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RETRIEVAL OF A SET FROM ITS LATTICE OF TOPOLOGIES

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Abstract : In this paper we show that the set X can be retrieved from T (X), the lattice of all topologies on the set X. We also establish topological and lattice theoretic characterization of T (X).

Keywords: Lattice of topologies, atoms, anti atoms, hull, co-hull, lattice joins and meets.

1. INTRODUCTION

Birkhoff (Birkhoff, 1936) proved that the collection $T(X)$ of all topologies defined on a set X is a complemented, non-distributive and non-modular lattice. The structure of $T(X)$ is complex, however we can characterize certain topological properties in terms of lattice of topologies. If $\tau_1, \tau_2 \in T(X)$, we can define an order relation by letting $\tau_1 < \tau_2$ if and only if $\tau_1 \subset \tau_2$. Then $<$ is a partial order on $T(X)$. The least upper bound of τ_1 and τ_2 denoted by $\tau_1 \vee \tau_2$ is the topology generated by $\{G_1 \cap G_2 : G_1 \in \tau_1 \text{ and } G_2 \in \tau_2\}$ and the greatest lower bound denoted $\tau_1 \wedge \tau_2$ by is the topology $\tau_1 \cap \tau_2$. The largest and smallest element of $T(X)$ are the discrete and indiscrete topologies on X. $T(X)$ possesses atoms and anti atoms. Atoms are of the form $\{\phi, A, X\}$ where $A \neq \phi$, $A \neq X, A \subset X$. Anti atoms are of the form $\tau(x, \Upsilon) = P(X \setminus \{x\}) \cup \Upsilon$, where $x \in X$ and Υ is an ultra filter on X, $\Upsilon \neq \Upsilon_{(x)}$ the principal ultra filter generated by x.

2. RETRIEVAL OF *X* THROUGH ANTI ATOMS AND ATOMS

Definition 2.1. Let $\tau \in T(X)$. Then the set $H(\tau) = \{\tau_d \in D : \tau_d \geq \tau\}$ is called the hull of τ, where D is the set of all anti atoms of $T(X)$ and A is its atoms.

2.1 RECOVERING *X* FROM *T* (*X*) THROUGH ANTI ATOMS

Let $\tau_1, \tau_2 \in D$. Define $\tau_1 \sim \tau_2$ if and only if hull of $\tau_1 \wedge \tau_2$ contains atmost two elements. Then $'\sim'$ is an equivalence relation on D. Therefore, $D = [\tau(x, \xi)] \cup [\tau(y, \beta)] \cup [\tau(z, \zeta)] \cup \ldots$ Each equivalence class represents a point. $D/\sim = X$.

2.2 RECOVERING *X* FROM *T* (*X*) THROUGH ATOMS

Definition 2.2. Let $\tau \in T(X)$. Then the set $co-H(\tau) = \{\tau_a \in A : \tau_a \leq \tau\}$ is called the co-hull of τ.

Describe a subset A_1 of A as follows.

$A_1 = \{\tau \in A :$ When ever $\tau_1 \in A$ then either $\tau_1 \prec \tau$ or $\tau_1 \prec \tau_c\}$

Say $\tau_1 \prec \tau$(comparable) if there exist a third distinct τ_2 such that $\tau_1 \vee \tau$, $\tau_2 \vee \tau$, $\tau_1 \vee \tau_2$, have exactly two atoms in their respective co-hulls. τ_c is a unique atom in $T(X)$ such that

(i) $\tau_1 \vee \tau_c$ have exactly two atoms in its co-hull

(ii) $\tau_C \nprec \tau$(not comparable).

In A_1 if τ_1 and τ_2 are two elements such that $\tau_1 = (\tau_2)_c$, then identify them, say $\tau_1 \sim \tau$.

$$A_1/\sim = X$$

From 2.1 and 2.2, we get

Corollary: 2.1 $T(X) \backsimeq T(Y)$ if and only if $|X| = |Y|$ where $|X|$ denotes the cardinal number of X (Chlatson, 1994).

3. TOPOLOGIES HAVING UNIQUE MEET REPRESENTATIONS

A meet representation (Birkhoff, 1967) of a topology τ, we mean the representation of τ as a meet of a family of finer topologies with atmost one accumulation point in such a way that no two topologies have the same accumulation point. Every topology has atleast one representation. Some have many. If there are two such representation, it is natural to call the first smaller than the second, if every member of the first is coarser than some member of the second. We can prove that every topology τ on X has a smallest meet-representation. This smallest one is given by the family $\{\tau_x : x \in X\}$ where for each $x \in X$. The topology τ_x is defined as $\{V \subset X :$ if $x \in V,$ then $V \in \tau\}$.

This representation is useful in characterizing certain topological properties. For example, we have that τ is Fre-ehet if and only if $T(X)$ is sequential. There is another way of looking at the above property. Consider the families of topologies that form a meet representation for some topology. The category with these as objects and naturally defined morphisms can be proved to be isomorphic to the category of all Cech-closure spaces. A Cech-closure space is defined similar to a topological space, but ommiting the idempotency of the closure operation. A meet representation for a topology then corresponds to a Cech-closure whose topological modification is this topology. This helps in answering some natural questions concerning meet representation.

We consider only one question in this context, the most natural one; What are the topologies having a unique-meet representaion? It turns out from the following result that nice topologies such as the metrizable ones have this property(Steiner, 1966).

Theorem 3.1. The following are equivalent to for a topology τ on a set X.

1. τ has a unique meet represenattion.
2. τ_x has a unique complement in $[\tau) = \{\tau' : \tau' \geq \tau\}$.
3. τ_A has a unique complement in $[\tau)$ for each $A \subset X$.
4. τ_A is a maximal complement of $\tau_X \setminus A$ in $[\tau)$ for all $A \subset X$.
5. τ_x is a maximal complement of $\tau_X \setminus \{x\}$ for all $x \in X$.

Remarks 1: $\tau_A = \{V \cup B : V \in \tau, V$ disjoint from $A\}$.

4. PRINCIPAL DUAL IDEALS

If τ is a topology on X, then the notation $[\tau) = \{\tau' : \tau' \geq \tau\}$ represents the principal dual ideal(ultra space) of $T(X)$ determined by τ. In section 3, we have seen that some topological properties can be characterized in terms of lattice theoretic properties. In the reverse direction we take some good lattice theoretic properties of $[\tau)$ and ask for the corresponding implications on the topology τ. The most natural lattice property is of course the distributivity. This has been considered in Larson(Larson and Andima, 1975). We can now supplement this result with some more elegant statements. Thus we have the following.

Theorem 4.1. The following statements are equivalent for a space (X, τ).

a. τ is distributive.

b. τ is modular.

c. Every closure operation finer than that of τ is idempotent.

d. Every finer topology has a unique meet representation.

e. When ever $\tau_1, \tau_2 \geq \tau$, then the τ closure of τ_2 closed set is τ_2 closed.

f. When ever $\tau_1, \tau_2 \geq \tau$, the $\tau_1 \wedge \tau_2$ closure of any subset is nothing but $\tau 1$ closure of its τ_2 closure.

g. When ever τ_1, $\tau_2 \geq \tau$, the union of a τ_1 neighbourhood and a τ_2 neighbourhood of a point becomes a $\tau_1 \wedge \tau_2$ neighbourhood of that point.

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