Indian Journal of Mathematics and Mathematical Sciences Vol. 12, No. 1, (January. 2016)

CHARACTERIZATION OF DISTRIBUTIVE LATTICE IN A SPACE 'X'

K. V. R. Srinivas*

ABSTRACT

The concept of codistributive pair was introduced by Repritskii (2) in a general lattice. He also introduces the concept of d-prime ideal. K.V.R. Srinivas (5) obtained a characterization for a lattice to be distributive interms of d-prime ideals. In this paper, a characterization for a lattice to be distributive in a space 'X' of d-prime ideals together with the well known topology $\{X_a : a \in L\}$. where $X_a = \{P \in X: a \notin P\}$ as an open sub-base is obtained.

Key words: *Codistributive pair, d-prime Ideal, T_1 - Space,*

INTRODUCTION

In his paper entitled "Representation of lattices by sub semi-group lattice of bands" Repritskii (1995) had described those finite lattices which are embeddable in sub-semi group lattices of a rectangular band. "On distributive prime ideals of a lattice", K.V.R. Srinivas obtained a characterization for a lattice to be distributive in terms of d-prime ideals (4). In this paper we obtained a characterization for a lattice to be a distributive in a space X, of d-prime Ideals together with the well known topology $\{X_a: a \in L\}$ Where $X_a=\{P \in X:a \notin P\}$ as an open sub-base. It is also obtained in this paper that "Every d-prime ideal of a bounded lattice 'L' is maximal iff'X' is T_1-Space". An interesting example that if every d-prime ideal of a lattice 'L' is maximal, then the lattice need not even be modular is also obtained.

First we start with the following preliminaries.

1.1. Definition (2): Let 'L', be a lattice. We call a pair (a,b) of elements form 'L' codistributive if, for any element $z \in L$,

$(a \Lambda b) \vee z = (a \vee z) \Lambda (b \vee z)$. Otherwise we call (a, b) as non–codistributive.

1.2 Definition (2): An ideal 'P' of lattice 'L' is called a distributively prime (or briefly d-prime) if for any codistributive pair $(a, b) \in L^2$ the condition $a \Lambda b \in P$ implies $a \in P$ or $b \in P$.

1.3 Definition (3): Let $(L, \vee, \Lambda)$ be a lattice. An ideal 'M' of lattice 'L' is called a maximal ideal if 'M' is maximal among the proper ideals of L, equivalently to say that $M \neq L$ and if 'N' is an ideal of 'L' such that $M \subseteq N$ then N=M or N=L.

* G.V.P. College of Engineering, Visakhapatnam-41.

1.4 Definition (6): A Lattice (L, $\vee$, Λ) is said to be a distributive lattice if it satisfy the following conditions.

(1) For any triplet of elements a, b, c of the lattice a Λ (b $\vee$ c) = (a Λ b) $\vee$ (a Λ c).

(2) For any triplet of elements a, b, c of the lattice a $\vee$ (b Λ c) = (a$\vee$b) Λ (a $\vee$ c).

1.5 Definition (6): A Lattice (L, $\vee$, Λ) is said to be a modular lattice, if for any triplet of elements a, b, c of a Lattice satisfying a $\vee$ c, the identity

$$a\vee (b \Lambda c) = (a\vee b) \Lambda c \text{ holds.}$$

1.6 Definition (6): A Topological space 'X' is called a T_1–space if and only if every single element subset of 'X' is closed.

1.7 Definition (1): The topological space 'A' is called compact iff every family F_1 of closed sets $\cap F_1 \neq \emptyset$, for all finite $F_1 \subseteq F$ then $\cap F \neq \emptyset$.

Theorem 1.8: $\{x_a$: a $\in$ L$\}$ form base for some topology on X iff every d-prime ideal is prime iff 'L' is distributive.

Proof: Suppose $\{x_a$: a $\in$L$\}$ form a base for some topology on X. Let 'P', be any d-prime ideal of 'L' such that a$\notin$p and b$\notin$p so that p$\in X_a$ and P$\in X_b$ and hence P$\in X_a \cap X_b$ therefore there exists c$\in$L such that p$\in X_c \subseteq X_a \cap X_b$. Hence $X_c \subseteq X_a$ and $X_c \subseteq X_b$. So that c $\leq$ a and c $\leq$ b as a $\leq$ b iff $X_a \subseteq X_b$ (4). Therefore c $\leq$ a $\wedge$b. If a $\wedge$ b $\in$ P, then c $\in$ P (since 'P' is an ideal of L) which is a contradiction as P$\in X_c$ and hence a $\wedge$b $\notin$p. Therefore 'P' is a prime ideal of 'L'.

Conversely if every d-prime ideal of a lattice 'L' is prime, then obviously $\{X_a$: a$\in$L$\}$ form base for some topology on X. But by using theorem (4) 'L' is distributive.

If 'L' is a lattice with '0' and '1'. In the following theorem. We have obtained a necessary and sufficient condition for a bounded lattice to be maximal is X is T_1 –Space.

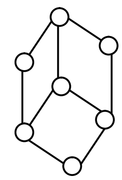
Theorem 1.9: Every d-prime ideal of a bounded lattice 'L' is maximal if and only if 'X' is T_1–Space.

Proof: Assume that every d-prime ideal of 'L' is maximal. Let 'P', be any d-prime ideal of lattice 'L'. Suppose Q$\in$X-{P} so that Q $\neq$ P and since P $\not\subseteq$ Q, and P, Q are maximal ideals of L, there exists a $\in$ P and a $\in$ Q. Hence Q $\in X_a \subseteq$ X-{P} (since a $\in$ P) so that X-{P} is open and hence {P} is closed. Therefore 'X' is T_1–Space.

Conversely suppose 'X' is T_1 – Space and let 'L', be a lattice with '0' and '1'. By using zorn's lemma, every proper ideal of 'L' is contained in a maximal ideal. Let P $\in$ X be a d-prime ideal of 'L' such that P $\subseteq$ M, where 'M' is a maximal ideal of 'L'. It remains to show that P = M. Let M $\in X_{a1} \cap X\, a_2 \cap \ldots \cap X_{an}$ be a basic neighbourhood of M, so that $a_1 \notin M$, $a_2 \notin M$… an $\notin$ M and hence $a_1 \notin P$, $a_2 \notin P$… $a_n \notin P$ (since $P \subseteq M$). Therefore $P \in X_{a1} \cap X_{a2} \cap \ldots \cap x_{an}$, so that every basic neighbourhood of 'M' contains P. Hence $M \in \{\bar{P}\} = \{P\}$ and hence M = P, therefore 'P' is a maximal ideal.

The following is an Interesting example that if every d-prime I deal of a lattice 'L' is maximal, then the lattice need not even be modular.

Ex 1.10:



In this {0} is not d-prime Ideal as (a,b) is Codistributive

{0,a} is not d-prime ideal as (c,d) is codistributive.

{0,b} is not d-prime ideal as (d,e) is codistributive.

{0,a,c}, {0,b,e}, {0,b,d}, {0,a,d} are d-prime ideals which are all maximal, but the lattice is not even modular.

The following is a necessary sufficient condition for a space 'X' to be compact is hat the lattice 'L' has greatest element '1' and $X = X_1$

Theorem 1.11

Suppose 'L' is a lattice and X, be the set o all proper d-prime Ideals of L. Then a space 'X' is compact if and only if 'L' has greatest element '1' and $X = X_1$

Proof: Let 'L' has greatest element '1', then $\bigcup_{a \ni L} X_a = X$. Since, $X_a \subseteq X$ so that

$\bigcup_{a \in L} X_a \subseteq X$, and for $P \in X$, there exists $a \in L$ such that $a \notin P$. Hence $P \in X_a$, so

that $X \subseteq \bigcup_{a \in L} X_a$ there fore $\cup X_a = X = X_1$. Now, suppose 'X' is compact, so that

X_a compact open, We have $X = X_{a1} \cup X_{a2} \cup \ldots \cup X_{an} = X_{a1 \vee a2 \vee \ldots \vee \vee an}$ (by using Lemma (3.45) (5)). It remains to show that $a_1 \vee a_2 \vee \ldots \vee a_n$ is the greatest element of L. For this Let $a \in L$, so that $X_a \subseteq X$, but $X = X_{a1 \vee \ldots \vee an}$ and hence $X_a \subseteq X_{a1 \vee \ldots \vee an}$ Therefore $a \leq a_1 \vee \ldots \vee_{an}$ (by lemma 3.39) (5) for any $a \in L$. Hence $a_1 \vee \ldots \vee_{an}$ is the greatest element of L. If '**1**' is the greatest element of L, then $X_1 \subseteq X \rightarrow$ (1). If $P \in X$, then $P \in X_1$, otherwise $1 \in P$, but 'P' is a proper d-prime ideal of L and hence $X \subseteq X_1 \rightarrow$ (2) From (1) & (2), we have $X = X_1$.

REFERENCES

[1] Gratzer, G., General Lattice theory, Academic press, New York, Sanfrancisco (1975).

[2] Repritskii V. B., On the representation of lattices by subsemigroup lattices of bands, semigroup form, Vol. 51, 379-389, (1995).

[3] Sankappanavar, A course in universal Algebra, springer Verlag (1981).

[4] Srinivas K. V. R., On distributive prime ideals of a lattice Bulletein of Calcutta Mathematical Society 97(1) 79-94, (2005).

[5] Srinivas K. V. R., On characterization of partial orders on regular semigroups and d-prime ideals in lattices Ph.D. thesis, Dept. of Maths, A.U. Visakhapatnam (2004).

[6] SZA SZG: Introduction to lattice theory, academic press, New York and London 1963.