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AN INTRODUCTION TO GRAPH EXTENSION

G. Suresh Singh* and Sunitha Grace Zacharia*

Abstract

In abstract algebra, field extensions are one of the main objects of study in field theory [5]. The general idea is to start with a base field and construct in some manner a larger field that contains the base field and satisfies additional properties. In this paper we study the extension of a graph G by adding edges in a particular manner. Extension in trees, cycles and neighbourhood graphs are also considered. Some graphs can be extended up to a complete graph. Further we characterise completely extendable graphs.

Keywords: *Extendable Graph, Completely Extendable Graph, Order of Extension.*

1. INTRODUCTION

Let $G = (p, q)$ be a finite, non trivial, undirected simple graph which is not complete. Extension on G is defined as adding edges to non adjacent pair of vertices in such a way that in the first extension add one edge to G and denoted as G^1, in the second extension add two edges in G^1, denoted by G^2, $G^2 = G \cup \{e_1, e_2, e_3\}$ and so on until no such an extension remains. Certain graphs become complete after a finite number of extensions. Then, the given graph is said to be completely extendable. For basic definitions and results in graph theory we follow [2] and [3].

Definition 1: Let G be a simple (p, q) graph. Extension on G is defined as follows; in the first extension, add one edge to G, denoted as G^1, $G^1 = G \cup \{e_1\}$. In the second extension add two edges on G^1 denoted by G^2, $G^2 = G \cup \{e_1, e_2, e_3\}$ and so on until no such an extension remains.

Theorem 2: Let G be a (p, q) graph and let $G^k = G \cup \{e_1, e_2, ..., e_m\}$. If G^k is the k^{th} extension of G, then $m = \dfrac{k(k+1)}{2}$.

* Department of Mathematics, University of Kerala, Kariavattom. Thiruvananthapuram, Kerala, India

** Corresponding Author, Department of Mathematics, University of Kerala, Kariavattom, Thiruvananthapuram, Kerala, India. *Email: sunithajosie@gmail.com*

Proof: We shall prove this theorem by induction on k, where G^k is the k^{th} extension of G. For $k = 1$, $G^1 = G \cup \{e_1\}$, $m = 1 \times 2/2$

Assume that result is true for $k = n$. We want to prove that it is true for $k = n + 1$.

For $k = n + 1$,

$$G^{n+1} = G \cup \left\{ e_1, e_2, \ldots, e_{\frac{n(n+1)}{2}}, e_{\frac{n(n+1)}{2}+1}, \ldots, e_{\frac{n(n+1)}{2}+(n+1)} \right\}$$

That is $G^{n+1} = G \cup \{e_1, e_2, ..., e_m\}$ where $m = \dfrac{(n+1)(n+2)}{2}$ which implies theorem is true for $k = n + 1$.

Hence if $G^k = G \cup \{e_1, e_2, ..., e_m\}$ is the k^{th} extension of G then $m = \dfrac{k(k+1)}{2}$.

Definition 3: If $G^k \cong K_p$, then G is said to be a completely extendable graph and k is known as the order of extension.

Example:

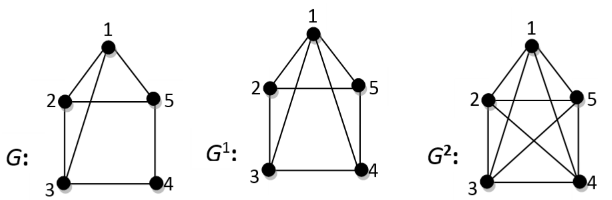


Figure 1

In Figure 1, G is a simple graph with 5 vertices and 7 edges. Number of edges added to G to get K_5 is 3.That is $m = 2 \times 3/2$. In the first extension G^1 add the edge (1, 4).That is

$$G^1 = G \cup \{(1, 4)\}.$$

To get G^2 add 2 edges in G^1, (2, 5) (3, 4). That is $G^2 = G \cup \{(1, 4), (2, 5), (3,4)\}$. Then G became a complete graph after the second extension. Therefore G is a completely extendable graph and order of extension is 2.

Theorem 4: Every tree is completely extendable.

Proof: Let T be a tree with p vertices ($p \geq 3$). If T is completely extendable then there exist a k such that $T^k \cong K_p$. Since T is a tree with p vertices $|E(T)| = p - 1$. But

$$|E(K_p)| = \frac{p(p-1)}{2}.$$

Therefore number of edges added to T to get

$$K_p \text{ is} = \frac{p(p-1)}{2} - (p-1) = \frac{p^2-3p+2}{2}. \qquad (1)$$

When $p = 3$ (1) becomes $\frac{9-9+2}{2} = 1$. Then it is clear that for $p = 3$ there exist $k = 1$ such that $T^1 \cong K_3$

When $p = 4$ (1) becomes $\frac{16-12+2}{2} = 3$. Then it is clear that for $p = 4$ there exist a $k = 2$ such that $T^2 \cong K_4$.

In general for $p = r$, $r \geq 3$ (1) becomes $\frac{r^2-3r+2}{2} = \frac{(r-2)(r-1)}{2}$.

That is number of edges added is equal to $m = 1+2+3+..., (r-2) = \frac{(r-2)(r-1)}{2}$

For every $p = r$ there exist a $k = r - 2$ such that $T^{r-2} \cong K_r$.

Therefore every tree is completely extendable and order of extension is $(p-2)$.

Theorem 5: Cycle C_p is not completely extendable.

Proof: Let G be a cycle with p vertices ($p \geq 4$). If G is completely extendable then there exist a k such that $G^k \cong K_p$. Since G is a cycle with p vertices $|E(G)| = p$. We know that $|E(K_p)| = \frac{p(p-1)}{2}$.

$$\text{Number of edges added to } G \text{ to get } K_p \text{ is} = \frac{p(p-1)}{2} - p = \frac{p^2-3p}{2} \qquad (1)$$

For $p = 4$ (1) becomes $\frac{16-12}{2} = 2$.We cannot find a k such that $G^k \cong K_p$.

Therefore C_p is not completely extendable.

Definition 6: Chord is an edge joining two non adjacent vertices in a cycle.

Theorem 7: Any Cycle C_p with $p - 3$ chords is completely extendable.

Proof: Let C_p be a cycle with p vertices ($p \geq 4$). Consider a graph G, $G = C_p + (p-3)$ chords. If G is completely extendable then there exist a k such that $G^k \cong K_p$. Since G is a cycle with p vertices and $(p-3)$ chords then $|E(G)| = p + p + -3 = 2p - 3$.

But $|E(K_p)| = p(p-1)/2$.

Number of edges added to G to get K_p is $= p(p-1)/2 - (2p-3)$

$$= \frac{p^2 - 5p + 6}{2} \quad (1)$$

For any value of p in (1), for example $p = 4$ (1) becomes (16-20 + 6)/2 =1. Then there exist a k (equal to1) such that $G^1 \cong K_p$.

For $p = 5$, (1) becomes (25-25 + 6)/2 = 3. Then there exist a k (equal to 2) such that $G^2 \cong K_p$.

In general for $p = r$, $r \geq 4$, (1) become $s(r - 3)(r - 2)/2$. There exist a $k = r - 3$ such that $G^{r-3} \cong K_p$ where $m = (r - 3)(r - 2)/2$. That is for every p there exist a k such that $G^k \cong K_p$.

Hence a cycle with p vertices and $(p - 3)$ chords is completely extendable.

Remark: Order of extension of any cycle with p vertices and $p - 3$ chords is $(p - 3)$.

Theorem 8: Let G be a (p, q) graph. If $q = pk - r$ where $r = k(k + 1)/2$ and $k < p$, then G is completely extendable.

Proof: We know that number of edges in K_p is $|E(K_p)| = p(p - 1)/2$

Number of edges in the given graph G is $|E(G)| = pk - k(k + 1)/2$.

Number of edges added to G to get K_p is $p(p - 1)/2 - pk + k(k + 1)/2$

$$= \frac{p^2 - (2k+1)P + \left(k^2 + k\right)}{2}. \quad (1)$$

For any value for p and k in (1), there exist a k such that $G^k \cong K_p$. Therefore G is completely extendable if $q = pk - r$, where $r = k(k + 1)/2$. That is $q = p - 1$, $2p - 3$, $3p - 6$, $4p - 10$, etc.

Note: For $k = 1$, $q = p - 1$ then $m = \frac{(p-2)(p-1)}{2}$. Order of extension is $p - 2$.

For $k = 2$, $q = 2p - 3$ then $m = (p - 3)(p - 2)/2$. Order of extension is p - 3.

For $k = 3$, $q = 3p$-6 then $m = (p - 4)(p - 3)/2$. Order of extension is p - 4 and so on.

In general order of extension of a graph G with $q = pk - r$ is $p - (k + 1)$, where $p > k$.

Theorem 9: If G is completely extendable graph, then size of G is $pk - r$, where k is any positive integer, $(k < p)$ and $r = k(k + 1)/2$.

Proof: Let G be a completely extendable graph and n be the order of extension of G. Then $G^n \cong K_p$. Number of edges added to G to get K_p is $n(n + 1)/2$.

That is $q + n(n + 1)/2 = p(p - 1)/2$.

$$q = [p(p - 1)/2 - [n(n + 1)/2] \tag{1}$$

For any value of p and n in (1), we get $q = pk - r$.

For example $p = 6$, $n = 3$ in (1) $q = 9$ and $9 = 2 \times 6 - 2 \times 3/2$, where $k = 2$, $p = 8$, $n = 5$ in (1) $q = 13$ and $13 = 2 \times 8 - 2 \times 3/2$, where $k = 2$.

That is for any completely extendable graph G, $q = pk - r$, where $k = 1, 2, ...$ and $r = k(k + 1)/2$.

Remark: From theorem 8 and theorem 9 we can give a characterization for completely extendable graphs as follows.

Theorem 10: Let G be a completely extendable graph if and only if size of G is $pk - r$, where k is any positive integer ($k < p$) and $r = k(k + 1)/2$.

Definition 11: (Neighbourhood Graphs) [4]. Let G be a connected graph . For any $v \in V(G)$, open neighbourhood of v is defined as the set of all vertices of G which are adjacent to v and is denoted by $N(v)$.

Construction: Let G be a connected graph with p vertices $v_1, v_2, ..., v_p$ and let $S_i = N(v_i)$, $i = 1, 2, ..., p$. Then the neighbourhood graph $N(G)$ of G is a graph with vertices are $S_1, S_2, ..., S_p$ such that two vertices S_i and S_j, $i \neq j$ are adjacent, if $S_i \cap S_j \neq \phi$.

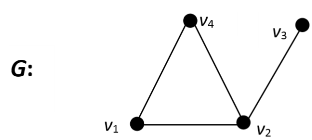
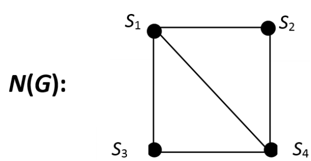


Figure 2

$S_1 = N(v_1) = \{v_2, v_4\}$ $S_2 = N(v_2) = \{v_1, v_3, v_4\}$ $S_3 = N(v_3) = \{v_2\}$ $S_4 = N(v_4) = \{v_1, v_2\}$



Theorem 12: $N(P_n)$ is not completely extendable.

Proof: Let P_n be a path with n vertices.

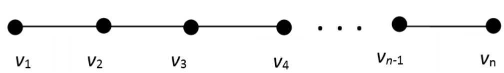


Figure 3

v_2 is common neighbourhood of v_1 and v_3. That is S_1 and S_3 are adjacent in $N(P_n)$, add one edge to $N(P_n)$. v_3 is common neighbourhood of v_2 and v_4. Therefore S_2 and S_4 are adjacent in $N(P_n)$. Again add one edge to $N(P_n)$. Continuing like this v_{n-1} is common neighbourhood of v_{n-2} and v_n. Therefore S_{n-2} and S_n are adjacent in $N(P_n)$, add one edge in $N(P_n)$. Total number of edges in $N(P_n) = n - 2$.

By Theorem 8, a graph G with $n - 2$ edges is not completely extendable.

Theorem 13: Neighbourhood graph of tree T with three pendant vertices is completely extendable.

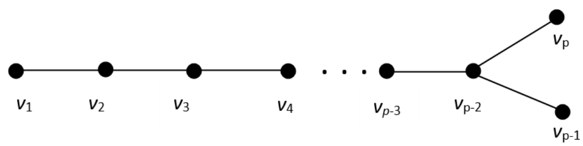


Figure 4

Proof: Let T be a tree with three pendant vertices. v_2 is the common neighbour of v_1 and v_3. Therefore $S_1$and S_3 are adjacent in $N(T)$. v_3 is the common neighbour of v_2 and v_4. Therefore S_2 and S_4 are adjacent in $N(T)$. Continuing like this v_{p-2} is common neighbour of v_{p-3}, v_{p-1} and v_p. Therefore S_{p-3} and S_{p-1} are adjacent, S_{p-3} and S_p are adjacent, S_{p-1} and S_p are adjacent in $N(T)$. Total number of edges in $N(T) = p - 1$. By Theorem 8, G with $p - 1$ edges is completely extendable.

Therefore $N(T)$ is completely extendable.

Theorem 14: Let G be a (p, q) graph. If G has $(p - 2)$ triangles contains all the edges then G is completely extendable.

Proof: Let G be a graph with p vertices, q edges and $(p - 2)$ triangles. We have to show that G is completely extendable. For, it is enough to show that G has $pk - r$ edges (by theorem 8). This can be proved by the method of mathematical induction on number of vertices p $(p \geq 3)$.

For $p = 3$, $t = 1$, $q = 3 = 2 \times 3 - 3$, where $k = 2$, $r = 2 \times 3/2$

Assume that result is true for $p = k$ and $t = k - 2$.

We want to show that result is true for $p = k + 1$ and $t = k - 1$. Let G be a graph with $k + 1$ vertices and $k - 1$ triangles. Remove one vertex of degree 2 from G, which causes removal of two edges. Then the resulting graph say G^* has k vertices and $k - 2$ triangles. Then by assumption

$$|E(G^*)| = 2k - 3.$$

But we have $E(G^*) = E(G) - 2$

Thus, $2k - 3 = E(G) - 2$

$$E(G) = 2k - 1 = 2(k + 1) - 3.$$

Thus the result holds for $p = k + 1$.

Thus for any G with p vertices and $p - 2$ triangles, number of edges $= pk - r$.

Remark: For any graph G with p vertices and $p - 2$ triangles, number of edges is $2p - 3$.

Theorem 15: Let G be a (p, q) graph. If $p = q$ then, G is not completely extendable.

Proof: By Theorem 8, G with $q = pk - r$ where $r = k(k + 1)/2$ is completely extendable. But in this graph $k = 1$ and $r = 0$. Therefore G is not completely extendable.

Notation 1: $\sum_i d_{G^k}(v_i)$ denotes sum of the degree of vertices of G^k where G^k is the k^{th} extension of G.

2. $\left[\sum_i d_{G^k}(v_i) : \sum_i d_G(v_i)\right]$ denotes the difference between sum of the degree of vertices of G^k and sum of the degree vertices of G.

Theorem 16: Let G be a (p ,q) graph. G^1, G^2, ..., G^k are the extension graphs of G. Then

$$\left[\sum_i d_{G^k}(v_i) : \sum_i d_G(v_i)\right] = \left[\sum_i d_{G^k}(v_i) : \sum_i d_{G^{k-1}}(v_i)\right]$$
$$+ \left[\sum_i d_{G^{k-1}}(v_i) : \sum_i d_{G^{k-2}}(v_i)\right] + \ldots \left[\sum_i d_{G^1}(v_i) : \sum_i d_G(v_i)\right].$$

Proof: Let G be a graph with p vertices and q edges. G^1, G^2, ..., G^k be the extensions of G. Number of edges added to G to get G^k is $k(k + 1)/2$. By fundamental theorem of graphs, sum of degree of vertices is twice the number of edges. Sum of the degree of vertices of $G^k = 2[q + k(k + 1)/2]$. We want to show that

$$\text{LHS} = \text{RHS}$$

$$\left[\sum_i d_{G^k}(v_i) : \sum_i d_G(v_i)\right] = 2[q + k(k + 1)/2] - 2q = k(k + 1) \tag{1}$$

$$\left[\sum_i d_{G^k}(v_i) : \sum_i d_{G^{k-1}}(v_i)\right] = 2[q + k(k + 1)/2] - 2[q + (k - 1)k/2] = 2k$$

$$\left[\sum_i d_{G^{k-1}}(v_i):\sum_i d_{G^{k-2}}(v_i)\right] = 2[q + k(k-1)k/2] - 2[q + (k-2)(k-1)/2]$$

$$= 2(k-1)$$

$$\left[\sum_i d_{G^2}(v_i):\sum_i d_{G^1}(v_i)\right] = 2[q+3] - 2[q+1] = 4 = 2\times 2$$

$$\left[\sum_i d_{G^1}(v_i):\sum_i d_G(v_i)\right] = 2[q+1] - 2\,q = 2 = 2\times 1$$

By adding the above equations, we get

$$\left[\sum_i d_{G^k}(v_i):\sum_i d_{G^{k-1}}(v_i)\right] + \left[\sum_i d_{G^{k-1}}(v_i):\sum_i d_{G^{k-2}}(v_i)\right] + \ldots \left[\sum_i d_{G^1}(v_i):\sum_i d_G(v_i)\right].$$

$$= 2[1 + 2 + \ldots\ldots + (k\text{-}1) + k]$$

$$= 2[k(k+1)/2] = k\,(k+1) \qquad (2)$$

Equating (1) and (2) LHS = RHS.

Theorem 17: Let $A(G)$ be the $p\times p$ adjacency matrix of the graph G with p vertices. If total number of zeros in $A\,(G) = r\,(r+1) + p$, then G is completely extendable and order of extension is r, where r is any positive integer.

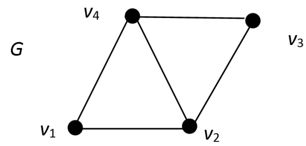
Proof: In $A(G)$ if v_{ij} is zero, then v_{ji} is also zero. If $A(G)$ has $r\,(r+1)+p$ zeros implies it has $[2\,r\,(r+1)/2]+p$ zeros. That is G has $r(r+1)/2$ non adjacent vertices. If number of edges added to G to get K_p is $r(r+1)/2$, then G is completely extendable. v_{ii} are zero for all i in $A(G)$. Therefore if G is completely extendable then total number of zeros in $A(G)$ is $r\,(r+1)+p$.

Definition 18: (*Laplace Matrix of a graph*) [1]. The Laplace Matrix of G is the matrix L indexed by the vertex set of G ,with zero row sums, where $L_{xy} = -A_{xy}$ for $x \neq y$. If D is the diagonal matrix, indexed by the vertex set of G such that D_{xx} is the degree of x, then

$L = D - A$. The elements of Laplace matrix are given by

$$\mathrm{L}_{ij} = \begin{cases} \deg v_i & \text{if } i=j \\ -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \end{cases}$$

Example: Consider the following graph *G,*



The Laplace Matrix of *G* is,

$$L(G) = \begin{pmatrix} \cdot & v_1 & v_2 & v_3 & v_4 \\ v_1 & 2 & -1 & 0 & -1 \\ v_2 & -1 & 3 & -1 & -1 \\ v_3 & 0 & -1 & 2 & -1 \\ v_4 & -1 & -1 & -1 & 3 \end{pmatrix}$$

Theorem 19: Laplace matrix of a completely extendable graph has $r\,(r + 1)$ zeros, where r is any positive integer.

Proof: Laplacian Matrix has $r\,(r + 1)$ zeros implies G has $r\,(r + 1)/2$ non adjacent vertices. Number of edges added to G to get K_p is $r(r + 1)/2$ which implies G is a completely extendable graph and r is the order of extension.

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