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# SOME NEW RESULTS ON INNER AND OUTER HAAR MEASURE IN LOCALLY COMPACT GROUPS

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**Abstract :** Every regular measure μ on a measure space gives integration. In fact, the integration process in linear but a linear functional is not necessarily an integral. Indeed, the relationships among the linear functional, integrals and measure much deeper but our objective is not to discuss that but only to explore the idea of integration on topological groups, specially on locally compact Hausdroff topological groups and obtained various results.

**Key Words :** Haar Measure, Haar Integral, Locally, Compact Groups.
Locally Compact Topological Group.

**Introduction :** The relationships among the linear functional, integrals and measure. Here we explore the idea of integration on topological groups and obtained some new results on locally compact topological group and locally compact Hausdroff topological group.

## 1 TRANSLATES AND INVARIANT FUNCTIONS

### 1.1 Definition

Let E be any non-empty set ant G be any algebraic group. Let $f : G \to E$ be any function. let $a \in G$ be a fixed point. Then the function

$$L_{fa} : G \to E$$

defined by

$$L_{fa}(x) = f(ax) \text{ for all } x \in G$$

Is called the left translate of f by a. Similarly, the function

$$R_{fa} : G \to E$$

defined on G is called the right translate of f by a if

$$R_{fa}(x) = f(xa) \text{ for all } x \in G.$$

Let f* be a function defined on G such that $f^*(x) = f(x^{-1})$ for all $x \in G$. Then f* is called inverse function.

### 1.2 Definition

Let F be a family of functions defined on any algebraic group G. suppose that $f \in F$ and $a \in G$ imply $L_{fa} \in F$. Let η be any function on F such that

$$\eta(f) = \eta(L_{fa})$$

for all $f \in F$ and $a \in G$. then the function $\eta$ is said to be a left invariant. This is also called invariant under left translations. Similarly, if

$$\eta(f) = \eta(R_{fa})$$

for all $f \in F$ and $a \in G$ where $f \in F$ and $a \in G$ imply $R_{fa} \in F$ then $\eta$ is said to be a right invariant or invariant under right translations.

A function $\eta$ defined on F is said to be two-sided invariant if it is both left and right invariant.

Suppose that $f \in F \Rightarrow f^* \in F$. If $\eta(f) = \eta(f)^*$ for all $f \in F$, then $\eta$ is said to be inversion invariant or invariant under inversion.

Any constant function $f \in F$ is not only two sided invariant but inversion invariant also. Such a trivial function does not helps much.

**1.3 Definition**

Let G be a group (not necessarily topological). Let A be a non-empty family of subset of G and E be any non-empty set. Let $\lambda : A \rightarrow E$ be any function.

Suppose that $A \in A$ and $g \in G$ simply $gA \in A$.

$$\text{If } \lambda(gA) = \lambda(A)$$

for all $g \in G$ and $A \in A$, then $\lambda$ is called left invariant. Similarly, if $A \in A$ and $g \in G \Rightarrow Ag \in A$ and

$$\lambda(Ag) = \lambda(A)$$

for all $g \in G$ and $A \in A$, then $\lambda$ is said to be right invariant. A function $\lambda : A \rightarrow E$ is said to be inversion invariant if

$$A \in A \Rightarrow A^{-1} \in A$$

and

$$\lambda(A^{-1}) = \lambda(A) \text{ for all } A \in A.$$

Now we mention a useful Lemma available in the literature.

**1.4 Lemma**

Let $f \in \not{C}_{00}^{+}(G)$ where G is a lcH space (G is not necessarily a group). Let $F = \{x \in G : f(x) \neq 0\}$. Then $\bar{F}$ is compact.

**Proof**

Since $f \in \not{C}_{00}^{+}(G)$, hence there exists a compact set E in G such that $f(E') = \{0\}$. Obviously $F \subset E$. Since E is a compact subset of a Hausdroff space G, hence E is closed. Therefore, $E = \bar{E}$ and accordingly $\bar{F} \subset \bar{E} = E$ which shows that $\bar{F}$ is a closed subset of a compact set E. Hence $\bar{F}$ is compact.

**2 INVARIANT FUNCTIONALS ON $f \in \not{C}_{00}^{+}(G)$**

**2.1 Proposition**

Let G be a locally compact topological group. Let $f, \psi \in \not{C}_{00}^{+}(G)$ where $\psi \neq 0$. Then there exists real numbers $\alpha_i \geq 0$ and elements $x_i \in G$ (i = 1, 2, ……, n) such that

$$f \leq \sum_{i=1}^{n} \alpha_i l_{\psi} x_i,$$

That is

$$f(x) \leq \sum_{i=1}^{n} \alpha_i l_{\psi} x_i(x) = \sum_{i=1}^{n} \alpha_i \psi(x_i\, x)$$

for all $x \in G$.

**Proof**

Since $\psi \neq 0$, hence there exists $g \in G$ such that $\psi(g) \neq 0$. Let $\beta = \frac{1}{2}\psi(g)$. Since $\psi \in \not{C}_{00}^{+}(G)$, hence $\beta > 0$. Since $\psi$ is continuous on G, hence there exists an open nhood V of the identity $e \in G$ such that $\psi(x) > \beta$ for all $x \in gV$. Since $f \in \not{C}_{00}^{+}(G)$, hence there is a compact subset F of G such that $f(F') = \{0\}$ that is, $f(t) = 0$ for all $t \in F'$. Since F is compact hence it can be covered by finite number of open sets $a_iV$ (i = 1, 2, ……, n) where $a_i \in F$. thus $F \subset \bigcup_{i=1}^{n}(a_iV)$. if $x \in a_iV$, then

$$(ga_i^{-1})\, x \in (ga_i^{-1})\, a_iV = gV$$

and hence $\psi\,(ga_i^{-1}x) > \beta$. Suppose $ga_i^{-1} = x_i$. then $x_i \in G$ and $\psi\,(x_ix) > \beta$ where $\psi\,(x_ix) = l\psi_{xi}\,(x)$. Since the set $\{f(t) : t \in F\}$is the image of a compact set F in the set of real numbers by a continuous function f, hence f is bounded for all $t \in F$. But $f(t) = 0$ for all $t \in F'$. Hence f is bounded for all $t \in G$. thus there exists a non-negative real $M = \|f\|_u$. Hence $f(x) \le M$ for all $x \in G$. Thus

$$f(x) \le M \le \sum_{i=1}^{n} \frac{M}{\beta}\, \psi\,(x_ix)$$

for all $x \in F$ as $x \in a_iV$ for some $a_i$. Since $f(t) = 0$ for $t \in F'$ hence

$$f(x) \le \sum_{i=1}^{n} \frac{M}{\beta}\, \psi\,(x_ix) = \sum_{i=1}^{n} \alpha_i\, \psi\,(x_ix),\ \left(\alpha_i = \frac{M}{\beta} \ge 0\right)$$

$$= \sum_{i=1}^{n} \alpha_i\, l_{\psi\,(xi)}\,(x)$$

for all $x \in G$. Thus $f \le \sum_{i=1}^{n} \alpha_i\, l_{\psi xi}$.

This completes the proof.

**2.2 Definition**

Let f and $\psi$ be real valued functions on a group G such that $\psi \ne 0$. If $\alpha_i$ are non-negative real numbers and $x_i \in G$ (i = 1, 2, ……, n)

Such that $f(x) \le \sum_{i=1}^{n} \alpha_i\, \psi\,(x_ix)$

for all $x \in G$, then we define

$$(f : \psi) = \begin{cases} \inf \sum_{i=1}^{n} \alpha_i f \le \sum_{i=1}^{n} \alpha_i\, l_{\psi xi}. \\ \infty \text{ if no such } \alpha_i \text{ and } x_i \text{ exist.} \end{cases}$$

**2.3 Remarks**

If G is a locally compact topological group and f, $\psi \in \not\subset_{00}{}^{+}(G)$, where $\psi \ne 0$, then the previous proposition shows that $(f : \psi)$ exists and

$$(f:\psi) \le \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \frac{M}{\beta}.\, n = \frac{\| f \|_u}{\beta}.n = \frac{n}{\beta} \| f \|_u \qquad \dots (1)$$

since $f(x) \le \sum_{i=1}^{n} \alpha_i \psi(x_i x)$

$\Rightarrow \| f \|_u \le \sum_{i=1}^{n} \alpha_i \| \psi \|_u$

$\Rightarrow \frac{\| f \|_u}{\| \psi \|_u} \le \sum_{i=1}^{n} \alpha_i$ for all such $\alpha_i$

hence $\frac{\| f \|_u}{\| \psi \|_u} \le \inf\{\sum_{i=1}^{n} \alpha_i \; f(x) \sum_{i=1}^{n} \alpha_i \; \psi(x_i x)\}$

$= (f:\psi) \le \frac{n}{\beta} \| f \|_u$

Thus

$$\frac{\| f \|_u}{\| \psi \|_u} \le (f:\psi) \le \frac{n}{\beta} \| f \|_u \qquad \dots (2)$$

Let $f \neq 0$. Then $\| f \|_u > 0$. Since $\psi \neq 0$, hence $\| \psi \|_u > 0$. Applying the process of precious proposition it can be easily seen, as we have done for f, that $\psi$ is bounded on g. Hence $\| \psi \|_u < \infty$. Thus

$$\| f \|_u > 0 \text{ and } 0 < \| \psi \|_u < \infty \qquad \dots (3)$$

Using (3) in (2), we see that

$$\text{if } f \neq 0, \text{ then } (f:\psi) \ge \frac{\| f \|_u}{\| \psi \|_u} > 0 \qquad \dots (4)$$

Let $a \in G$, Then since $f(x) \le \sum_{i=1}^{n} \alpha_i \; \psi(x_i x)$ for all $x \in G$

Hence $I_{fa}(x) = f(ax) \le \sum_{i=1}^{n} \alpha_i \; \psi(x_i \, a\, x)$ for all $a\, x \in G$,

$= \sum_{i=1}^{n} \alpha_i \; \psi(t_i \; x)$ for all $x \in a^{-1} G = G$,

Where $t_i = x_i$, $a \in G$. accordingly, for any $f \in \not\subset_{00}{}^{+}(G)$ and for any a $\in$ G we have

$$(L_{fa}:\psi) = (f:\psi) \qquad \dots (5)$$

Since $f(x) \le \sum_{i=1}^{n} \alpha_i \; \psi(x_i x)$ for all $a \in G$

$= \sum_{i=1}^{n} \alpha_i \; \psi(a(a^{-1} x_i)x)$ for any $a \in G$

$= \sum_{i=1}^{n} \alpha_i \; \psi(a\, t_i x)$, $t_i = a^{-1} x_i \in G$

$$= \sum_{i=1}^{n} \alpha_i \ l_{\psi a} \ (t_i x) \text{ for all } x \in G$$

Where $\alpha_i \geq 0$ and $t_i = a^{-1}x_i \in G$.

Hence $(f : \psi) = (I : l_{\psi a})$ ...(6)

It is quite obvious that

$$(\alpha f : \ \psi) = \alpha \ (f : \psi) \text{ for all } \alpha \geq 0 \quad ...(7)$$

as

$$(\alpha f : \ \psi) = \inf \sum_{i=1}^{n} \alpha\alpha_i = \alpha \ (\inf \sum_{i=1}^{n} \alpha_i) = \alpha(f : \ \psi).$$

Let $f_1$, $f_2 \in \not{C}_{00}{}^{+}(G)$ where G is the locally compact topological group. Then there exists some reals $\alpha_i \geq 0$ and $\beta_j \geq 0$ (I = 1, 2, ..., n; j =1, 2, ..., m) and some $x_i$ $t_j \in G$ such that

$$f_1(x) \leq \sum_{i=1}^{n} \alpha_i \ \psi \ (x_i x)$$

and

$$f_2(x) \leq \sum_{j=1}^{n} \beta_j \ \psi \ (t_j x)$$

for all $x \in G$. Hence we have

$$(f_1 + f_2) \ (x) = f_1(x) + f_2(x) \leq \sum_{i=1}^{n} \alpha_i \ \psi \ (x_i x) + \sum_{j=1}^{n} \beta_j \ \psi \ (t_j x)$$

Taking infima successively on $\{\alpha_i\}$ and $\{\beta_j\}$ we have

$$(f_1 + f_2 : \psi) \leq (f_1 : \psi) + (f_2 : \psi) \quad ...(8)$$

For all $f_1$, $f_2$, $\psi \in \not{C}_{00}{}^{+}(G)$ where $\psi \neq 0$.

Again let f, $\theta$, $\psi \in \not{C}_{00}{}^{+}(G)$ where $\theta \neq 0$, $\psi \neq 0$. If a topological group G is locally compact, then there exist some real members $\alpha_h \geq 0$ and $\beta_k \geq 0$ and elements $x_h$, $t_k \in G(h = 1, 2, ..., p; k =1, 2, ..., q)$

Such that

$$f(x) \leq \sum_{h=1}^{p} \alpha_h \ \theta \ (x_h x) \text{ and } \theta(x) \leq \sum_{k=1}^{p} \beta_k \ \psi \ (t_j x) \text{ for all } x \in G.$$

Obviously $\alpha_h \beta_k$ is a non-negative real and $t_k$, $x_h$, $\in G$ for each value of k and h. Also,

$$f(x) \leq \sum_{h=1}^{p} \alpha_h\ \theta(x_h x) \leq \sum_{h=1}^{p} \alpha_h\left(\sum_{k=1}^{q} \beta_k\ \psi(t_k(x_h x))\right)$$
$$= \sum_{h=1}^{p} \sum_{k=1}^{q} \alpha_h \beta_k\ \psi((t_k x_h)x)$$

For all $x \in G$. hence by definition of $(f : \psi)$, we have

$$(f : \psi) \leq \sum_{h=1}^{p} \sum_{k=1}^{q} \alpha_h \beta_k = \left(\sum_{h=1}^{p} \alpha_h\right)\left(\sum_{k=1}^{q} \beta_k\right)$$

For all such possible $\alpha_h$ and $\beta_k$. Taking infima successively, we have

$$(f : \psi) \leq (f : \theta) : (\theta : \psi) \qquad \ldots(9)$$

It is quite obvious that

$$\text{If } f_1 \leq f_2 \text{ then } (f_1 : \psi) \leq (f_2 : \psi) \qquad \ldots(10)$$

Thus we have the following:

**2.4 Proposition**

Let G be a locally compact topological group. Let $f, f_1, f_2, \theta, \psi \in \not{C}_{00}{}^{+}(G)$ where $\theta$ and $\psi$ are non – zero functions. Then

(i) $(f : \psi)$ exists;

(ii) $(f : \psi) = \frac{\| f \|_u}{\| \psi \|_u} > 0$ for all $f \neq 0$;

(iii) $(f : \psi) = (l_{fn} : \psi) = (f : l_{\psi a})$ for all $a \in G$;

(iv) $(\alpha f : \psi) = \alpha(f : \psi)$ if $\alpha \geq 0$;

(v) $(f_1 + f_2 : \psi) \leq (f_1 : \psi) + (f_2 : \psi)$;

(vi) $(f : \psi) \leq (f : \theta) \leq (\theta : \psi)$;

(vii) $(f_1 : \psi) \leq (f_2 : \psi)$ if $f_1 \leq f_2$

**2.5 Definition**

Let G be a locally compact topological group and $f, \theta, \psi \in \not{C}_{00}{}^{+}(G)$, where $\theta$ and $\psi$ are non-zero functions. Then we define

$$\eta_{\theta\psi}(f) = \frac{(f : \psi)}{(\theta : \psi)}$$

This is well defined as $(\theta : \psi) > 0$.

If $f = 0$, then $f(x) = 0$ for all $x \in G$ and $(f : \psi) = 0$, hence $\eta_{\theta\psi}(0) = 0$.

## 2.6 Proposition

Let G be a locally compact topological group. Let $a \in G$ and f, $f_1$, $f_2$, θ, ψ $\in \mathcal{C}_{00}{}^{+}(G)$ where θ and ψ are non – zero functions. Then

(i) $\eta_{\theta\psi}(L_{fa}) = \eta_{\theta\psi}(f)$;

(ii) $\eta_{\theta\psi}(\alpha f) = \alpha\eta_{\theta\psi}(f)$ for all $\alpha \geq 0$;

(iii) $\eta_{\theta\psi}(f_1 + f_2) \leq \eta_{\theta\psi}(f_1) + \eta_{\theta\psi}(f_2)$;

(iv) $\eta_{\theta\psi}(f_1) \leq \eta_{\theta\psi}(f_2)$ if $f_1 \leq f_2$;

(v) $(f : \theta) \geq \eta_{\theta\psi}(f) \geq \frac{1}{(\theta : f)}$ if $f \neq 0$.

**Proof:**

(i), (ii) and (iii) are the immediate consequences of (iii), (iv) and (v) of the previous proposition, (iv) is obvious.

Using (vi) of the previous proposition we see that if $f \neq 0$, then

$$\eta_{\theta\psi}(f) = \frac{(f: \psi)}{(\theta : \psi)} \leq \frac{(f: \theta)(\theta: \psi)}{(\theta : \psi)} = (f : \theta)$$

and

$$\eta_{\theta\psi}(f) = \frac{(f: \psi)}{(\theta : \psi)} \geq \frac{(f : \psi)}{(\theta : f)\,(f : \psi)} = \frac{1}{(\theta : f)}$$

Thus (v) follows. This completes the proof.

## 2.7 Proposition

Let $f_j$ be non-zero functions in $\mathcal{C}_{00}{}^{+}(G)$ where G is a locally compact $T_0$ topological group. Let α be any positive number and θ, ψ are non-zero functions in $\mathcal{C}_{00}{}^{+}(G)$. Then for any $\in > 0$, there exits an open nhood of U of e such that

$$\sum_{j=1}^{n} \alpha_j\ \eta_{\theta\psi}(f_j) - \eta_{\theta\psi}\left(\sum_{j=1}^{n} \alpha_j f_j\right) \leq \in$$

If $(U') = \{ 0 \}$ and $0 \leq \alpha_j \leq \alpha$

**Proof**

Since $f_j \in \not{C}_{00}^+(G)$, hence there exists a compact subset F of G such that $f_j(F') = \{ 0 \}$ for j = 1, 2, ..., n. Since G is locally compact hence there exists an open nhhod V of e such that $\overline{V}$ is compact.

Since F and $\overline{V}$ are compact hence f $(F\overline{V})$ is a compact in G. Since G is $T_0$ hence it is $T_2$.

Thus G is lcH topological group. Therefore by Urysohn's lemma, there exists a function f such that

$$f \in \not{C}_{00}^+(G),\ f(F\overline{V}) = \{ 1 \} \text{ and } f(G) \subset [0, 1].$$

Suppose that

$$m = \max\{ \| f_1 \|_u \| f_n \|_u\} \quad ...(1)$$

And M = $M_{ax}$m. if $f_j$ and f are left uniformly continuous. Therefore, there is a symmetric open nhood U of e in G such that $U \subset V$ and

$$t^{-1} x \in U \Rightarrow |f_j(t) - f_j(x)| < \epsilon_1 \quad ...(2)$$

and

$$|f(t) - f(x)| < \epsilon_2 \quad ...(3)$$

For every $\epsilon_1 > 0, \varepsilon_2 > 0$. Let $\beta > 0$ such that $\beta < M$.

Let $\psi \in \not{C}_{00}^+(G)$ and $\psi \neq 0$ such that $\psi(U') = \{0\}$.

Let $\alpha_j \geq 0$ for j = 1, 2, ..., n such that $\alpha_j \leq \alpha$. Let

$$\phi \textstyle\sum_{j=1}^{n} \alpha_j f_j + \beta f \quad ...(4)$$

and $$\phi_j \quad (x) \begin{cases} \dfrac{\alpha_j f_j(x)}{\phi(x)} & \text{if } x \in F, \\ 0 \ \text{if } x \in F', \end{cases} \quad ...(5)$$

Obviously $\phi \in \not{C}_{00}^+(G)$, and $\phi\phi_j = \alpha_j f_j$ for all j = 1, 2, ..., n.

Also,

$$\sum_{j=1}^{n} \phi_j(x) = \begin{cases} \frac{1}{\phi(x)} \sum_{j=1}^{n} \alpha_j f_j(x) & \text{if } x \in F, \\ 0 & \text{if } x \in F', \end{cases}$$

$$= \begin{cases} \frac{1}{\phi(x)} (\phi(x) - \beta f(x)) & \text{if } x \in F, \\ 0 & \text{if } x \in F', \end{cases}$$

Where $\beta > 0$, $\phi(x) > 0$ and $f(x) \geq 0$, hence

$$\frac{1}{\phi(x)} (\phi(x) - \beta f(x)) = 1 - \beta \frac{f(x)}{\phi(x)} \leq 1 \text{ for all } x \in F$$

Thus $\sum_{j=1}^{n} \phi_j(x) \leq 1$ for all $x \in G$.

From (2), (3) and (4) we see that if $t^{-1} x \in U$, then

$$|\phi(t) - \phi(x)| = | \sum_{j=1}^{n} \alpha_j (f_j(t) - fj(x)) + \beta(f(t) - f(x)) |$$

$$\leq \sum_{j=1}^{n} |\alpha_j| \, |f_j(t) - fj(x)| + |\beta| \, |f(t) - f(x)|$$

$$< \sum_{j=1}^{n} \alpha_j \in_1 + \beta \in_2$$

$$= \in_1 \sum_{j=1}^{n} \alpha_j + \beta \in_2, \text{ where } \alpha_j < \alpha$$

$$\leq n. \in_1 \alpha + \beta \in_2 \quad \ldots(6)$$

If $x \notin F\overline{V}$ and $t^{-1} x \in U$, then $t \notin F$. For, if $t \in F$ then $x = (tt^{-1})x = t(t^{-1}x) \in FU \subset FV \subset F\overline{V}$, a contradiction. Similarly, if $t \notin F\overline{V}$ and $t^{-1}x \in U$, then $t \notin F$.

For, if $x \in F$ and $t^{-1} x \in U$, then $t = x(x^{-1}t) = x = t(t^{-1}x) \in FU^{-1} + FU$ (as U is symmetric) $\subset FV \subset F\overline{V}$, a contradiction.

Obviously, $\| \phi \|_u \leq \sum_{j=1}^{n} \alpha_j \| f_j \|_u + \| f \|_u$

$$\leq \sum_{j=1}^{n} \alpha . m + \beta . 1 = n. \alpha. m + \beta = M + \beta$$

$$< 2M \quad \ldots (7)$$

Let $t^{-1}x \in U$ such that $x, t \subset F\overline{V}$ also, then $x\, t \in F$ and hence $f_j(x) = 0 = f_j(t)$. Also $f(t) = 1 = f(x)$. Hence from (4), $\phi(t) = \beta\, \phi(x)$, and from (1), (6), (7) and (2) we have

$$\left| \phi_j(t) - \phi_j(x) \right| = \left| \frac{\alpha_j f_j(x)}{\phi(t)} - \frac{\alpha_j f_j(x)}{\phi(x)} \right|$$

$$= \left| \frac{\alpha_j (f_j(t)\phi(x) - (f_j(x)\phi(t))}{\phi(t)\phi(x)} \right|$$

$$= \left| \frac{\alpha_j (f_j(t)\phi(x) - f_j(t)\phi(t) + f_j(t)\phi(t) - f_j(x)\phi(t)}{\phi(t)\phi(x)} \right|$$

$$= \left| \alpha_j \frac{(f_j(t)\phi(x) - f_j(t)\phi(t)) + f_j(t) - f_j(x))\phi(t)}{\phi(t)\phi(x)} \right|$$

$$\leq \alpha_j \frac{\|f_j\|_u \ \phi(x)\phi(t) \ \|\phi\|_u \ f_j \ (t)\phi(x)}{\beta.\beta}$$

$$< \frac{\alpha}{\beta^2}\{m.(n \in_1 \alpha + \beta \in_2) + 2M . \in_1\} \qquad ...(8)$$

We can take $\in_1 = \frac{\beta^3}{4Mn\alpha}$ and $\in_2 = \frac{\beta^2}{4M}$ ...(9)

Then $n \in_1 \alpha + \beta \in_2 = \frac{\beta^3}{2M}$, and hence

$$\frac{\alpha}{\beta^2}\{m\ (n \in_1 \alpha + \beta \in_2) + 2M . \in_1\} = \frac{\alpha}{\beta^2}\left(m \frac{\beta^3}{2M} + \frac{\beta^3}{2n\alpha}\right)$$

$$= \frac{\alpha}{\beta^2}\left(m \frac{\beta^3}{2mn\alpha} + \frac{\beta^3}{2n\alpha}\right) \frac{\beta}{n}$$

Therefore from (6) and (8) we see that if $t^{-1} \in U$

Such that $x, t \in F\overline{V}$, then

$$|\phi(t)\phi(x)| < \frac{\beta^3}{2M} \qquad ...(10)$$

and

$$|\phi_j(t)\phi_j(x)| < \frac{\beta}{n} \qquad ...(11)$$

Since $\psi, \phi \in \not{C}_{00}{}^{+}(G)$, where $\phi \neq 0$ hence by proposition 6.2.1, there exists some positive real numbers $c_k$ and elements $t_k \in G$ ($k = 1, 2, ..., p$) such that

$$\phi(x) \leq \sum_{k=1}^{p} c_k\ (f_j x) \text{ for all } x \in G \qquad ...(12)$$

Since $\psi(U') = \{0\}$, hence $\psi(t_k x) = 0$ if $t_k x \in U'$

Therefore if $t_k x \in U'$, then $\psi(t_k x)$ contribute nothing in summation of (12). Hence for a fixed x we take only those $t_k x$ which belongs to U, that is where $x \in t_k^{-1}U = tU$ (here $t = t_k^{-1}$). Hence we restrict the summation of (12) only to those k for which $t_k^{-1} \in U$, where $t = t_k^{-1}$. From (11) and (12), we have

$$\alpha_j f_j(x) = \phi(x)\phi_j(x) \leq \sum_{k=1}^{p} c_k \left(\phi_j(x)+\frac{\beta}{n}\right)\psi(t_k x),\ t = t_k^{-1}$$

$$= \sum_{k=1}^{p} c_k \left(\phi_j(t_k^{-1})+\frac{\beta}{n}\right)\psi(t_k x) \qquad ...(13)$$

For all $x \in G$. this shows that

$$(\alpha_j f_j : \psi) \leq \sum_{k=1}^{p} c_k \left(\phi_j(t_k^{-1})+\frac{\beta}{n}\right) \qquad ...(14)$$

$$\therefore \sum_{j=1}^{n}(\alpha_j f_j : \psi) \leq \sum_{k=1}^{p} c_k\{\sum_{j=1}^{n}[\phi_j(t_k^{-1})+\frac{\beta}{n}]\} \qquad ...(15)$$

Since, $\sum_{j=1}^{n}\phi_j(x) \leq 1$ for all $x \in G$ hence $\sum_{j=1}^{n}\phi_j(t_k^{-1}) \leq 1$.

Accordingly,

$$\sum_{j=1}^{n}(\alpha_j f_j : \psi) \leq \sum_{k=1}^{p} c_k(l + n.\frac{\beta}{n}) = (1+\beta)\sum_{k=1}^{p} c_k \qquad ...(16)$$

From (12), $(\phi : \psi) = \inf(\sum_{k=1}^{p} c_k)$, hence taking infimum on (16), we have

$$\sum_{j=1}^{n}(\alpha_j f_j : \psi) \leq (1+\beta)(\phi : \psi) \qquad ...(17)$$

Dividing (1&) by $(\theta : \psi)$, we have

$$\sum_{j=1}^{n}\frac{(\alpha_j f_j : \psi)}{(\theta : \psi)} \leq (1+\beta)\frac{(\phi : \psi)}{(\theta : \psi)}$$

Or $\quad \sum_{j=1}^{n}\eta_{\theta\psi}(\alpha_j f_j) \leq (1+\beta)\,\eta_{\theta\psi}(\phi)$

Or $\quad \sum_{j=1}^{n}\alpha_j\eta_{\theta\psi}(f_j) \leq (1+\beta)\,\eta_{\theta\psi}(\sum_{j=1}^{n}(\alpha_j f_j \ +:\beta f))$

$$\leq (1+\beta)\{\eta_{\theta\psi}(\sum_{j=1}^{n}(\alpha_j f_j) + \eta_{\theta\psi}(\beta f)\}$$

$$= \eta_{\theta\psi}(\sum_{j=1}^{n}\alpha_j f_j) + \beta\eta_{\theta\psi}\beta\eta_{\theta\psi}(\sum_{j=1}^{n}\alpha_j f_j) + (1+\beta)\beta\eta_{\theta\psi}(f)$$

$$\leq \eta_{\theta\psi}(\sum_{j=1}^{n}\alpha_j f_j) + \beta\sum_{j=1}^{n}\alpha_j\,\eta_{\theta\psi}(f_j) + (1+\beta)\beta\eta_{\theta\psi}(f)$$

$$\leq \eta_{\theta\psi}(\sum_{j=1}^{n} \alpha_j f_j) + \alpha\beta \sum_{j=1}^{n} \eta_{\theta\psi}(f_j) + (1+\beta)\beta\eta_{\theta\psi}(f)$$

$$\leq \eta_{\theta\psi}(\sum_{j=1}^{n} \alpha_j f_j) + \alpha\beta \sum_{j=1}^{n} (f_j : \theta) + (1+\beta)\beta\eta_{\theta\psi}(f)$$

$$\therefore \sum_{j=1}^{n} \alpha_j \eta_{\theta\psi}(f_j) - \eta_{\theta\psi}(\sum_{j=1}^{n}(\alpha_j f_j))$$

$$\leq \alpha\beta \sum_{j=1}^{n} (f_j : \theta) + (1+\beta)\beta(f : \theta) < \in,$$

If we choose $\beta$ sufficiently small.

This completes the proof.

**2.8 Proposition**

Let $f \in \not\subset_{00}^{+}(G)$, where G is a lcH topological group. Let W be a nhood of the identity $e \in G$. Let $F = \{x \in G : f(x) \neq 0\}$. Then there exists some $x_j \in \bar{F}$ and $f_j \in \not\subset_{00}^{+}(G)$( $j = 1, 2, 3, ..., n$) such that

$$f = \sum_{j=1}^{n} f_j$$

and if $F_j = \{x \in G : f_j(x) \neq 0\}$, then $\bar{F}_j \subset Wx_j$.

**Proof**

Let $W^0$ be the interior of W.$W^0$ is an open nhood of e. There exists an open nhood V of e such that $\bar{V} \subset W^0$ and $\bar{V}$ is compact. By lemma 6.1.4, $\bar{F}$ is compact. Since each right translation of G onto G is a homeomorphism, hence $W^0x$ is open and $\bar{V}x$ is compact for every $x \in G$. Also $\bar{V}x \subset W^0x \subset Wx$. Obviously, if x runs through $\bar{F}$ then $W^0x$ is an open cover of a compact set $\bar{F}$, hence there exists some $x_j \in \bar{F}$ such that $\bar{F} \subset \bigcup_{j=1}^{n} W^0 x_j$. Since $\bar{V}x_j$ is compact and $W^0x$ is open such that $\bar{V}x_j \subset W^0x_j$.

Hence by Urysohn's lemma, there exists a continuous function,

$\phi_j : G \to [0, 1]$ such that

$$\phi_j(x) = \begin{cases} 1 \text{ if } x \in \bar{V}x_j \\ 0 \text{ if } x \in (W^0 x_j)' \end{cases}$$

Put $\phi(x) = \sum_{j=1}^{n} \phi_j(x)$. Then $\phi(x) \geq 1$ for all $x \in \bar{F}$.

Now we may define

$$f_j(x) = \begin{cases} \frac{f(x)\phi_j(x)}{\phi(x)} \text{ if } x \in \overline{F}, \\ 0 = f(x) \text{ if } x \not\subset \overline{F}. \end{cases}$$

Obviously $f_j$ is continuous on G and $fj(\overline{F}') = \{ 0 \}$ where is $\overline{F}$ is compact. Hence $f_j \in \not\subset_{00}{}^{+}(G)$. Also $\sum_{j=1}^{n} f_j(x) = f(x)$ for all $x \in G$. Hence

$$\sum_{j=1}^{n} f_j = f.$$

$f_j(x) \neq 0$ only when $f(x)$ and $\phi_j(x)$ both are non-zero, that is, only when

$$x \in F \cap \overline{V}x_j.$$

Hence $F_j \subset \overline{V}x_j \subset Wx_j$. Therefore

$$\overline{F}_j \subset \overline{V}x_j = \overline{V}x_j \subset Wx_j$$

as $\overline{V}x_j$, being a compact subset of a Hausdroff space G, is closed.

**2.9 Corollary**

For every nhood W of e of a lcH topological group G, there exist $x_j \in G$ and $f_j \in \not\subset_{00}{}^{+}(G)$ $(j = 1, 2, \ldots, n)$ such that

$$f_j x = o \text{ for all } x \in (Wx_j)' \subset (W^0 x_j)'$$

and

$$\sum_{j=1}^{n} f_j(x) = 1 \text{ for all } x \in \overline{F}$$

Where $F = \{ x \in G : f(x) \neq 0 \}$ and $f \in \not\subset_{00}{}^{+}(G)$.

**Proof**

If we slightly amend the definition of $f_j$ in the proof of the proposition as below:

$$f_j(x) = \begin{cases} \frac{\phi_j(x)}{\phi(x)} \text{ if } x \in \overline{F}, \\ f(x) \text{ if } x \notin \overline{F}. \end{cases}$$

the $\sum_{j=1}^{n} f_j \ (x) = 1$ for all $x \in \bar{F}$ and $f_j(x) \neq 0$ only when $x \in \bar{V}x_j \subset Wx_j$. Thus $f_j(x) = 0$ for all $x \in (Wx_j)'$. It is obvious that $f_j \in \not{C}_{00}^{+}(G)$.

**2.10 Note**

Using the left translation in place of right translation in the proof, we can easily see that there exist some $x_j \in \bar{F}$ and $f_j \in \not{C}_{00}^{+}(G)$ $(j = 1, 2, ..., n)$ such that

$\sum_{j=1}^{n} f_j = f$ and $\bar{F}_j \subset x_jW$;

And in particular $\sum_{j=1}^{n} f_j \ (x) = 1$ for all $x \in \bar{F}$ and

$f_j(x) = =0$ for all $x \ (Wx_j)'$.

**2.11 Proposition**

Let G be a lcH topological group. Let $f \ (\neq 0) \in \not{C}_{00}^{+}(G)$. Let $\alpha > 0$ be any real number. Let U be an open nhood of the identity $e \in G$ such that whenever $x, y \in G$ implies $y^{-1}x \in U$, then $|f(x) - f(y)| < \alpha$ (such U exists as f being an element of $\not{C}_{00}^{+}(G)$ is left uniformly continuous. Let F be a compact subset of G such that $f(F')=\{0\}$. Let $\psi \ (\neq 0) \in \not{C}_{00}^{+}(G)$ such that $\psi \ (U') = \{0\}$. Let $\delta$ be a real number such that $\delta > \alpha$, then there exist some $t_j \in F^{-1}$ and real numbers

$\alpha_j \geq 0 \ (j = 1, 2, ..., n)$ such that

$$|f(g) - \sum_{j=1}^{n} \alpha_j \psi \ (t_j \, g)| \leq \delta$$

For all $g \in G$.

**Proof**

If $t, x \in G$ such that $t^{-1}x \in U$ then

$|f(x) - f(t)| < \alpha$; that is

$$f(x) - \alpha < f(t) < f(x) + \alpha \qquad ...(1)$$

and hence

$$(f(x) - \alpha)\psi \ (t^{-1}x) \leq f(t) \ \psi \ (t^{-1}x) \leq (f(x) + \alpha) \ \psi \ (t^{-1}x) \qquad ...(2)$$

If t, x ∈ G such that $t^{-1}x \notin U$ then $\psi(t^{-1}x) = 0$ as $\psi(U') = \{0\}$. Corresponding to each $\psi \in \not{C}_{00}^{+}(G)$, there is $\psi^*$ defined on G such that $\psi^*(x) = \psi(x^{-1})$ for all x ∈ G. Let V be a nhood of e such that $\overline{V}$ is compact. Since $\psi \in \not{C}_0(G)$ hence ψ is right uniformly continuous. Accordingly, if x, y ∈ G such that $xy^{-1} \in V$ then $\{\psi(x) - \psi(y)| < \beta$ for any $\beta > 0$. We choose β so that

$$(f : \psi^*)\, \beta < \delta - \alpha \qquad ...(3)$$

Since $f(F') = \{0\}$, hence there exist $x_j \in F$ (j = 1, 2, ..., n) and hence tj $= x_j^{-1} \in F^{-1}$ such that

$$\{x \in G : f(x) \neq 0\} \subset F \subset \cup_{j=1}^{n}(x_j V) \qquad ...(4)$$

By previous proposition there exist continuous functions $\phi_j \in \not{C}_{00}^{+}(G)$ such that

$$\left.\begin{aligned} &\phi_j(((xjV)') = \{0\} \ (j = 1, 2, ..., n) \\ \text{and} \quad &\textstyle\sum_{j=1}^{n} \phi_j(x) = 1 \text{ if } f(x) \neq 0 \end{aligned}\right\} \qquad ...(5)$$

for every x, g ∈ G we have $x_j^{-1}g \in G$ and $x^{-1}g \in G$ where $(x_j^{-1}g)(x^{-1}g)^{-1} = x_j^{-1}x$. Hence if $x_j^{-1}x \in V$, that is, if $x \in x_jV$, then

$$|\psi(x_j^{-1}g) - \psi(x^{-1}g)| < \beta \qquad ...(6)$$

And if $x_j^{-1}x \notin V$, then $x \notin x_jV$ and hence $x \in (x_jV)'$

Which implies $\phi_j(x) = 0$ ...(7)

From (6) we have

$$\psi(x^{-1}g) - \beta < (x_j^{-1}g) < \psi(x^{-1}g) + \beta$$

if $x_j^{-1}x \in V$. Hence, $x_j^{-1}x \in V$ then for each j

$$\begin{aligned} \phi_j(x)f(x)\,[\psi(x^{-1}g) - \beta] &\leq \phi_j(x)f(x)\,\psi(x_j^{-1}g) \\ &\leq \phi_j(x)f(x)\,[\psi(x_j^{-1}g) + \beta] \qquad ...(8) \end{aligned}$$

For all x, g ∈ G. Giving the values of j in (8) and then adding we get

$$\sum_{j=1}^{n} \phi_j(x)\, f(x)\, [\psi(x^{-1}g) - \beta] \le \sum_{j=1}^{n} \phi_j(x)\, f(x)\, \psi(x_j^{-1}g)$$

$$\le \sum_{j=1}^{n} \phi_j(x)\, f(x)\, [\psi(x_j^{-1}g) + \beta]$$

Or $$f(x)\, [\psi(x^{-1}g) - \beta] \sum_{j-1}^{n} \phi_j(x) \le f(x) \sum_{j=1}^{n} \phi_j(x)\, \psi(x_j^{-1}g)$$

$$\le f(x)\, [\psi(x_j^{-1}g) + \beta] \sum_{j=1}^{n} \phi_j \quad .$$

If $x \in G$ and $f(x) \neq 0$, then using (5), we get

$$f(x)\, [\psi(x^{-1}g) - \beta] \le f(x) \sum_{j=1}^{n} \phi_j(x)\, \psi(x_j^{-1}g)$$

$$\le f(x)\, [\psi(x_j^{-1}g) + \beta]$$

if $f(x) = 0$, then it holds trivially, Thus for all $x, g \in G$,

We have

$$f(x)\, \psi(x^{-1}g) - \beta f(x) \le f(x) \sum_{j=1}^{n} \phi_j(x)\, \psi(x_j^{-1}g)$$

$$\le f(x)\, \psi(x_j^{-1}g) + \beta f(x) \qquad \ldots(9)$$

If $x^{-1}g \in U$ then using (2) in (9), we have

$$(f(g) - \alpha)\psi(x^{-1}g) - \beta f(x) \le f(x)\, \psi(x^{-1}g) - \beta f(x)$$

$$\le f(x) \sum_{j=1}^{n} \phi_j(x)\, \psi(x_j^{-1}g)$$

$$\le f(x)\, \psi(x^{-1}g) + \beta f(x)$$

$$\le (f(g) + \alpha)\, \psi(x^{-1}g) + \beta f(x)$$

Or $$(f(g) - \alpha)\psi(x^{-1}g) - \beta f(x) \le f(x) \sum_{j=1}^{n} \phi_j(x)\, \psi(t_j g)$$

$$\le (f(g) + \alpha)\, \psi(x^{-1}g) + \beta f(x)$$

(as $t_j = x_j^{-1}$)

Or $$(f(g) - \alpha)\psi^*(g^{-1}x) - \beta f(x) \le \sum_{j=1}^{n} \psi(t_j g)(\phi_j f)(x)$$

$$\le (f(g) + \alpha)\, \psi^*(g^{-1}x) + \beta f(x)$$

Or $$(f(g) - \alpha)\, l_{\psi * g^{-1}}(x) - \beta f(x) \le \psi(t_j g)(\phi_j f)(x)$$

$$\le (f(g) + \alpha)\, l_{\psi * g^{-1}}(x) + \beta f(x)$$

Or $$([f(g) - \alpha]\, l_{\psi * g^{-1}} - \beta f)(x) \le \sum_{j=1}^{n} (\psi(t_j g)\phi_j f)(x)$$

$$\le ([f(g) + \alpha])\, l_{\psi * g^{-1}} + \beta f)(x)$$

$\therefore \quad [f(g)-\alpha]\, l_{\psi * g^{-1}} - \beta f \leq \sum_{j=1}^{n} \psi(t_j g)\phi_j f$

$\leq [f(g)+\alpha]\, l_{\psi * g^{-1}} + \beta f \qquad \ldots(10)$

Let $\theta$, $f_0 \in \not{C}_{00}{}^{+}(G)$ such that $\theta \neq 0$, $f_0 \neq 0$. Then from the first inequality of (10) we have

$[f(g)-\alpha]\, l_{\psi * g^{-1}} \leq \beta f + \sum_{j=1}^{n} \psi(t_j g)\phi_j f$

And hence from proposition 6.2.6, we have

$\eta_{\theta f_0}([f(g)-\alpha]\, l_{\psi * g^{-1}}) \leq \eta_{\theta f_0}(\beta f + \sum_{j=1}^{n} \psi(t_j g)\phi_j f)$

$\leq \eta_{\theta f_0}(\beta f) + \eta_{\theta f_0}(\sum_{j=1}^{n} \psi(t_j g)\phi_j f)$

$= \beta \eta_{\theta f_0}(f) + \eta_{\theta f_0}(\sum_{j=1}^{n} \psi(t_j g)\phi_j f)$

$\therefore \eta_{\theta f_0}([f(g)-\alpha]\, l_{\psi * g^{-1}}) - \beta \eta_{\theta f_0}(f) \leq \eta_{\theta f_0}(\sum_{j=1}^{n} \psi(t_j g)\phi_j f) \quad \ldots(11)$

And similarly the second inequality of (10) implies

$\eta_{\theta f_0}(\sum_{j=1}^{n} \psi(t_j g)\phi_j f \leq [f(g)-\alpha]\eta_{\theta f_0}(l_{\psi * g^{-1}}) + \beta \eta_{\theta f_0}(f) \qquad \ldots(12)$

Where $\eta_{\theta f_0}(l_{\psi * g^{-1}}) = \eta_{\theta f_0}\psi^*$. From (11) and (12), we have

$[f(g)-\alpha]\, \eta_{\theta f_0}(\psi^*) - \beta \eta_{\theta f_0}(f) \leq \eta_{\theta f_0}(\sum_{j=1}^{n} \psi(t_j g)\phi_j f)$

$\leq [f(g)-\alpha]\, \eta_{\theta f_0}(\psi^*) + \beta \eta_{\theta f_0}(f) \qquad \ldots(13)$

Since $\dfrac{\eta_{\theta f_0}(f)}{\eta_{\theta f_0}(\psi^*)} = \dfrac{\frac{(f:f_0)}{(\theta:f_0)}}{\frac{(\psi^*:f_0)}{(\theta:f_0)}} = \dfrac{(f:f_0)}{(\psi\psi:f_0)} \leq \dfrac{(f:\psi^*)(\psi^*:f_0)}{(\psi^*:f_0)}$

Hence $\dfrac{\eta_{\theta f_0}(f)}{\eta_{\theta f_0}(\psi^*)} \leq (f : \psi *) \leq \dfrac{\delta-\alpha}{\beta}$ (from (3)) $\qquad \ldots(14)$

Dividing (13) by $\eta_{\theta f_0}(\psi^*)$ and using proposition 6.2.6, we have

$[f(g)-\alpha] < (\delta - \alpha) + \eta_{\theta f_0}\left[\dfrac{\sum_{j=1}^{n} \psi(t_j g)\phi_j f}{\eta_{\theta f_0}(\psi^*)}\right]$

And

$\eta_{\theta f_0}\left[\dfrac{\sum_{j=1}^{n} \psi(t_j g)\phi_j f}{\eta_{\theta f_0}(\psi^*)}\right] < [f(g)+\alpha] + (\delta - \alpha) = f(g) + \delta$

$$\therefore f(f) - \delta < \eta_{\theta f_0}\left[\frac{\sum_{j=1}^{n}\psi(t_j g)\phi_j f}{\eta_{\theta f_0}(\psi *)}\right] < [f(g) + \delta$$

That is, $f(g) - \epsilon_1 \leq \eta_{\theta f_0}\left[\sum_{j=1}^{n}\frac{\psi(t_j g)\phi_j f}{\eta_{\theta f_0}(\psi *)}\right] \leq f(g) + \epsilon_1$ ...(15)

For some $\epsilon_1 < \delta$. For,

$$\frac{\sum_{j=1}^{n}\psi(t_j g)\phi_j f}{\eta_{\theta f_0}(\psi *)} = \sum_{j=1}^{n}\frac{\psi(t_j g)}{\eta_{\theta f_0}(\psi *)}\phi_j f,$$

Put $\epsilon > 0$, $\alpha > 0$, $0 \leq c \leq \alpha$.

By proposition 6.2.7, there exists a nhood W of e

Such that if $\theta(W') = \{0\}$, then

$$\sum_{j=1}^{n}\frac{\psi(t_j g)}{\eta_{\theta f_0}(\psi *)}\eta_{\theta f_0}(\psi *) \leq \eta_{\theta f_0}\left[\sum_{j=1}^{n}\frac{\psi(t_j g)}{\eta_{\theta f_0}(\psi *)}(\phi_j f)\right] + \varepsilon \quad ...(16)$$

Applying (ii) and (iii) of proposition 6.2.6 on (16) and using (15) we have

$$f(g) - \varepsilon_1 < \eta_{\theta f_0}\left[\sum_{j=1}^{n}\frac{\psi(t_j g)}{\eta_{\theta f_0}(\psi *)}(\phi_j f)\right] \leq \sum_{j=1}^{n}\frac{\psi(t_j g)}{\eta_{\theta f_0}(\psi *)}\eta_{\theta f_0}(\phi_j f)$$

$$\leq \eta_{\theta f_0}\left[\sum_{j=1}^{n}\frac{\psi(t_j g)}{\eta_{\theta f_0}(\psi *)}(\phi_j f)\right] + \varepsilon \; f(g) + \varepsilon_1 + \varepsilon \quad ...(17)$$

If we put $\frac{\eta_{\theta f_0}(\phi_j f)}{\eta_{\theta f_0}(\psi *)} = \alpha_j$, then from (17) we have

$$f(g) - \varepsilon_1 + \varepsilon < f(g) - \varepsilon_1 \leq \sum_{j=1}^{n}\alpha_j\,\psi(t_j g) \leq f(g) + \varepsilon_1 + \varepsilon$$

or $\quad f(g) - \delta < \sum_{j=1}^{n}\alpha_j\,\psi(t_j g) \leq f(g) + \delta$

$$\therefore |f(g) - \sum_{j=1}^{n}\alpha_j\,\psi(t_j g)| \leq \delta$$

For all $g \in G$ and for some $t_j \in F^{-1} \subset G$.

This completes the proof.

## 2.12 Proposition

Let G be a lcH topological group. Let $f \in \not{C}_{00}^{+}(G)$ be a non-zero function. Then for a given real $\delta > 0$, there exist non-zero functions $\psi$, $f_0 \in \not{C}_{00}^{+}(G)$, some real $\alpha_j \geq 0$ and $t_j \in G$ (j=1, 2, ..., n)such that

$$|f(g) - \sum_{j=1}^{n} \alpha_j \, \psi(t_j g)| \leq \delta \, f_0(g)$$

For all g, that is

$$|f - \sum_{j=1}^{n} \alpha_j \, \psi t_j| \leq \delta \, f_0$$

**Proof**

Since G is locally compact, hence there exists an open nhood $U_0$ of the identity $e \in G$ such that $\bar{U}_0$ is compact. Let $\theta \in \not{C}_{00}^{+}(G)$ such that $\theta \neq 0$. Then $f + \theta \in \not{C}_{00}^{+}(G)$ and $f + \theta \neq 0$. Let

$$A = \{x \in G : (f + \theta)(x) \neq 0\} \quad ...(1)$$

Put $F = \bar{A}\,\bar{U}_0$. Then F is a closed and compact. Let $f_0 \in \not{C}_{00}^{+}(G)$ such that $f_0(F) = \{1\}$. Such $f_0$ exists by Urysohn's lemma. Since $f \in \not{C}_{00}^{+}(G) \subset \not{C}_0$ (G), hence f is left uniformly continuous. Therefore, there exists an open nhood $V_0$ of e such that

$$|f(x) - f(y)| < \frac{\delta}{2} \quad ...(2)$$

provided $x, y \in G \Rightarrow y^{-1}x \in W_0$.

Similarly there exists an open nhood $W_0$ of e such that

$$|\theta(x) - \theta(y)| < \frac{\delta}{2} \quad ...(3)$$

provided $x, y \in G \Rightarrow y^{-1}x \in V_0$.

Let $W = U_0 \cap V_0 \cap W_0$. Then W is an open nhood of e such that whenever $x, y \in G$ and $y^{-1}x \in W$ then (2) and (3) will hold, that is]

$$|f(x) - f(y)| < \frac{\delta}{2} \text{ and } |\theta(x) - \theta(y)| < \frac{\delta}{2} \quad ...(4)$$

We can choose a function $\psi \in \not{C}_{00}^{+}(G)$ such that $(W') = \{0\}$.

Suppose that

$A_0 = \{x \in G : (f\,x) \neq 0\}$ ...(5)

Put $\bar{A}_0$ = E. Then $\bar{A}_0 \subset$ A and hence $\bar{E} = \bar{A}_0 \subset \bar{A} \subset \bar{A}\ \ \bar{U}_0$= F.

Since E is a closed subset of a compact set F, hence E is compact. Also we have from (5)

f(x) = for all x $\in$E'.

$\therefore$ f(E′) = {0} ...(6)

Applying previous proposition there exist some $t_j \in E^{-1}$ and real number

$\alpha_j \geq$ (j=1, 2, ..., n)such that

$|f(g) - \sum_{j=1}^{n} \alpha_j\, \psi(t_j g)| \leq \delta$ ...(7)

For all g $\in$ G.

Since E $\subset$ F and f (E′) = {0}, hence f (F′) = {0}.

$\psi(t_j g) \neq 0$ only when $t_j g \in W$.

But $t_j g \in W \Leftrightarrow g \in t_j^{-1}W \in \overleftarrow{E}W = \bar{A}_0 W \subset \bar{A}_0\, \bar{U}_0 \subset \bar{A}\bar{U}_0$ = F

Thus $\psi(t_j g) = 0$ for all g $\in$ F′.

Since $f_0(g) = 1$ for all g $\in$ F, hence from (7),

We have

$|f(g) - \sum_{j=1}^{n} \alpha_j\, \psi(t_j g)| \leq \delta\, f_0(g)$ ...(8)

For all g $\in$ F. Since $f_0(g) \neq 0$. f(g) = 0 and $\psi(t_j g) = 0$ for all g $\in$ F′.

Hence $|f(g) - \sum_{j=1}^{n} \alpha_j\, \psi(t_j g)| = 0 \leq \delta\, f_0(g)$ ...(9)

For all g $\in$ G. From (8) and (9), we have

$|f(g) - \sum_{j=1}^{n} \alpha_j\, \psi(t_j g)| \leq \delta\, f_0(g)$ ...(10)

For all g $\in$ G. Thus there exist $\psi$, $f_0 \in \not\subset_{00}{}^{+}(G)$ and real $\alpha_j \geq 0$, $t_j$ = G (j = 1, 2, ..., n) such that

$|f(g) - \sum_{j=1}^{n} \alpha_j 1\, \psi t_j| \leq \delta\, f_0$ ...(11)

This completes the proof.

Let G be a lcH topological group. Let U be an open nhood of the identity e ∈ G. Let $\phi_U$ be a function in $\not{c}_{00}{}^+(G)$ such that $\phi_U \neq 0$ and $\phi_U(U') = \{0\}$. Let $\phi U_1$ and $\phi U_2$ be two functions in $\not{c}_{00}{}^+(G)$ corresponding to open nhood $U_1$ and $U_2$ of e, where $U_1 \subset U_2$. Then we write $\phi U_1 \leq \phi U_2$. It can be easily shown that $\{\phi_U\}$ is a directed set with the partial ordering '≤'.

**2.13 Remarks**

Let G be a lcH topological group. Let $\theta$ be a non-zero function in $\not{c}_{00}{}^+(G)$. then by proposition 6.2.6 we see that for every non-zero function $\psi \in \not{c}_{00}{}^+(G)$,

$$\frac{1}{(\theta:f)} \leq \eta_{\theta\psi}(f) \leq (f:\theta) \text{ if } f \neq 0,$$

$$\eta_{\theta\psi}(l_{fa}) = \eta_{\theta\psi}(f),$$

$$\eta_{\theta\psi}(\alpha f) = \alpha\eta_{\theta\psi}(f), \text{ for } \alpha \geq 0,$$

$$\eta_{\theta\psi}(f_1 + f_2) \leq \eta_{\theta\psi}(f_1) + \eta_{\theta\psi}(f_2)$$

For all a ∈ G and for all f, $f_1$ and $f_2 \in \not{c}_{00}{}^+(G)$, But by proposition 6.2.7, we see that $f_1$ and $f_2 \in \not{c}_{00}{}^+(G)$ then for every $\in > 0$,

$$\eta_{\theta\psi}(f_1) + \eta_{\theta\psi}(f_2) \leq \eta_{\theta\psi}(f_1 + f_2) + \in.$$

Now, let U be an open nhood of the identity e ∈ G. Then

$$0 \leq \eta_{\theta\phi_U}(f) \text{ if } f \neq 0, \quad \ldots(1)$$

$$\eta_{\theta\phi_U}(l_{fa}) = \eta_{\theta\phi_U}(f), \quad \ldots(2)$$

$$\eta_{\theta\phi_U}(\alpha f) = \eta_{\theta\phi_U}(f), \text{ for } \alpha \geq 0 \quad \ldots(3)$$

And

$$\eta_{\theta\phi_U}(f_1 + f_2) \leq \eta_{\theta\phi_U}(f_1) + \eta_{\theta\phi_U}(f_2) \leq \eta_{\theta\phi_U}(f_1 + f_2) + \in. \quad \ldots(4)$$

For all a ∈ G and for all f, $f_1$ and $f_2 \in \not{c}_{00}{}^+(G)$ where $f \neq 0$.

Taking limits and applying the previous proposition 2.13, we have the following:

**2.14 Proposition**

Let G be a lcH topological group. Then

(i) $\eta_\theta(f)$ is real and non-negative if $f \neq 0$,

(ii) $\eta_\theta(l_{fa}) = \eta_\theta(f)$,

(iii) $\eta_\theta(\alpha f) = \alpha\eta_\theta(f)$, for $\alpha \geq 0$,

(iv) $\eta_\theta(f_1 + f_2) = \eta_\theta(f_1) + \eta_\theta(f_2)$

For all $a \in G$ and for all f, $f_1$ and $f_2 \in \not\subset_{00}{}^+(G)$.

Thus it follows that we have constructed a function $\eta_\theta$ on $\not\subset_{00}{}^+(G)$ which is nontrivial, left-invariant, non-negative and positive homogeneous where G is lcH topological group.

**3 Haar integrals on $\not\subset_{00}{}^+(G)$ and its uniqueness**

**3.1 Definition**

A function $\eta_\theta$ corresponding to a given non-zero $\theta \in \not\subset_{00}{}^+(G)$ as constructed above which satisfies (i) to (iv) of previous proposition 2.14 is called a left Haar integral on $\not\subset_{00}{}^+(G)$.

A function $\eta_\theta$ satisfying (i), (iii) and (iv) and also having the property

$$\eta_\theta\,(r_{fa}) = \eta_\theta(f)$$

can be constructed on $\not\subset_{00}{}^+(G)$, where G is lcH topological group. this function will be called a right Haar integral on $\not\subset_{00}{}^+(G)$.

The following proposition shows that a left (right) Haar integral is unique up to a multiplicative constant, that is, if there are two left (right) Haar integrals then they differ only by a constant factor.

**3.2 Proposition**

Let G be a lcH topological group. Let $\theta$ be a non-zero function in $\not{c}_{00}{}^{+}(G)$. Let $\eta_\theta$ and $\xi_\theta$ be two left Haar integrals defined on $\not{c}_{00}{}^{+}(G)$. Then there exists a real $\alpha > 0$ such that

$$\eta_\theta(f) = \alpha\xi_\theta(f)$$

For all $f \in \not{c}_{00}{}^{+}(G)$. Similar results holds for right Haar integral.

**Proof**

Let $f \in \not{c}_{00}{}^{+}(G)$. Proposition 2.1 shows that for every non-zero function $\psi \in \not{c}_{00}{}^{+}(G)$, there exist some real $\alpha_j \geq 0$ and elements $x_j \in G$ such that

$$f \leq \sum_{j=1}^{n} \alpha_j l_{\psi x_j} \quad \text{...(1)}$$

Since $\xi_\theta$ is a left Haar integral on $\not{c}_{00}{}^{+}(G)$, hence

$$\xi_\theta(f) \leq \xi_\theta\left(\sum_{j=1}^{n} \alpha_j l_{\psi x_j}\right)$$

$$= \sum_{j=1}^{n} \xi_\theta(\alpha_j l_{\psi x_j}) = \sum_{j=1}^{n} \alpha_j \xi_\theta(l_{\psi x_j})$$

$$= \sum_{j=1}^{n} \alpha_j \xi_\theta(\psi)$$

$$= \left(\sum_{j=1}^{n} \alpha_j\right) \xi_\theta(\psi) \quad \text{...(2)}$$

When f = 0, the case is trivial. Hence we assume that $f \neq 0$. For non-zero f, $\sum_{j=1}^{n} \alpha_j > 0$. Hence from (2), we have

$$\xi_\theta(\psi) \leq \frac{\xi_\theta(f)}{\sum_{j=1}^{n} \alpha_j} > 0 \quad \text{...(3)}$$

For all non-zero $\psi \in \not{c}_{00}{}^{+}(G)$.

Let $\phi$ be a non-zero function in $\not{c}_{00}{}^{+}(G)$. Then again by proposition 6.2.1, there exist real $\beta_j \geq 0$ and $t_j \in G$ (j = 1, 2, ..., m) such that

$$f \leq \sum_{j=1}^{n} \beta_j l_{\phi t_j} \quad \text{...(4)}$$

$\therefore \quad \xi_\theta(\mathrm{f}) \le (\sum_{j=1}^{n} \beta_j)\xi_\theta(\phi)$

$\therefore \quad \sum_{j=1}^{n} \beta_j \ge \frac{\xi_\theta(\mathrm{f})}{\xi_\theta(\phi)}$ for all possible $\beta_j$ ...(5)

Taking infimum on (5) we have

$$(\mathrm{f} : \phi) \ge \frac{\xi_\theta(\mathrm{f})}{\xi_\theta(\phi)} \quad ...(6)$$

For all $\mathrm{f} \in \not\subset_{00}{}^{+}(G)$ and all non-zero $\phi \in \not\subset_{00}{}^{+}(G)$.

Let $\in > 0$ be given. Then by proposition 6.2.12, there exist non-zero functions $\mathrm{f}, \omega \in \not\subset_{00}{}^{+}(G)$, real $c_j \ge 0$ and $t_j \in G$ ($j = 1, 2, ..., p$) such that

$$|\,\psi - \sum_{j=1}^{p} c_j l_{\omega t_j}\,| \le \in f_0 \quad ...(7)$$

$$\therefore \quad \sum_{j=1}^{p} c_j . l_{\omega t_j} \le \psi + \in . f_0 \quad ...(8)$$

and

$$\psi \le \in . f_0 + \sum_{j=1}^{p} c_j l_{\omega t_j} \quad ...(9)$$

From (8), we have

$$\xi_\theta(\sum_{j=1}^{p} c_j l_{\omega t_j}) \le \xi_\theta(\psi + \in . f_0)$$

Or $\quad \sum_{j=1}^{p} c_j . \xi_\theta(l_{\omega t_j}) \le \xi_\theta(\psi) + \xi_\theta(\in . f_0)$

Or $\quad (\sum_{j=1}^{p} c_j)\xi_\theta(\psi) \le \xi_\theta(\psi) + \in . \xi_\theta(f_0)$ ...(10)

Again from (9), we have

$$\begin{aligned}(\psi : \phi) &\le (\in . f_0 + \sum_{j=1}^{n} c_j . l_{\omega t_j} : \phi)\\ &\le \in . (f_0 + \phi) + (\sum_{j=1}^{n} c_j . l_{\omega t_j} : \phi)\\ &\le \in . (f_0 + \phi) + \sum_{j=1}^{n}(c_j . l_{\omega t_j} : \phi)\\ &= \in . (f_0 + \phi) + \sum_{j=1}^{n} c_j(l_{\omega t_j} : \phi)\\ &= \in . (f_0 + \phi) + \sum_{j=1}^{n} c_j(\omega : \phi)\\ &= \in . (f_0 + \phi) + (\sum_{j=1}^{n} c_j)(\omega : \phi) \quad ...(11)\end{aligned}$$

For every non-zero $\phi \in \mathcal{C}_{00}^{+}(G)$.

Taking $\omega$ in place of $\phi$ in (6) and (11) we have

$$(f : \omega) \geq \frac{\xi_\theta (f)}{\xi_\theta (\omega)} \text{ for all } f \neq 0. \qquad ...(12)$$

And

$$(\psi : \omega) \leq (\in . (f_0 + \omega) + (\textstyle\sum_{j=1}^{n} c_j) \qquad ...(13)$$

From (10) and (13) , we have

$$\xi_\theta (\psi) + \in \xi_\theta (f_0) \geq (\textstyle\sum_{j=1}^{n} c_j)\, \xi_\theta (\omega) \geq [(\psi : \omega) - \in . (f_0 : \omega)]\, \xi_\theta (\omega)$$

$$= \left[1 - \in . \frac{(f_0 : \omega)}{(\psi : \omega)}\right](\psi : \omega)\, \xi_\theta (\omega)$$

$$\geq \left[1 - \in . \frac{(f_0 : \omega)(\psi : \omega)}{(\psi : \omega)}\right](\psi : \omega)\, \xi_\theta (\omega)$$

$$= [1 - \in . \ (f_0 : \omega)](\psi : \omega)\, \xi_\theta (\omega) \qquad ...(14)$$

As $\xi_\theta (f) > 0$ for $f \neq 0$, we have from (14),

$$\frac{\xi_\theta (\psi)}{\xi_\theta (f)} + \in \frac{\xi_\theta (f_0)}{\xi_\theta (f)} \geq [1 - \in . (f_0 : \psi)](\psi : \omega)\, \xi_\theta \frac{\xi_\theta (\omega)}{\xi_\theta (f)}$$

$$\geq [1 - \in . (f_0 : \psi)](\psi : \omega)\, \xi_\theta \frac{\xi_\theta (f)}{(f : \omega)\, \xi_\theta (f)} \quad \text{(using (12))}$$

$$= [1 - \in . (f_0 : \psi)] \frac{(\psi : \omega)}{(f : \omega)}$$

$$= [1 - \in . (f_0 : \psi)] \frac{\frac{(\psi : \omega)}{(\theta : \omega)}}{\frac{(f : \omega)}{(\theta : \omega)}}$$

$$= [1 - \in . (f_0 : \psi)] \frac{\eta_{\theta_\omega}(\psi)}{\eta_{\theta_\omega}(f)} \qquad ...(15)$$

Taking limit in (15) over $\omega \in \{ \phi_U \}$ we have (as $\varepsilon \to 0$)

$$\frac{\xi_\theta (\psi)}{\xi_\theta (f)} \geq \frac{\eta_\theta(\psi)}{\eta_\theta(f)} \qquad ...(16)$$

Or

$$\frac{\eta_\theta(\psi)}{\eta_\theta(f)} \geq \frac{\xi_\theta (\psi)}{\xi_\theta (f)} \qquad ...(17)$$

From (16) and (17), we have

$$\frac{\xi_\theta (\psi)}{\xi_\theta (f)} = \frac{\eta_\theta(\psi)}{\eta_\theta(f)}$$

Or $\eta_\theta(f) = \frac{\eta_\theta(\psi)}{\xi_\theta(\psi)} \xi_\theta(f) = \alpha \,.\, \xi_\theta(f)$

Where $\alpha = \frac{\eta_\theta(\psi)}{\xi_\theta(\psi)}$

This completes the proof.

## 4 Extension of a function

Let X be a non-empty set. Let $L_r(X)$ be a real linear space of real-valued functions on X such that $f \in L_r(X)$ implies $|f| \in L_r(X)$. Suppose that $L_r^+(X)$ denotes the of all non-negative function in $L_r(X)$. We have

### 4.1 Proposition

Let $\mu$ be a real-valued function on $L_r^+(X)$ such that

$$\mu(f + g) = \mu(f) + \mu(g)$$

and

$$\mu(\alpha f) = \alpha\mu(g)$$

for all $f, g \in L_r^+(X)$ and for all $\alpha \geq 0$. Then $\mu$ can be extended uniquely so as to be a linear functional on $L_r(X)$.

**Proof**

If $f \in L_r(X)$, then we define

$$f^+ = \frac{1}{2}(|f| + f) \text{ and } f^- = \frac{1}{2}(|f| - f).$$

Obviously, $f^+ \geq 0$, $f^- \geq 0$; $f^+, f^- \in L_r(X)$ and

$$f = f^+ + f^-$$

This shows that every function $f \in L_r(X)$ can be written as difference of two functions in $L_r^+(X)$. If $\phi = \phi_1 - \phi_2$ where $\phi \in L_r(X)$ and $\phi_1, \phi_2 \in L_r^+(X)$ then we define

$$\mu_r(\phi) = \mu(\phi_1) - \mu(\phi_2).$$

If $\phi = \phi_1 - \phi_2 = \psi_1 - \psi_2$ where $\phi_1, \phi_2, \psi_1, \psi_2 \in L_r^+(X)$ then $= \phi_1 + \psi_2 = \phi_2 + \psi_1$. Therefore,

$\mu(\phi_1 + \psi_2) = \mu(\phi_2 + \psi_1)$

Or $\mu(\phi_1) + \mu(\psi_2) - \mu(\phi_2) - \mu(\psi_1)$

Or $\mu(\phi_1) - \mu(\phi_2) = \mu(\psi_1) - \mu(\psi_2)$

Which shows that $\mu_r$ is defined uniquely on $L_r(X)$ and is an extension of $\mu$. It is not difficult to verify that $\mu_r$ is a linear functional on $L_r(X)$.

**4.2 Proposition**

Let X be a non-empty set. Let $L_k(X)$ be a complex linear space of complex – valued functions of X. Let $R_X$ and $R_X^+$ be the set of real valued and non-negative real valued functions in $L_k(X)$, respectively. If $f \in L_k(X)$ implies $\bar{f} \in L_k(X)$ and that every $f \in R_X$ can be written as difference of two functions on $R_X^+$. If $\eta$ be any complex – valued function defined on $R_X^+$ such that

$\eta(f + g) = \eta(f) + \eta(g)$

and

$\eta(\alpha f) + \alpha\eta(f)$

for all $f, g \in R_X^+$and for all $\alpha \geq 0$. Then $\eta$ can be extended uniquely so as to be a linear functional on $L_k(X)$.

**Proof**

By previous proposition, $\eta$ can be extended uniquely to a function $\eta_r$ on $R_X$. The function $\eta_r$ is additive as well as homogeneous. Let $f \in L_k(X)$. If we define

$f_1 = \frac{1}{2}(f + \bar{f})$ and $f_2 = \frac{1}{2i}(f - \bar{f})$

Then $f = f_1 + if_2$ and $f_1, f_2 \in R_X$. Let us define

$\eta_c(f) = \eta_r(f_1) + i\eta_r(f_2)$

It is obvious that

$\eta_c(f + g) = \eta_r(f_1) + \eta_r(g)$

and

$$\eta_c(\beta f) = \beta \eta_r(f)$$

For all f, g ∈ $L_k(X)$ and for all β ∈ K.

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