# DECOMPOSITIONS OF COMPLETE GRAPHS INTO ISOMORPHIC BIPARTITE GRAPHS

***S. Somasundaram, A. Nagarajan & G. Mahadevan***

***ABSTRACT:*** *In this paper we deal with the decomposition of a complete graph into isomorphic bipartite graphs using* α*- valuation.*

## 1. INTRODUCTION

In this paper, we consider only simple graphs. Our notation and terminology are as in [1]. Rosa introduced in [3] (see also [1]) the β- and β-valuations of a graph $G$ as follows: Let $|E(G)| = \in$ and let $f$ be a one to one mapping of $V$(G) into $\{0, 1, 2, ..., \in\}$. Then $f$ is called a β-valuation of $G$ if $\{|f(u) - f(v)|: uv \in E(G)\} = \{1, 2, 3, ..., \in\}$. A β-valuation $f$ is called an α-valuation if there exists a non-negative number λ such that for every $uv \in E(G)$ with $f(u) < f(v), f(u) \leq 1 < f(v)$. If a graph $G$ admits an α-valuation, then $(X, Y)$, where $X = \{u: f(u) \leq \lambda\}$ and $Y = \{v : \lambda < f(v)\}$, forms a bipartition of $G$. For any positive integer $n$, $Q_n(G) = Gx\underbrace{K_2 x ... x K_2}_{n-1 \text{ times}}$ is the $n$-dimensional $G$-cube. Note that the graph $Q_n(K_2)$ is the ordinary $n$-dimensional cube. $Q_n(G)$ has $n2^{n-1}$ vertices and $[2 \in + (n - 1)\nu]2^{n-2}$ edges where ν and ∈ denote number of vertices and edges of $G$ respectively.

In [1] R. Balakrishnan and R. Sampathkumar show that the graphs $Q_n(K_{3,3})$, $Q_n(K_{4,4}]$ and $Q_n(P_k)$ admit α-valuations. In this paper we prove that $Q_n(K_{2,3})$ admits an α-valuation.

## 2. SOME RESULTS

**Theorem 2.1:** [1] For every positive integer $n$ there exists an α-valuation of $Q_n(K_2)$.

**Theorem 2.2:** [1] For every positive integer $n$ there exists an α-valuation of $Q_n(K_{3,3})$.

**Theorem 2.3:** [1] For every positive integer $n$ there exists an α-valuation of $Q_n(K_{4,4})$.

**Theorem 2.4:** [1] For every positive integer $n$ there exists an α-valuation of $Q_n(P_k)$.

In [3] Rosa has proved the following Theorem.

**Theorem 2.5:** If a graph $G$ with ∈ edges has an α-valuation, then for any positive integer $c$ there exists a cyclic decomposition of the edges of the complete graph $K_{2c \in +1}$ into subgraphs isomorphic to $G$.

## 3. MAIN RESULTS

R. Balakrishnan and R. Sampathkumar suggested the following open problem in [1].

**Problem:** Does there exist an α-valuation for $Q_n(K_{r,r})$, $r \geq 5$ and $n \geq 2$.

In this connection we propose the conjecture, "There is no α-valuation for $Q_2(K_{5,5})$ and $Q_2(K_{6,6})$". The following computer program in Pascal gives a result which is the basis of our conjecture 1.

```
PROGRAM K55 (INPUT, OUTPUT);
LABEL 80;
VAR
    I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I,J,K : INTEGER;
    A: ARRAY [1..25] OF INTEGER;
    BEGIN
            I1:=0;
             I10:=25;
    FOR I2:= 1 TO 24 DO
    FOR I3:=I2+1 TO 24 DO
    FOR I4:=I3+1 TO 24 DO
    FOR I5:=I4+1 TO 24 DO
    BEGIN
            A[1]:=I10-I1;
            A[2]:= I10-I2;
            A[3]:= I10-I3;
            A[4]:= I10-I4;
            A[5]:= I10-I5;
    FOR I6:=I5+1 TO 24 DO
    BEGIN
            A[6]:=I6-I1;
            A[7]:=I6-I2;
            A[8]:=I6-I3;
```

```
        A[9]:=I6-I4;
        A[10]:=I6-I5;
FOR I7:=I6+1 TO 24 DO
BEGIN
        A[11]:=I7-I1;
        A[12]:=I7-I2;
        A[13]:=I7-I3;
        A[14]:=I7-I4;
        A[15]:=I7-I5;
FOR I8:=I7+1 TO 24 DO
BEGIN
        A[16]:=I8-I1;
        A[17]:=I8-I2;
        A[18]:=I8-I3;
        A[19]:=I8-I4;
        A[20]:=I8-I5;
FOR I9:=I8+1 TO 24 DO
BEGIN
        A[21]:=I9-I1;
        A[22]:=I9-I2;
        A[23]:=I9-I3;
        A[24]:=I9-I4;
        A[25]:=I9-I5;
FOR I:=1 TO 24 DO
FOR J:=I+1 TO 25 DO
IF ABS(A[I])=ABS(A[J]) THEN GOTO 80;
WRITELN(LST, I1:4, I2:4, I3:4, I4:4,I5:4);
WRITELN(LST, I6:4, I7:4, I8:4, I9:4,I10:4);
```

```
WRITELN;
WRITELN;
80: ;
END;
END;
END;
END;
END;
END.
```

The output of the above program is

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 5 | 10 | 15 | 20 | 25 |
| 0 | 5 | 10 | 15 | 20 |
| 21 | 22 | 23 | 24 | 25 |

From the output we have the following result :

The only possible α-valuations of $K_{5,5}$ are of the following form: The vertex labels have a difference of five in one set of the bipartition and a difference of one in the other.

We believe the following is true.

**Conjecture 1:** There is no a-valuation for $Q_2(K_{5,5})$.

However we present below a spanning subgraph of $K_{5,5}$ for which the situation is entirely different. Consider a bipartite graph $G$ which is a subgraph of $K_{5,5}$.

**Theorem 3.1:** For every positive integer $n$, there exists an $a$-valuation of $Q_n(G)$ where $G$ is given above in Figure 1.

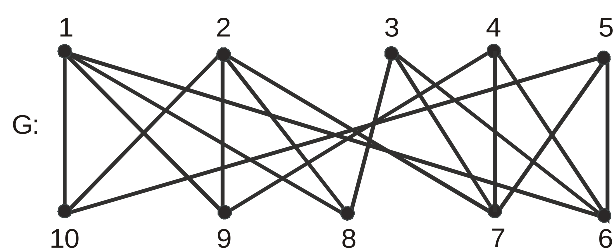


**Figure 1.**

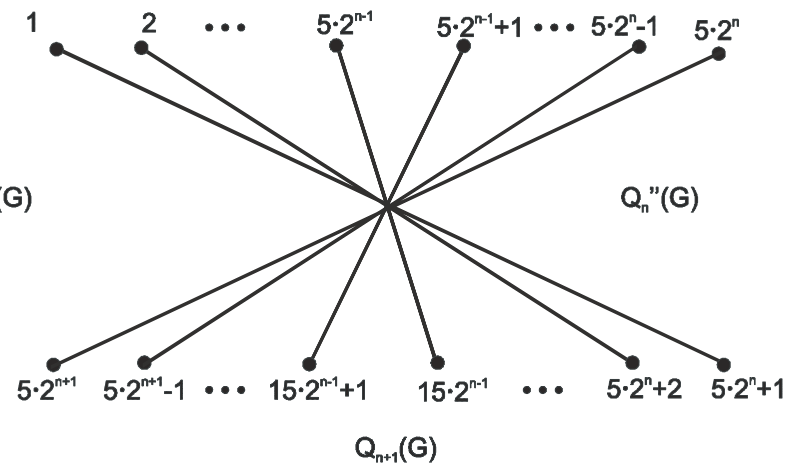


**Figure 2**

**Proof:** The proof will be by induction on $n$. The vertices of $Q_n(G)$ are labelled as 1, 2, 3, ..., $5.2^n$. The value assigned to a vertex $i$ (respectively an edge $e$) of $Q_n(G)$ in the $\alpha$-valuation which is to be constructed will be denoted by $f_n(i)$ (respectively $\overline{f}_n(e)$). $Q_n(G)$ is a bipartite graph with bipartition $(X, Y)$ where $X = \{1, 2, 3, ...., 5.2^{n-1}\}$ and $Y = \{5.2^{n-1} + 1, ..., 5.2^n\}$. We will choose the labelling of the vertices in $Q_{n+1}(G)$ corresponding to the centrally symmetric scheme shown in Fig 2.

In Figure 2, only the edges of $Q_{n+1}(G)$ that link the two isomorphic copies $Q_n'(G)$, $Q_n''(G)$ of $Q_n(G)$ are indicated, edges within $Q_n'(G)$ and $Q_n''(G)$ are omitted. We show that for every positive integer $n$, an $\alpha$-valuation $f_n$ of $Q_n(G)$ can be constructed with the following property:

$\{|f_n(u) - f_n(v)|: uv \in E(Q_n(G))\} = \{1, 2, 3, \ldots, (5n + 12).2^{n-1}\}$ and $0 = f_n(1) < f_n(2) < \ldots. < f_n(5.2^{n-1}) = 1 < f_n(5.2^{n-1} + i)$, $1 \leq i \leq 5.2^{n-1}$ … (1). Condition (1) is trivially satisfied for $Q_1(G) = G$ upon putting $f_1(1) = 0, f_1(2) = 3, f_1(3) = 5, f_1(4) = 7, f_1(5) = 9, f_1(6) = 11, f_1(7) = 10, f_1(8) = 15, f_1(9) = 16, f_1(10) = 17$. Assume that $f_n$ has already been constructed. We construct $f_{n+1}$ as follows :

For $1 \leq i \leq 5.2^{n-1}$, put

$$\begin{aligned} f_{n+1}(i) &= f_n(i) \\ f_{n+1}(5.2^{n-1} + i) &= f_n(i) + k_n \\ f_{n+1}(5.2^n + i) &= f_n(5.2^{n-1} + i) + k_n \\ f_{n+1}(15.2^{n-1} + i) &= f_n(5.2^{n-1} + i) + k_{n+1} \end{aligned}$$

where $k_1 = 11$ and for $n \geq 1$, $k_{n+1} = |E(Q_{n+1}(G)| - |E(Q_n(G))| = (22 + 5n)2^{n-1}$

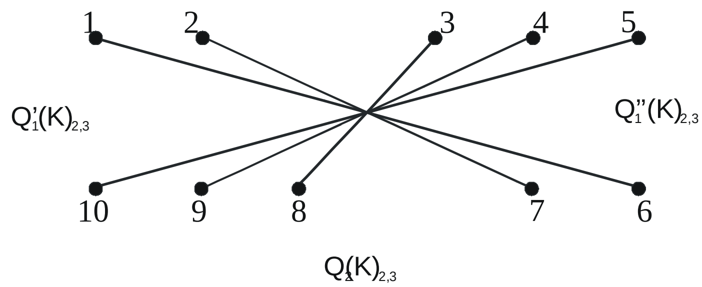


**Figure 3**

Since the values of $\overline{f_n}$ form the integer interval $[1, (5n + 12)2^{n-1}]$, the values of $\overline{f_{n+1}}$ corresponding to the edges in $Q_n'(G)$ form the integer interval $[(22 + 5n)2^{n-1} + 1, (22 + 5n)2^{n-1} + (5n + 12)2^{n-1}] = [(22 + 5n)2^{n-1} + 1, (10n + 34)2^{n-1}] = [(22 + 5n)2^{n-1} + 1, (5n + 17)2^n]$ and the values of $\overline{f_{n+1}}$ corresponding to the edges in $Q_n''(G)$ form the integer interval $[1, (5n + 12)2^{n-1}]$. By induction, one can verify that the values of $\overline{f_{n+1}}$ at the edges indicated in Figure 2 will form the integer interval $[(5n + 12)2^{n-1} + 1, (22 + 5n)2^{n-1}]$. From (1) and from the definition of $f_{n+1}$ it is clear that $f_{n+1}(1) < f_{n+1}(2) < \ldots < f_{n+1}(5.2^n) = \lambda_1 < f_{n+1}(5.2^n + i)$, $1 \leq i \leq 5.2^n$. Hence we have $f_{n+1}(u) \leq \lambda_1 < f_{n+1}(v)$ for every edge $uv$ in $E(Q_{n+1}(G))$, as $Q_{n+1}(G)$ is a bipartite graph with bipartition $(X, Y)$ where $X = \{1, 2, 3, \ldots, 5.2^n\}$ and $Y = \{5.2^n + 1, 5.2^n + 2, \ldots, 5.2^{n+1}\}$. This completes the proof of 3.1.

**Corollary 3.2:** For every positive integer $n$, there exists an $\alpha$-valuation of $Q_n(K_{2,3})$.

**Proof:** If $n = 1$, then the labelling $f_1$ is given for $Q_1(K_{2,3})$ as follows: $f_1(1) = 0, f_1(2) = 3, f_1(3) = 4, f_1(4) = 5, f_1(5) = 6$. The labelling of the vertices of $Q_2(G)$ corresponding to the centrally symmetric scheme is shown in Figure 3. The labelling $f_2$ is given as below: $f_2(1) = 0, f_2(2) = 3, f_2(3) = 5, f_2(4) = 7, f_2(5) = 9, f_2(6) = 11, f_2(7) = 10, f_2(8) = 15, f_2(9) = 16, f_2(10) = 17$. This is the labelling for $G$ in 3.1. Hence by 3.1, for every positive integer $n$ there exists an $\alpha$-valuation for $Q_n(K_{2,3})$.

A graph $G$ is said to be $H$-decomposable if $G$ is the edge-disjoint union of subgraphs of $G$ each of which is isomorphic to $H$. This is denoted by $H \mid G$.

2.5 and 3.2 combine to give the following corollary.

**Corollary 3.3:** Let $n$ and $c$ be integers $\geq 1$. Then $Q_n(K_{2,3})|K_m$, where $m = c(5n + 7)2^{n-1} + 1$.

The following computer program in Pascal gives a result which is the basis of our conjecture 2.

```
PROGRAM K66 (INPUT, OUTPUT);
```

```
LABEL 80;
VAR
    I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11, I12, I,J,K : INTEGER;
    A: ARRAY [1..36] OF INTEGER;
    BEGIN
            I1:=0;
            I12:=36;
    FOR I2:= 1 TO 35 DO
    FOR I3:=I2+1 TO 35 DO
    FOR I4:=I3+1 TO 35 DO
    FOR I5:=I4+1 TO 35 DO
    FOR I6:=I5+1 TO 35 DO
    BEGIN
            A[1]:=I12-I1;
            A[2]:= I12-I2;
            A[3]:= I12-I3;
            A[4]:= I12-I4;
            A[5]:= I12-I5;
            A[6]:= I12-I6;
    FOR I7:=I6+1 TO 35 DO
    BEGIN
            A[7]:=I7-I1;
            A[8]:=I7-I2;
            A[9]:=I7-I3;
            A[10]:=I7-I4;
            A[11]:=I7-I5;
            A[12]:=I7-I6;
    FOR I8:=I7+1 TO 35 DO
    BEGIN
            A[13]:=I8-I1;
            A[14]:=I8-I2;
```

```
        A[15]:=I8-I3;
        A[16]:=I8-I4;
        A[17]:=I8-I5;
        A[18]:=I8-I6;
FOR I9:=I8+1 TO 35 DO
BEGIN
        A[19]:=I9-I1;
        A[20]:=I9-I2;
        A[21]:=I9-I3;
        A[22]:=I9-I4;
        A[23]:=I9-I5;
        A[24]:=I9-I6;
FOR I10:=I9+1 TO 35 DO
BEGIN
        A[25]:=I10-I1;
        A[26]:=I10-I2;
        A[27]:=I10-I3;
        A[28]:=I10-I4;
        A[29]:=I10-I5;
        A[30]:=I10-I6;
FOR I11:=I10+1 TO 35 DO
BEGIN
        A[31]:=I11-I1;
        A[32]:=I11-I2;
        A[33]:=I11-I3;
        A[34]:=I11-I4;
        A[35]:=I11-I5;
        A[36]:=I11-I6;
FOR I:=1 TO 35 DO
FOR J:=I+1 TO 36 DO
IF ABS(A[I])=ABS(A[J]) THEN GOTO 80;
```

```
WRITELN(LST, I1:4, I2:4, I3:4, I4:4,I5:4, I6:4);
WRITELN(LST, I7:4, I8:4, I9:4, I10:4,I11:4, I12:4);
80: ;
END;
END;
END;
END;
END;
END;
END.
```

The output of the above program shows the only possible α-valuations of $K_{6,6}$ are of the following form:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 6 | 12 | 18 | 24 | 30 | 36 |
| 0 | 1 | 2 | 6 | 7 | 8 |
| 9 | 12 | 21 | 24 | 33 | 36 |
| 0 | 1 | 2 | 9 | 10 | 11 |
| 12 | 15 | 18 | 30 | 33 | 36 |
| 0 | 1 | 2 | 18 | 19 | 20 |
| 21 | 24 | 27 | 30 | 33 | 36 |
| 0 | 1 | 4 | 5 | 8 | 9 |
| 10 | 12 | 22 | 24 | 34 | 36 |
| 0 | 1 | 6 | 7 | 12 | 13 |
| 14 | 16 | 18 | 32 | 34 | 36 |
| 0 | 1 | 12 | 13 | 24 | 25 |
| 26 | 28 | 30 | 32 | 34 | 36 |
| 0 | 2 | 4 | 6 | 8 | 10 |
| 11 | 12 | 23 | 24 | 35 | 36 |
| 0 | 2 | 4 | 18 | 20 | 22 |
| 23 | 24 | 29 | 30 | 35 | 36 |
| 0 | 2 | 12 | 14 | 24 | 26 |
| 27 | 28 | 31 | 32 | 35 | 36 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 3 | 6 | 9 | 12 | 15 |
| 16 | 17 | 18 | 34 | 35 | 36 |
| 0 | 3 | 6 | 18 | 21 | 24 |
| 25 | 26 | 27 | 34 | 35 | 36 |
| 0 | 3 | 12 | 15 | 24 | 27 |
| 28 | 29 | 30 | 34 | 35 | 36 |
| 0 | 6 | 12 | 18 | 24 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 |

As before in this case also we believe the following is true:

**Conjecture 2:** There is no $\alpha$-valuation for $Q_2(K_{6,6})$.

However the situation is different for the subgraph $Q_2(K_{3,3})$ of $K_{6,6}$.

**Theorem 3.4:** Let $G = Q_2(K_{3,3})$. There exists an $\alpha$-valuation of $Q_n(G)$ for any positive integer $n$.

**Proof:** The Proof follows from 2.2.

## REFERENCES


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**S. Somasundaram**
Department of Mathematics,
Manonmaniam Sundaranar University,
Tirunelveli-627 012.

**A. Nagarajan**
Department of Mathematics,
V.O. Chidambaram College,
Tuticorin-628 008.

**G. Mahadevan**
Department of Mathematics,
Gandhigram Rural University,
Gandhigram-624 302.