^t \int\limits_{-(v_2-v_1)/\sqrt{2}}^{(v_2-v_1)/\sqrt{2}} p_s(v)\,dv\,ds.$$

Therefore, setting

$$\widetilde{\Delta} u_{(k)} := \frac{u_{(k)} - u_{(k-1)}}{\sqrt{2}},$$

we obtain that

$$\lim_{n\to\infty} \sum_{k=2}^{n} \mathbf{E}\sigma_k = \lim_{n\to\infty} \sum_{k=2}^{n} \int\limits_0^t \int\limits_{-\widetilde{\Delta} u_{(k)}}^{\widetilde{\Delta} u_{(k)}} p_s(v)\,dv\,ds$$

$$= \lim_{n\to\infty} \sum_{k=2}^{n} \int\limits_0^t 2\widetilde{\Delta} u_{(k)} \left(\frac{1}{2\widetilde{\Delta} u_{(k)}} \int\limits_{-\widetilde{\Delta} u_{(k)}}^{\widetilde{\Delta} u_{(k)}} p_s(v)\,dv \right) ds = \int\limits_0^t \frac{\sqrt{2}u}{\sqrt{2\pi s}}\,ds = \frac{2u\sqrt{t}}{\sqrt{\pi}}.$$

The theorem is proved. □

Corollary 4.3. *We have*

$$\mathbf{E}\nu_t([0;u]) = 1 + \frac{u}{\sqrt{\pi t}}.$$

Proof. Since $\nu_t([0;u])$ (and so $\mathbf{E}\nu_t([0;u])$) is non-increasing with respect to t, the required equality follows immediately from (4.4) and Theorem 4.2. □

Remark 4.4. Note that the mean value of $\nu_t([0;u])$ can be computed easily using the one-point density of the Pfaffian point process formed by the clusters of the Arratia flow. Indeed, it was proved in [15] that the kernel of this Pfaffian point process has the form

$$K_t(v_1, v_2) := \frac{1}{\sqrt{t}} K\left(\frac{v_1}{\sqrt{t}}, \frac{v_2}{\sqrt{t}}\right),$$

where

$$K(v_1, v_2) := \begin{pmatrix} -F''(v_2 - v_1) & -F'(v_2 - v_1) \\ F'(v_2 - v_1) & \operatorname{sign}(v_2 - v_1) \cdot F(|v_2 - v_1|) \end{pmatrix}$$

with the function F given by

$$F(z) := \frac{1}{\sqrt{\pi}} \int\limits_z^{+\infty} e^{-r^2/4}\, dr \,(\equiv F_1(z)), \quad z \in \mathbb{R}$$

(for the corresponding definitions see [15] and references therein; for the proof of the existence of the n-point density see [11]). In particular, this implies that

$$\mathbf{E}\nu_t([0;u]) = 1 + \mathbf{E}N_t([0;u]) = 1 + \frac{1}{\sqrt{t}} \int\limits_0^u \mathrm{Pf}\left(K\left(\frac{v}{\sqrt{t}}, \frac{v}{\sqrt{t}}\right)\right) dv = 1 + \frac{u}{\sqrt{\pi t}}.$$

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