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# AN ASYMPTOTIC COMPARISON OF TWO TIME-HOMOGENEOUS PAM MODELS

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Abstract. Both Wick-Itô-Skorokhod and Stratonovich interpretations of the Parabolic Anderson model (PAM) lead to solutions that are real analytic as functions of the noise intensity $\varepsilon$, and, in the limit $\varepsilon \to 0$, the difference between the two solutions is of order $\varepsilon^2$ and is non-random.

## 1. Introduction

Let $W = W(x),\ x \in [0, \pi]$ be a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. With no loss of generality, we assume that all realizations of $W$ are in $\mathcal{C}^{1/2-}((0, \pi))$, that is, Hölder continuous of every order less than 1/2.

Consider the equations

$$\begin{aligned} &\frac{\partial u_\diamond(t,x;\varepsilon)}{\partial t} = \frac{\partial^2 u_\diamond(t,x;\varepsilon)}{\partial x^2} + \varepsilon u_\diamond(t,x;\varepsilon) \diamond \dot{W}(x),\ t > 0,\ 0 < x < \pi, \\ &u_\diamond(t,0;\varepsilon) = u_\diamond(t,\pi;\varepsilon) = 0,\ u_\diamond(0,x;\varepsilon) = \varphi(x), \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\frac{\partial u_\circ(t,x;\varepsilon)}{\partial t} = \frac{\partial^2 u_\circ(t,x;\varepsilon)}{\partial x^2} + \varepsilon u_\circ(t,x;\varepsilon) \circ \dot{W}(x),\ t > 0,\ 0 < x < \pi, \\ &u_\circ(t,0;\varepsilon) = u_\circ(t,\pi;\varepsilon) = 0,\ u_\circ(0,x;\varepsilon) = \varphi(x). \end{aligned} \tag{1.2}$$

Equation (1.1) is the Wick-Itô-Skorokhod formulation of the parabolic Anderson model with potential $\varepsilon\dot{W}$; equation (1.2) is the corresponding Stratonovich (or geometric rough path) formulation. These equations, with $\varepsilon = 1$, are studied in [1] and [2], respectively.

The objective of the paper is to show that

- The solutions of (1.1) and (1.2) are real-analytic functions of $\varepsilon$: with suitable functions $u_\diamond^{(n)}$, and $u_\circ^{(n)}$, the equalities

$$u_\diamond(t,x;\varepsilon) = u_\diamond(t,x;0) + \sum_{n=1}^{\infty} \varepsilon^n u_\diamond^{(n)}(t,x) \tag{1.3}$$

$$u_\circ(t,x;\varepsilon) = u_\circ(t,x;0) + \sum_{n=1}^{\infty} \varepsilon^n u_\circ^{(n)}(t,x) \tag{1.4}$$

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hold for all $t > 0$, $x \in [0, \pi]$, $\varepsilon > 0$, and every realization of $W$;
- The first two terms in (1.3) and (1.4) are the same so that

$$|u_\diamond(t, x; \varepsilon) - u_\circ(t, x; \varepsilon)| = O(\varepsilon^2), \ \varepsilon \to 0, \tag{1.5}$$

for all $t > 0$ and $x \in [0, \pi]$, and every realization of $W$.

Equalities (1.3) and (1.4) are in the spirit of [5]. Equality (1.5) is similar to [9, Proposition 4.1]; see also [8].

The precise statement of the main result is in Section 2, and the proof is in Sections 3, 4, and 5.

## 2. The Main Result

Denote by $p = p(t, x, y)$ the heat semigroup on $[0, \pi]$ with zero boundary conditions:

$$p(t, x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t} \sin(kx) \sin(ky), \ t > 0, \ x, y \in [0, \pi]. \tag{2.1}$$

Let $\varphi = \varphi(x)$ be a continuous function on $[0, \pi]$, and let $u = u(t, x)$ be the solution of the heat equation

$$\begin{gathered} \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2}, \ t > 0, \ 0 < x < \pi, \\ u(t, 0) = u(t, \pi) = 0, \ u(0, x) = \varphi(x), \end{gathered} \tag{2.2}$$

that is,

$$u(t, x) = \int_0^\pi p(t, x, y) \varphi(y) dy. \tag{2.3}$$

Next, define the function $\mathfrak{u} = \mathfrak{u}(t, x)$ by

$$\mathfrak{u}(t, x) = \int_0^t \int_0^\pi p(t - s, x, y) u(s, y) \, dW(y) \, ds. \tag{2.4}$$

That is, $\mathfrak{u}$ is the mild solution of

$$\begin{gathered} \frac{\partial \mathfrak{u}(t, x)}{\partial t} = \frac{\partial^2 \mathfrak{u}(t, x)}{\partial x^2} + u(t, x) \dot{W}(x), \ t > 0, \ 0 < x < \pi, \\ \mathfrak{u}(t, 0) = \mathfrak{u}(t, \pi) = \mathfrak{u}(0, x) = 0. \end{gathered} \tag{2.5}$$

Because $u$ is non-random, no stochastic integral is required to define $\mathfrak{u}$.

**Proposition 2.1.** *If $\varphi \in \mathcal{C}((0, \pi))$, then $\mathfrak{u}$ is a continuous function of $t$ and $x$ for all $t > 0$ and $x \in [0, \pi]$.*

*Proof.* This follows by the Kolmogorov continuity criterion: $\mathfrak{u}$ is a Gaussian random field and direct computations show

$$\mathbb{E}\big(\mathfrak{u}(t + \tau, x + h) - \mathfrak{u}(t, x)\big)^2 \leq C(t) \big(\tau^2 + h^2\big)^{1/4} \max_{x \in [0, \pi]} |\varphi(x)|;$$

cf. [1, Sections 6 and 7]. □

Next, define the functions $u_\diamond^{(n)} = u_\diamond^{(n)}(t,x)$, $n = 0,1,2\ldots$, $t \ge 0$, $x \in [0,\pi]$, by $u_\diamond^{(0)}(t,x) = u(t,x)$, and, for $n \ge 1$, $u_\diamond^{(n)}$ is the mild solution of

$$\begin{aligned} \frac{\partial u_\diamond^{(n)}(t,x)}{\partial t} &= \frac{\partial^2 u_\diamond^{(n)}(t,x)}{\partial x^2} + u_\diamond^{(n-1)}(t,x) \diamond \dot{W}(x),\ t>0,\ 0<x<\pi, \\ u_\diamond^{(n)}(t,0) &= u_\diamond^{(n)}(t,\pi) = u_\diamond^{(n)}(0,x) = 0. \end{aligned} \tag{2.6}$$

In other words,

$$u_\diamond^{(n)}(t,x) = \int_0^t \int_0^\pi p(t-s,x,y) u_\diamond^{(n-1)}(s,y) \diamond dW(y)\, ds, \quad n \ge 1, \tag{2.7}$$

and, in particular, $u_\diamond^{(1)} = \mathfrak{u}$.

Similarly, define the functions $u_\circ^{(n)} = u_\circ^{(n)}(t,x)$, $n = 0,1,2\ldots$, $t \ge 0$, $x \in [0,\pi]$, by $u_\circ^{(0)}(t,x) = u(t,x)$, and, for $n \ge 1$, $u_\circ^{(n)}$ is the mild solution of

$$\begin{aligned} \frac{\partial u_\circ^{(n)}(t,x)}{\partial t} &= \frac{\partial^2 u_\circ^{(n)}(t,x)}{\partial x^2} + u_\circ^{(n-1)}(t,x) \circ \dot{W}(x),\ t>0,\ 0<x<\pi, \\ u_\circ^{(n)}(t,0) &= u_\circ^{(n)}(t,\pi) = u_\circ^{(n)}(0,x) = 0. \end{aligned} \tag{2.8}$$

In other words,

$$u_\circ^{(n)}(t,x) = \int_0^t \int_0^\pi p(t-s,x,y) u_\circ^{(n-1)}(s,y) \circ dW(y)\, ds, \quad n \ge 1, \tag{2.9}$$

and, in particular, $u_\circ^{(1)} = \mathfrak{u}$.

The main result of the paper can now be stated as follows.

**Theorem 2.2.** *Let $\varphi \in \mathcal{C}((0,\pi))$. Then*

(1) *Equality* (1.3) *holds with $u_\diamond^{(n)}$ from* (2.7).
(2) *Equality* (1.4) *holds with $u_\circ^{(n)}$ from* (2.9).
(3) *Equality* (1.5) *holds and*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2}\Big(u_\circ(t,x;\varepsilon) - u_\diamond(t,x;\varepsilon)\Big) = \int_0^\pi p^{(3)}(t,x,z)\varphi(z)dz, \tag{2.10}$$

*where*

$$p^{(3)}(t,x,z) = \int_0^\pi \int_0^t \int_0^s p(t-s,x,y)p(s-r,y,y)p(r,y,z)\, dr\, ds\, dy.$$

The proof is carried out in the following three sections.

## 3. The Wick-Itô-Skorokhod Case

The objective of this section is the proof of (1.3).

The solution of (1.1) is defined as a chaos solution (cf. [6, Theorems 3.10]). It is a continuous in $(t,x)$ function (cf. [1, Sections 6 and 7]) and has a representation as a series

$$u_\diamond(t,x;\varepsilon) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} u_{\boldsymbol{\alpha}}(t,x;\varepsilon)\xi_{\boldsymbol{\alpha}}, \tag{3.1}$$

where

$$\mathcal{J} = \left\{ \boldsymbol{\alpha} = (\alpha_k,\, k \geq 1) : \alpha_k \in \{0,1,2,\ldots\},\ \sum_k \alpha_k < \infty \right\},$$

$$\xi_{\boldsymbol{\alpha}} = \prod_k \left( \frac{\mathrm{H}_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right),\ \mathrm{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

$$\xi_k = \int_0^\pi \mathfrak{m}_k(x)\, dW(x),\ \mathfrak{m}_k(x) = \sqrt{2/\pi}\, \sin(kx),$$

and, with $|\boldsymbol{\alpha}| = \sum_k \alpha_k$, $|(\mathbf{0})| = 0$, $|\boldsymbol{\epsilon}(k)| = 1$,

$$u_{(\mathbf{0})}(t,x;\varepsilon) = u(t,x),$$

$$u_{\boldsymbol{\alpha}}(t,x;\varepsilon) = \varepsilon \sum_k \sqrt{\alpha_k} \int_0^t \int_0^\pi p(t-s,x,y) u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(k)}(s,y;\varepsilon)\mathfrak{m}_k(y)\, dyds;$$

see [1, Section 3] for details. In particular,

$$\sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t,x;1)\xi_{\boldsymbol{\alpha}} = \sum_{|\boldsymbol{\alpha}|=n-1} \int_0^t \int_0^\pi p(t-s,x,y) u_{\boldsymbol{\alpha}}(s,y;1)\xi_{\boldsymbol{\alpha}} \diamond dW(y)\, ds. \quad (3.2)$$

Comparing (2.7) with (3.2) shows that

$$u_\diamond^{(n)}(t,x) = \sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t,x;1)\xi_{\boldsymbol{\alpha}}. \quad (3.3)$$

In other words, (1.3) is equivalent to (3.1).

Next,

$$\mathbb{E}|u_\diamond^{(n)}(t,x)| = \mathbb{E}\left| \sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t,x;1)\xi_{\boldsymbol{\alpha}} \right| \leq \left( \mathbb{E}\left( \sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t,x;1)\xi_{\boldsymbol{\alpha}} \right)^2 \right)^{1/2}$$

$$= \left( \sum_{|\boldsymbol{\alpha}|=n} |u_{\boldsymbol{\alpha}}(t,x;1)|^2 \right)^{1/2} \leq C^n(t) n^{-n/4} \sup_{x\in(0,\pi)} |\varphi(x)|^{1/2},$$

where the last inequality follows by [1, Theorem 4.1]. As a result,

$$\sum_{n\geq 0} \varepsilon^n \mathbb{E}\big|u_\diamond^{(n)}(t,x)\big| < \infty,$$

that is, the series converges absolutely with probability one for all $t > 0,\ x \in [0,\pi]$, and $\varepsilon \in \mathbb{R}$.

*This concludes the proof of* (1.3).

## 4. The Stratonovich Case

The objective of this section is to prove (1.4). To simplify the presentation, we use the following notations:

$$\Lambda = (-\mathbf{\Delta})^{1/2}, \quad H^{\theta} = \Lambda^{-\theta}\big(L_2((0,\pi))\big), \quad \|\cdot\|_{\theta} = \|\Lambda^{\theta}\cdot\|_{L_2((0,\pi))}, \quad \theta \in \mathbb{R},$$

$$p * g(t,s,x) = \int_0^{\pi} p(t-s,x,y)g(s,y)\,dy,$$

where $\mathbf{\Delta}$ is the Laplace operator on $(0,\pi)$ with zero boundary conditions and $p$ is the heat kernel (2.1).

By direct computation,

$$\|p*g\|_{\gamma}(t,s) \le C_{T,\theta}\,(t-s)^{-\theta/2}\|g\|_{\gamma-\theta}(s),\ \theta>0, \quad \gamma\in\mathbb{R}, \quad t\in(s,T]; \tag{4.1}$$

cf. [3, Lemma 7.3].

Consider the equation

$$\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} + v(t,x)\circ \dot W(x) + f(t,x)\circ \dot W(x),\ t>0, \tag{4.2}$$

which includes (1.2) as a particular case. By definition, a solution (classical, mild, generalized, etc.) of (4.2) is a suitable limit, as $\epsilon \to 0$, of the corresponding solutions of

$$\frac{\partial v_\epsilon(t,x)}{\partial t} = \frac{\partial^2 v_\epsilon(t,x)}{\partial x^2} + v_\epsilon(t,x)V_\epsilon(x) + f(t,x)V_\epsilon(x),\ t>0, \tag{4.3}$$

where $V_\epsilon,\ \epsilon>0$ are smooth functions on $[0,\pi]$ such that

$$\sup_{\epsilon}\left\|\int V_\epsilon\right\|_{\mathcal{C}^{1/2}} < \infty,\ \lim_{\epsilon\to 0}\sup_{x\in[0,\pi]}\left|\int_0^x V_\epsilon(y)dy - W(x)\right| = 0.$$

By [2, Theorem 3.5],

- The generalized solution of (4.2) is the same as the generalized solution of the equation
$$v_t = \Big(v_x + W(x)v + W(x)f\Big)_x - W(x)v_x - W(x)f_x; \tag{4.4}$$
the subscripts $t$ and $x$, as in $f_x$, denote the corresponding partial derivatives;
- The mild solution of (4.2) is the solution of the integral equation

$$\begin{aligned} v(t,x) = &\int_0^t p*\big((f+v)W\big)_x(t,s,x)\,ds - \int_0^t p*\big((f+v)_x W\big)(t,s,x)\,ds \\ &+ \int_0^{\pi} p(t,x,y)\varphi(y)dy. \end{aligned} \tag{4.5}$$

On the one hand, mild and generalized solutions of (4.2) are the same: just use $\mathfrak{m}_k$ as the test functions. On the other hand, different definitions of the solution lead to different regularity results.

By standard parabolic regularity, if $\varphi \in H^0$ and $f \in L_2\big((0,T);H^\gamma\big)$, $\gamma \in (1/2, 1]$, then there is a unique generalized solution of (4.4) in the normal triple $(H^1, H^0, H^{-1})$ and

$$v \in L_2\big((0,T);H^1\big) \bigcap \mathcal{C}\big((0,T);H^0\big) \tag{4.6}$$

for every realization of $W$; cf. [4, Theorem 3.4.1]. Note that we cannot claim $v \in \mathcal{C}\big((0,T);H^\gamma\big)$ even if $\varphi \in H^\gamma$. In fact, because $W \in \mathcal{C}^{1/2-}$ is a point-wise multiplier in $H^\gamma$ for $\gamma \in (-1/2, 1/2)$ [3, Lemma 5.2], an attempt to find a traditional regularity result for equation (4.4) in a normal triple $(H^{r+1}, H^r, H^{r-1})$ leads to an irreconcilable pair of restrictions on $r$: to have $Wf \in L_2((0,T);H^r)$ we need $r < 1/2$, whereas to have $Wf_x \in L_2((0,T);H^{r-1})$ we need $r - 1 > -1/2$ or $r > 1/2$.

Accordingly, to derive a bound on $\|v\|_\gamma(t)$ for $t > 0$, we use the mild formulation (4.5).

**Proposition 4.1.** *Let $\gamma \in (1/2, 1)$, $f \in L_2\big((0,T);H^\gamma\big)$, $\varphi \in H^0$, and let $v$ be the mild solution of (4.2) with $v|_{t=0} = \varphi$. Then, for every $T > 0$ and every realization of $W$, there exists a number $C_\circ$ such that*

$$\|v\|_\gamma(t) \le C_\circ \left( t^{-\gamma/2}\|\varphi\|_0 + \int_0^t (t-s)^{-\gamma}\|f\|_\gamma(s)\,ds \right). \tag{4.7}$$

*Proof.* Throughout the proof, $C$ denotes a number depending on $\gamma$, $T$, and the norm of $W$ in the space $\mathcal{C}^{1-\gamma}$. The value of $C$ can change from one instance to another. With no loss of generality, we assume that $\varphi$ and $f$ are smooth functions with compact support.

To begin, let us show that if $V$ is the mild solution of

$$\frac{\partial V(t,x)}{\partial t} = \frac{\partial^2 V(t,x)}{\partial x^2} + f(t,x) \circ \dot{W}(x),\ t > 0,$$

$V(t,0) = V(t,\pi) = 0$, $V|_{t=0} = \varphi$, then

$$\|V\|_\gamma(t) \le Ct^{-\gamma/2}\|\varphi\|_0 + C\int_0^t (t-s)^{-\gamma}\|f\|_\gamma(s)\,ds. \tag{4.8}$$

Indeed, by (4.5),

$$V(t,x) = \int_0^t p * (fW)_x(t,s,x)\,ds - \int_0^t p * \big(f_x W\big)(t,s,x)\,ds + \int_0^\pi p(t,x,y)\varphi(y)dy.$$

Using (4.1) with $\theta = \gamma$,

$$\left\| \int_0^\pi p(t,\cdot,y)\varphi(y)dy \right\|_\gamma \le Ct^{-\gamma/2}\|\varphi\|_0.$$

Then

$$\|V\|_\gamma(t) \le \int_0^t \|p*(fW)_x\|_\gamma(t,s)\,ds + \int_0^t \|p*\big(f_x W\big)\|_\gamma(t,s)\,ds + Ct^{-\gamma/2}\|\varphi\|_0. \tag{4.9}$$

To estimate the first term on the right hand side of (4.9), we use (4.1) with $\theta = 2\gamma$. Then

$$\|p * (fW)_x\|_\gamma(t,s) \le C(t-s)^{-\gamma}\|(fW)_x\|_{-\gamma}(s) \le C(t-s)^{-\gamma}\|fW\|_{1-\gamma}(s),$$

and, because $W \in \mathcal{C}^{1/2-}((0,\pi))$ is a (point-wise) multiplier in $H^{1-\gamma}$,

$$\|fW\|_{1-\gamma}(s) \le C_W \|f\|_{1-\gamma}(s);$$

recall that $0 < 1-\gamma < 1/2$. Finally, as $1-\gamma < \gamma$,

$$\|p * (fW)_x\|_\gamma(t,s) \le C(t-s)^{-\gamma}\|f\|_\gamma(s). \tag{4.10}$$

To estimate the second term on the right hand side of (4.9), we use (4.1) with $\theta = 1$. Then

$$\|p * \big(f_x W\big)\|_\gamma(t,s) \le \frac{C}{\sqrt{t-s}}\,\|f_x W\|_{\gamma-1}(s) \le \frac{C}{\sqrt{t-s}}\,\|f_x\|_{\gamma-1}(s),$$

that is,

$$\|p * \big(f_x W\big)\|_\gamma(t,s) \le C(t-s)^{-1/2}\|f\|_\gamma(s). \tag{4.11}$$

To establish (4.8), we now combine (4.9), (4.10), and (4.11), keeping in mind that $(t-s)^{-1/2} \le C(t-s)^{-\gamma}$ because $\gamma > 1/2$.

Next, (4.8) applied to (4.2) implies

$$\|v\|_\gamma(t) \le C\int_0^t (t-s)^{-\gamma}\|v\|_\gamma(s)\,ds + C\int_0^t (t-s)^{-\gamma}\|f\|_\gamma(s)\,ds + Ct^{-\gamma/2}\|\varphi\|_0,$$

and then a generalization of the Gronwall inequality (e.g. [10, Corollary 2]) leads to (4.7). □

**Corollary 4.2.** *If $\varphi \in H^0$ and $\gamma \in (1/2,1)$, then, for every $T > 0$, $a > 0$, and every realization of $W$, there exists a number $\tilde{C}_\circ$ such that the mild solution of (1.2) satisfies*

$$\sup_{|\varepsilon|<a} \|u_\circ(t,\cdot,\varepsilon)\|_\gamma \le \tilde{C}_\circ\, t^{-\gamma/2}\|\varphi\|_0,\ t \in (0,T]. \tag{4.12}$$

Next, define the functions $u_\circ^{(n),\varepsilon} = u_\circ^{(n),\varepsilon}(t,x)$, $n = 0,1,2\ldots$, $t \ge 0$, $x \in [0,1]$, $\varepsilon \in \mathbb{R}$, by $u_\circ^{(0),\varepsilon}(t,x) = u_\circ(t,x;\varepsilon)$ and, for $n \ge 1$,

$$\begin{aligned} \frac{\partial u_\circ^{(n),\varepsilon}(t,x)}{\partial t} &= \frac{\partial^2 u_\circ^{(n),\varepsilon}(t,x)}{\partial x^2} + \varepsilon u_\circ^{(n),\varepsilon}(t,x) \circ \dot{W}(x) + u_\circ^{(n-1),\varepsilon}(t,x) \circ \dot{W}(x), \\ u_\circ^{(n),\varepsilon}(t,0) &= u_\circ^{(n),\varepsilon}(t,\pi) = 0,\ u_\circ^{(n),\varepsilon}(0,x) = 0. \end{aligned} \tag{4.13}$$

In particular,

$$u_\circ^{(n),0}(t,x) = u_\circ^{(n)}(t,x).$$

Note that all equations in (4.13) are of the form (4.2).

**Proposition 4.3.** *If $\varphi \in H^0$, then, for every $\gamma \in (1/2,2/3)$ and every realization of $W$,*

$$\lim_{\varepsilon\to\varepsilon_0} \frac{1}{(\varepsilon-\varepsilon_0)^n}\Big\|u_\circ(t,\cdot;\varepsilon) - \sum_{k=0}^n (\varepsilon-\varepsilon_0)^k u_\circ^{(k),\varepsilon_0}(t,\cdot)\Big\|_\gamma = 0, \tag{4.14}$$

*$n \ge 0$, $\varepsilon_0 \in \mathbb{R}$, $t \ge 0$.*

*Proof.* Throughout the proof, $C$ denotes a number depending on $\gamma$, $T$, $\varepsilon_0$, and the norm of $W$ in the space $\mathcal{C}^{1-\gamma}$. Define

$$\mathfrak{v}_{\varepsilon}^{(n)}(t,x)=\frac{1}{(\varepsilon-\varepsilon_0)^n}\left(u_{\circ}(t,x;\varepsilon)-\sum_{k=0}^{n}(\varepsilon-\varepsilon_0)^k u_{\circ}^{(k),\varepsilon_0}(t,x)\right). \tag{4.15}$$

By (1.2),

$$\begin{aligned}\sum_{k=0}^{n}(\varepsilon-\varepsilon_0)^k\frac{\partial u_{\circ}^{(k),\varepsilon_0}(t,x)}{\partial t}&=\sum_{k=0}^{n}(\varepsilon-\varepsilon_0)^k\frac{\partial^2 u_{\circ}^{(k),\varepsilon_0}(t,x)}{\partial x^2}\\&\quad+\varepsilon_0\sum_{k=0}^{n}(\varepsilon-\varepsilon_0)^k u_{\circ}^{(k),\varepsilon_0}(t,x)\circ\dot{W}(x)\\&\quad+(\varepsilon-\varepsilon_0)\sum_{k=1}^{n}(\varepsilon-\varepsilon_0)^{k-1}u_{\circ}^{(k-1),\varepsilon_0}(t,x)\circ\dot{W}(x),\end{aligned} \tag{4.16}$$

so that

$$\begin{aligned}\frac{\partial\mathfrak{v}_{\varepsilon}^{(0)}(t,x)}{\partial t}&=\frac{\partial^2\mathfrak{v}_{\varepsilon}^{(0)}(t,x)}{\partial x^2}+\varepsilon_0\mathfrak{v}_{\varepsilon}^{(0)}(t,x)\circ\dot{W}(x)+(\varepsilon-\varepsilon_0)u_{\circ}(t,x;\varepsilon)\circ\dot{W}(x),\\\frac{\partial\mathfrak{v}_{\varepsilon}^{(n)}(t,x)}{\partial t}&=\frac{\partial^2\mathfrak{v}_{\varepsilon}^{(n)}(t,x)}{\partial x^2}+\varepsilon_0\mathfrak{v}_{\varepsilon}^{(n)}(t,x)\circ\dot{W}(x)+\mathfrak{v}_{\varepsilon}^{(n-1)}(t,x)\circ\dot{W}(x),\ n\ge 1,\end{aligned} \tag{4.17}$$

$\mathfrak{v}_{\varepsilon}^{(n)}(0,x)=0$, $n\ge 0$, and (4.14) becomes

$$\lim_{\varepsilon\to\varepsilon_0}\|\mathfrak{v}_{\varepsilon}^{(n)}\|_{\gamma}(t)=0,\quad n\ge 0,\ t\ge 0,\ \varepsilon_0\in\mathbb{R}. \tag{4.18}$$

Note that all equations in (4.17) are of the form (4.2), and (4.18) trivially holds for $t=0$. Accordingly, combining the second equation in (4.17) with (4.7),

$$\|\mathfrak{v}_{\varepsilon}^{(n)}\|_{\gamma}(t)\le C\int_0^t(t-s)^{-\gamma}\|\mathfrak{v}_{\varepsilon}^{(n-1)}\|_{\gamma}(s)\,ds,$$

$n\ge 1$, and then, for $t>0$, (4.18) follows by induction: for $n=0$, (4.12) yields

$$\begin{aligned}\|\mathfrak{v}_{\varepsilon}^{(0)}\|_{\gamma}(t)&\le|\varepsilon-\varepsilon_0|\,C\int_0^t(t-s)^{-\gamma}\|u_{\circ}(s,\cdot,\varepsilon)\|_{\gamma}\,ds\\&\le C|\varepsilon-\varepsilon_0|\,\|\varphi\|_0\int_0^t(t-s)^{-\gamma}s^{-\gamma/2}\,ds\le C|\varepsilon-\varepsilon_0|\,\|\varphi\|_0t^{1-(3/2)\gamma}\to 0,\ \varepsilon\to\varepsilon_0;\end{aligned}$$

similarly, for $n\ge 1$,

$$\|\mathfrak{v}_{\varepsilon}^{(n)}\|_{\gamma}(t)\le C^{(n)}|\varepsilon-\varepsilon_0|\,\|\varphi\|_0,$$

because $1-(3/2)\gamma>0$. □

**Proposition 4.4.** *If $\varphi\in H^0$ and $\gamma\in(1/2,1)$, then*

$$\lim_{n\to\infty}c^n\sup_{|\varepsilon|<a}\|u_{\circ}^{(n),\varepsilon}\|_{\gamma}(t)=0,\ t\ge 0, \tag{4.19}$$

*for all $c>0$, $a>0$, and every realization of $W$.*

*Proof.* Throughout this proof, $C$ denotes a number depending on $\gamma$, $T$, $a$, and the norm of $W$ in the space $\mathcal{C}^{1-\gamma}$.

Combining (4.13) and (4.7),

$$\|u_\circ^{(n),\varepsilon}]\|_\gamma(t) \le C \int_0^t (t-s)^{r-1} \|u_\circ^{(n-1),\varepsilon}\|_\gamma(s)\, ds.$$

By iteration and (4.12), with $r = 1-\gamma > 0$,

$$\begin{aligned}
&\sup_{|\varepsilon|<a} \|u_\circ^{(n),\varepsilon}\|_\gamma(t) \le C^n \|\varphi\|_0 \\
&\quad \times \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} (t-s_n)^{r-1}(s_n - s_{n-1})^{r-1} \cdots (s_2 - s_1)^{r-1} s_1^{-\gamma/2} ds_1 \cdots ds_n \\
&\quad = C^n \|\varphi\|_0 \, \frac{\big(\Gamma(r)\big)^n \Gamma(1-(\gamma/2))}{\Gamma\big(nr+1\big)}\, t^{nr-(\gamma/2)},
\end{aligned}$$

where $\Gamma$ is the Gamma function

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt.$$

Then (4.19) follows by the Stirling formula. □

Equality (1.4) now follows:

- By the Sobolev embedding theorem, every element, or equivalence, class from $H^\gamma$, $\gamma > 1/2$, has a representative that is a continuous function on $[0,\pi]$;
- By Proposition 4.3 and the Taylor formula,

$$u_\circ(t,x) = u(t,x) + \sum_{k=1}^n u_\circ^{(k),0}(t,x)\varepsilon^k + R_n(t,x);$$

- By Proposition 4.4,

$$\lim_{n\to\infty} R_n(t,x) = 0.$$

*This concludes the proof of* (1.4).

## 5. The Correction Term

The objective of this section is the proof of (2.10).

Using (1.3) and (1.4), and remembering that $u_\circ^{(1)} = u_\diamond^{(1)} = \mathfrak{u}$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-2}\Big(u_\circ(t,x;\varepsilon) - u_\diamond(t,x;\varepsilon)\Big) = u_\circ^{(2)}(t,x) - u_\diamond^{(2)}(t,x) \\
&\qquad = \int_0^t \int_0^\pi p(t-s,x,y)\big(\mathfrak{u}(s,y)\circ dW(y) - \mathfrak{u}(s,y)\diamond dW(y)\big)\, ds.
\end{aligned} \tag{5.1}$$

By definition,

$$\xi_k \diamond \xi_n = \begin{cases} \xi_k \xi_n, & k \ne n, \\ \xi_n^2 - 1, & k = n, \end{cases}$$

and therefore

$$\xi_k\xi_n - \xi_k \diamond \xi_n = \begin{cases} 0 & k \neq n, \\ 1, & k = n, \end{cases} \tag{5.2}$$

Then (5.2) and [7, Theorem 3.1.2] imply that, for a function $f = f(x)$ of the form

$$f(x) = \sum_{k=1}^{\infty} f_k(x)\xi_k,$$

with $f_k$ non-random and satisfying

$$\sum_k \int_0^{\pi} |f_k(x)|\, dx < \infty, \tag{5.3}$$

the following equality holds:

$$\int_0^{\pi} f(x) \circ dW(x) - \int_0^{\pi} f(x) \diamond dW(x) = \sum_{k=1}^{\infty} \left( \int_0^{\pi} f_k(x)\mathfrak{m}_k(x)\, dx \right). \tag{5.4}$$

Condition (5.3) ensures that the sum on the right-hand side of (5.4) converges absolutely.

Next, recall that, by (2.4),

$$\mathfrak{u}(s,y) = \sum_{k=1}^{\infty} \left( \int_0^{\pi} \int_0^{s} p(s-r,y,z)u(r,z)\mathfrak{m}_k(z)\, dr\, dz \right) \xi_k.$$

For fixed $s \in [0,T]$ and $y \in [0,\pi]$, define

$$g(z) = \int_0^{s} p(s-r,y,z)u(r,z)dr,\ \ g_k = \int_0^{\pi} g(z)\mathfrak{m}_k(z)\, dz.$$

Then

$$\mathfrak{u} = \sum_{k=1}^{\infty} g_k\xi_k,$$

and (5.3) in this case will follow from uniform, in $(s,y)$ convergence of

$$\sum_{k=1}^{\infty} |g_k|,$$

which, by Bernstein's theorem [11, Theorem VI.3-1], will, in turn, follow from

$$|g(z+h) - g(z)| \leq Ch^{\delta} \tag{5.5}$$

with $\delta \in (1/2, 1)$ and $C$ independent of $s, y, z$.

Recall that

$$u(r,z) = \sum_{k=1}^{\infty} \varphi_k e^{-k^2 r}\, \mathfrak{m}_k(z),\ \ \varphi_k = \int_0^{\pi} \varphi(x)\mathfrak{m}_k(x)\, dx,$$

and, by integral comparison,

$$\sum_{k=1}^{\infty} k^p e^{-k^2 t} \leq \frac{C(p)}{t^{(1+p)/2}},\ \ p \geq 0.$$

Also,

$$|\sin(k(z+h)) - \sin(kz)| \le k^\delta h^\delta,\ \ \delta \in (0,1),$$

and the maximum principle implies $|u(r,z)| \le C$. Then

$$p(s-r,y,z) \le \frac{C}{\sqrt{s-r}},\quad |p(s-r,y,z+h) - p(s-r,y,z)| \le \frac{Ch^\delta}{(s-r)^{(1+\delta)/2}},$$
$$|u(r,z+h) - u(r,z)| \le \frac{Ch^\delta}{r^{(1+\delta)/2}},$$

and (5.5) follows because

$$\int_0^s \frac{dr}{\big(r(s-r)\big)^{(1+\delta)/2}} < \infty$$

for $\delta \in (1/2, 1)$.

We now apply (5.4) to (5.1):

$$\int_0^t \int_0^\pi p(t-s,x,y)\big(\mathfrak{u}(s,y)\circ dW(y) - \mathfrak{u}(s,y)\diamond dW(y)\big)\,ds$$
$$= \int_0^t \int_0^\pi \sum_{n=1}^\infty \left(\int_0^\pi \left(\int_0^s p(s-r,y,z)u(r,z)\,dr\right)\mathfrak{m}_n(z)\,dz\right)\mathfrak{m}_n(y)p(t-s,x,y)\,dy\,ds$$
$$= \int_0^\pi \int_0^t \int_0^s p(t-s,x,y)p(s-r,y,y)u(r,y)\,dr\,ds\,dy,$$

which, in view of (2.3) and the Fubini theorem, is the same as (2.10).

*This concludes the proof of* (2.10).

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