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PARAMETRIC FAMILY OF SDES DRIVEN BY LÉVY NOISE

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Abstract. In this article we study the existence and uniqueness of strong solutions of a class of parameterized family of SDEs driven by Lévy noise. These SDEs occurs in connection with a class of stochastic PDEs, which take values in the space of tempered distributions $\mathcal{S}'$. These results extend a correspondence for diffusion processes, which had been proved earlier in the literature.

1. Introduction

Given a complete filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$ satisfying the usual conditions, consider a class of stochastic differential equations (SDEs) in $\mathbb{R}^d$, viz.

$$\begin{aligned} dU_t &= \bar{b}(U_{t-};\xi)dt + \bar{\sigma}(U_{t-};\xi)\cdot dB_t + \int_{(0<|x|<1)} \bar{F}(U_{t-},x;\xi)\,\widetilde{N}(dtdx) \\ &\quad + \int_{(|x|\geq 1)} \bar{G}(U_{t-},x;\xi)\,N(dtdx), \quad t\geq 0 \\ U_0 &= \kappa, \end{aligned} \tag{1.1}$$

and associated stochastic PDEs in the space of tempered distributions $\mathcal{S}'$ (more specifically, in the Hermite-Sobolev spaces $\mathcal{S}_p, p\in\mathbb{R}$, which are real separable Hilbert spaces, see Section 2), viz.

$$\begin{aligned} dY_t &= A(Y_{t-})\cdot dB_t + \widetilde{L}(Y_{t-})\,dt + \int_{(0<|x|<1)} \big(\tau_{F(Y_{t-},x)} - Id\big)\; Y_{t-}\,\widetilde{N}(dtdx) \\ &\quad + \int_{(|x|\geq 1)} \big(\tau_{G(Y_{t-},x)} - Id\big)\; Y_{t-}\,N(dtdx), \quad t\geq 0 \\ Y_0 &= \xi, \end{aligned} \tag{1.2}$$

where

(i) $\{B_t\}$ denotes an $\mathbb{R}^d$-valued standard Brownian motion (with components $B^1_t, B^2_t, \cdots, B^d_t$) and N a Poisson random measure driven by a Lévy measure ν. $\widetilde{N}$ denotes the corresponding compensated random measure.

(ii) The initial conditions κ and ξ are $\mathcal{F}_0$-measurable random variables taking values in $\mathbb{R}^d$ and $\mathcal{S}_p$ respectively.

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(iii) ξ, κ, B and N are independent of each other and the filtration $(\mathcal{F}_t)$ is generated by these random variables.
(iv) $\tau_x, x \in \mathbb{R}^d$ denote the translation operators (see Section 2).
(v) The coefficients $\bar{\sigma}, \bar{b}, \bar{F}$ and $\bar{G}$ in (1.1) are defined in terms of σ, b, F and G which are the coefficients in (1.2). Note that the coefficients are allowed to be $\mathcal{F}_0$ measurable. The notations and hypotheses on these coefficients are described in Section 3.
(vi) The operator $\widetilde{L}$ is a second order differential operator together with an integro-differential operator. The operator A is a first order differential operator. These operators are defined using certain coefficients σ, b and F (see [5] for details).

Special cases of these stochastic PDEs have already been investigated in [3, p. 524], [4, p. 170], [13, p. 237] and [2]. In particular, this model includes the well-known weak formulation of the Itô formula, viz.

$$\delta_{B_t} = \tau_{B_t}\delta_0 = \delta_0 - \sum_{i=1}^{d}\int_0^t \partial_i \delta_{B_s}\, dB_s^i + \frac{1}{2}\sum_{i=1}^{d}\int_0^t \partial_i^2 \delta_{B_s}\, ds.$$

We do not consider these stochastic PDEs in the present article and as such, we skip the explicit definitions of the operators A and $\widetilde{L}$.

In this article, we consider the existence and uniqueness of strong solutions of the SDEs (1.1). In [5], we use these results of the present article to prove existence and uniqueness results for the associated stochastic PDEs (1.2). In fact, we show that if $\{U_t\}$ solves (1.1) with initial condition $\kappa = 0$, then $Y_t := \tau_{U_t}\xi$ solves (1.2). One may study the ergodicity/stationarity properties of the stochastic PDEs (1.2) via the corresponding finite dimensional SDEs (1.1). This problem was investigated for the Gaussian noise case in [4].

Observe that the transformation $Y_t = \tau_{U_t}\xi$ reformulates the SDEs (1.1) into the stochastic PDEs (1.2). However, this allows us to study evolution systems involving a larger class of initial conditions than originally given in the SDE. For example, the stochastic PDEs (1.2) may model a multi-particle system where ξ represents the initial configuration of the system.

1.1. Main results. As already mentioned, in this article we consider the existence and uniqueness of strong solutions of the SDEs (1.1). A standard approach in proving the existence and uniqueness results for SDEs is to assume that the coefficients are Lipschitz (see [1, 6–8, 10–12] and the references therein). The goal of this article is to describe hypotheses, which include appropriate parameterized versions of Lipschitz regularity of the coefficients and prove in detail the existence and uniqueness results.

If κ and ξ are deterministic, then the problem is reduced to the usual existence and uniqueness problem for SDEs driven by a Lévy noise. As such, a method in obtaining the result for random κ and ξ would be through a 'conditioning' argument. However, such an approach would require checking the continuity properties of the process with respect to the initial parameters (κ, ξ) and the proof would need to be separated into steps where the parameter is bounded or square integrable in norm and the general case, as in our case (see, for example, Theorem 3.4). Indeed, we separate our argument into such steps, but take the usual approach of Picard iteration in proving the existence and a Gronwall inequality argument in proving the uniqueness. Another point to note is the notion of solution that we use. We only require the solution to be an adapted process with rcll path that satisfy the corresponding integral equation of (1.1). In the diffusion case, it is known via Ikeda-Watanabe argument (see [8, Chapter IV]), that the solution is a measurable function of the initial condition and the Brownian motion. In such a case uniqueness for the equation with random initial condition would follow from the case of fixed initial condition by a 'conditioning' argument. The extension to the case when the equation depends on an additional parameter ξ seems to be open and a 'conditioning' argument is one possible approach to uniqueness. In this paper, however we use a more direct approach as mentioned above.

1.2. Layout of the paper. We now describe the layout of the paper. In Section 2, we define the space of Schwartz class functions $\mathcal{S}$ and its dual, the space of tempered distributions $\mathcal{S}'$. We also recall definitions of the Hermite-Sobolev spaces $\mathcal{S}_p, p \in \mathbb{R}$.

In Section 3, we state the notation and hypotheses followed in the rest of the article. In Theorem 3.4, the existence and uniqueness result is proved for the reduced equation with 'global Lipschitz' coefficients and then in Theorem 3.5 the same is proved for the general case (i.e. involving the large jumps) by an interlacing technique. In Theorem 3.6, we prove the result for 'local Lipschitz' coefficients.

2. Topology on Schwartz Space

Let $\mathcal{S}$ be the space of rapidly decreasing smooth functions on $\mathbb{R}^d$ with dual $\mathcal{S}'$, the space of tempered distributions (see [9]). Let $\mathbb{Z}^d_+ := \{n = (n_1, \cdots, n_d) : n_i \text{ non-negative integers}\}$. If $n \in \mathbb{Z}^d_+$, we define $|n| := n_1 + \cdots + n_d$.

For $p \in \mathbb{R}$, consider the increasing norms $\|\cdot\|_p$, defined by the inner products

$$\langle f, g\rangle_p := \sum_{n\in\mathbb{Z}^d_+} (2|n| + d)^{2p} \langle f, h_n\rangle\langle g, h_n\rangle, \quad f, g \in \mathcal{S}. \tag{2.1}$$

In the above equation, $\{h_n : n \in \mathbb{Z}^d_+\}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d, dx)$ given by the Hermite functions and $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathcal{L}^2(\mathbb{R}^d, dx)$. For $d = 1$, $h_n(t) := (2^n n!\sqrt{\pi})^{-1/2} \exp\{-t^2/2\}H_n(t)$, where $H_n, t \in \mathbb{R}$ are the Hermite polynomials (see [9]). For $d > 1$, $h_n(x_1, \cdots, x_d) := h_{n_1}(x_1)\cdots h_{n_d}(x_d)$ for all $(x_1, \cdots, x_d) \in \mathbb{R}^d, n \in \mathbb{Z}^d_+$, where the Hermite functions on the right hand side are one-dimensional. We define the Hermite-Sobolev spaces $\mathcal{S}_p, p \in \mathbb{R}$ as the completion of $\mathcal{S}$ in $\|\cdot\|_p$. Note that the dual space $\mathcal{S}'_p$ is isometrically isomorphic with $\mathcal{S}_{-p}$ for $p \geq 0$. We also have $\mathcal{S} = \bigcap_p(\mathcal{S}_p, \|\cdot\|_p), \mathcal{S}' = \bigcup_{p>0}(\mathcal{S}_{-p}, \|\cdot\|_{-p})$ and $\mathcal{S}_0 = \mathcal{L}^2(\mathbb{R}^d)$.

For $x \in \mathbb{R}^d$, let τ_x denote the translation operators on $\mathcal{S}$ defined by $(\tau_x\phi)(y) := \phi(y - x), \forall y \in \mathbb{R}^d$. These operators can be extended to $\tau_x : \mathcal{S}' \to \mathcal{S}'$ by

$$\langle \tau_x\phi\,,\,\psi\rangle := \langle \phi\,,\,\tau_{-x}\psi\rangle\,,\,\forall\psi \in \mathcal{S}.$$

Proposition 2.1. *The translation operators $\tau_x, x \in \mathbb{R}^d$ have the following properties:*

(a) *([14, Theorem 2.1]) For $x \in \mathbb{R}^d$ and any $p \in \mathbb{R}$, $\tau_x : \mathcal{S}_p \to \mathcal{S}_p$ is a bounded linear map. In particular, there exists a real polynomial P_k of degree $k = 2(\lfloor |p| \rfloor + 1)$ such that*

$$\|\tau_x \phi\|_p \leq P_k(|x|)\|\phi\|_p, \ \forall \phi \in \mathcal{S}_p,$$

where $|x|$ denotes the Euclidean norm of x.

(b) *([15, Proposition 3.1]) Fix $\phi \in \mathcal{S}_p$ for some $p \in \mathbb{R}$. The map $x \in \mathbb{R}^d \mapsto \tau_x \phi \in \mathcal{S}_p$ is continuous.*

3. Main Results

3.1. Notations and hypotheses. We use the following notations throughout the paper.

- The set of positive integers will be denoted by $\mathbb{N}$. Recall that for $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. The transpose of any element $x \in \mathbb{R}^{n \times m}$ will be denoted by x^t.
- For any $r > 0$, define $\mathcal{O}(0, r) := \{x \in \mathbb{R}^d : |x| < r\}$. Then $\overline{\mathcal{O}(0, r)} = \{x \in \mathbb{R}^d : |x| \leq r\}$ and $\mathcal{O}(0, r)^c = \{x \in \mathbb{R}^d : |x| \geq r\}$.
- Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P\big)$ be a filtered complete probability space satisfying the usual conditions viz. $\mathcal{F}_0$ contains all $A \in \mathcal{F}$, s.t. $P(A) = 0$ and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, t \geq 0$.
- Let $p > 0$. Let $\sigma = (\sigma_{ij})_{d \times d}, b = (b_1, \cdots, b_d)^t$ be such that $\sigma_{ij}, b_i : \Omega \to \mathcal{S}_p$ are $\mathcal{F}_0$ measurable and
$$\beta := \sup\{\|\sigma_{ij}(\omega)\|_p, \|b_i(\omega)\|_p : \omega \in \Omega, 1 \leq i, j \leq d\} < \infty. \qquad (\sigma\mathbf{b})$$
- Define $\bar{\sigma} : \Omega \times \mathbb{R}^d \times \mathcal{S}_{-p} \to \mathbb{R}^{d \times d}$ and $\bar{b} : \Omega \times \mathbb{R}^d \times \mathcal{S}_{-p} \to \mathbb{R}^d$ by $\bar{\sigma}(\omega, z; y) := \langle \sigma(\omega), \tau_z y \rangle$ and $\bar{b}(\omega, z; y) := \langle b(\omega), \tau_z y \rangle$, where
$$(\langle \sigma(\omega), \tau_z y \rangle)_{ij} := \langle \sigma_{ij}(\omega), \tau_z y \rangle, \ (\langle b(\omega), \tau_z y \rangle)_i := \langle b_i(\omega), \tau_z y \rangle.$$
- Let $F : \Omega \times \mathcal{S}_{-p} \times \mathcal{O}(0,1) \to \mathbb{R}^d$ and $G : \Omega \times \mathcal{S}_{-p} \times \mathcal{O}(0,1)^c \to \mathbb{R}^d$ be $\mathcal{F}_0 \otimes \mathcal{B}(\mathcal{S}_p) \otimes \mathcal{B}(\mathcal{O}(0,1))/\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{F}_0 \otimes \mathcal{B}(\mathcal{S}_p) \otimes \mathcal{B}(\mathcal{O}(0,1)^c)/\mathcal{B}(\mathbb{R}^d)$ measurable respectively. Here $\mathcal{B}(\mathcal{K})$ denotes the Borel σ-field of set $\mathcal{K}$.
- Define $\bar{F} : \Omega \times \mathbb{R}^d \times \mathcal{O}(0,1) \times \mathcal{S}_{-p} \to \mathbb{R}^d$, $\bar{G} : \Omega \times \mathbb{R}^d \times \mathcal{O}(0,1)^c \times \mathcal{S}_{-p} \to \mathbb{R}^d$ by $\bar{F}(\omega, z, x; y) := F(\omega, \tau_z y, x)$, $\bar{G}(\omega, z, x; y) := G(\omega, \tau_z y, x)$.
- Let $\{B_t\}$ denote a standard Brownian motion and let N denote a Poisson random measure driven by a Lévy measure ν. $\widetilde{N}$ will denote the corresponding compensated random measure.

Consider the following SDE in $\mathbb{R}^d$,

$$\begin{aligned} dU_t &= \bar{b}(U_{t-}; \xi)dt + \bar{\sigma}(U_{t-}; \xi) \cdot dB_t + \int_{(0<|x|<1)} \bar{F}(U_{t-}, x; \xi)\, \widetilde{N}(dtdx) \\ &\quad + \int_{(|x| \geq 1)} \bar{G}(U_{t-}, x; \xi)\, N(dtdx), \quad t \geq 0 \\ U_0 &= \kappa, \end{aligned} \tag{3.1}$$

where ξ is an $\mathcal{S}_{-p}$-valued $\mathcal{F}_0$-measurable random variable and κ is an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variable. We also assume that ξ, κ, B and N are independent of each other and that the filtration $(\mathcal{F}_t)$ is generated by these random variables. In particular, $\mathcal{F}_0$ is generated by κ and ξ. Note that the i-th component of $\int_0^t \bar{\sigma}(U_{s-};\xi)\cdot dB_s$ is $\sum_{j=1}^d \int_0^t \bar{\sigma}_{ij}(U_{s-};\xi)\, dB_s^j$. We list some hypotheses.

(F1) For all $\omega \in \Omega$ and $x \in \mathcal{O}(0,1)$ there exists a constant $C_x \geq 0$ s.t.

$$|F(\omega, y_1, x) - F(\omega, y_2, x)| \leq C_x \|y_1 - y_2\|_{-p-\frac{1}{2}}, \forall y_1, y_2 \in \mathcal{S}_{-p}. \tag{3.2}$$

We assume C_x to depend only on x and independent of ω. Since $\|y\|_{-p-\frac{1}{2}} \leq \|y\|_{-p}, \forall y \in \mathcal{S}_{-p}$, we have

$$|F(\omega, y_1, x) - F(\omega, y_2, x)| \leq C_x \|y_1 - y_2\|_{-p}, \forall y_1, y_2 \in \mathcal{S}_{-p}.$$

(F2) The constant C_x mentioned above has the following properties, viz.

$$\sup_{|x|<1} C_x < \infty, \quad \int_{(0<|x|<1)} C_x^2\, \nu(dx) < \infty.$$

(F3) $\sup_{\omega\in\Omega, |x|<1} |F(\omega, 0, x)| < \infty$ and $\sup_{\omega \in \Omega} \int_{(0<|x|<1)} |F(\omega, 0, x)|^2\, \nu(dx) < \infty$.

(G1) The mapping $y \to G(\omega, y, x)$ is continuous for all $x \in \mathcal{O}(0,1)^c$ and $\omega \in \Omega$.

Remark 3.1. Examples of coefficients F and G satisfying the above hypotheses can be constructed. See [5, Example 3.1].

Lemma 3.2 ([5, Lemma 3.2]). *Assume **(F1)**, **(F2)** and **(F3)**. Then, for any bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$ the following are true.*

(i) $\sup_{\omega\in\Omega, y\in\mathcal{K}, |x|<1} |F(\omega, y, x)| < \infty$.

(ii) $\sup_{\omega\in\Omega, y\in\mathcal{K}} \int_{(0<|x|<1)} |F(\omega, y, x)|^2 \nu(dx) =: \alpha(\mathcal{K}) < \infty$.

(iii) $\sup_{\omega\in\Omega, y\in\mathcal{K}} \int_0^t \int_{(0<|x|<1)} |F(\omega, y, x)|^4 \nu(dx) ds < \infty$ *for all* $0 \leq t < \infty$.

Using the continuity result in Proposition 2.1 the next result follows.

Lemma 3.3 ([5, Lemma 3.3]). *Suppose **(G1)** holds. Then the map* $z \in \mathbb{R}^d \to \bar{G}(\omega, z, x; \xi(\omega)) = G(\omega, \tau_z\xi(\omega), x) \in \mathbb{R}^d$ *is continuous for all* $x \in \mathcal{O}(0,1)^c$ *and* $\omega \in \Omega$.

3.2. Global Lipschitz coefficients. In this subsection, we establish the existence and uniqueness of strong solutions of (3.1) under 'global Lipschitz' coefficients $\bar{\sigma}, \bar{b}, \bar{F}$. To do this we first study the same problem for the corresponding reduced equation, viz.

$$\begin{aligned} dU_t &= \bar{b}(U_{t-};\xi)dt + \bar{\sigma}(U_{t-};\xi)\cdot dB_t + \int_{(0<|x|<1)} \bar{F}(U_{t-}, x; \xi)\, \widetilde{N}(dtdx), \quad t \geq 0 \\ U_0 &= \kappa; \end{aligned} \tag{3.3}$$

with ξ and κ as in (3.1). Later, in Theorem 3.5 we prove the result for equation (3.1).

Theorem 3.4. *Let (σ**b**), **(F1)**, **(F2)** and **(F3)** hold. Suppose the following conditions are satisfied.*

(i) κ, ξ *are* $\mathcal{F}_0$ *measurable, as stated in* (3.1).

(ii) (Global Lipschitz in z, locally in y) For every bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$, there exists a constant $C(\mathcal{K}) > 0$ such that for all $z_1, z_2 \in \mathbb{R}^d,\ y \in \mathcal{K}$ and $\omega \in \Omega$

$$\begin{aligned}&|\bar{b}(\omega, z_1; y) - \bar{b}(\omega, z_2; y)|^2 + |\bar{\sigma}(\omega, z_1; y) - \bar{\sigma}(\omega, z_2; y)|^2 \\ &+ \int_{(0<|x|<1)} |\bar{F}(\omega, z_1, x; y) - \bar{F}(\omega, z_2, x; y)|^2\, \nu(dx) \leq C(\mathcal{K})\, |z_1 - z_2|^2.\end{aligned} \tag{3.4}$$

Then (3.3) has an $(\mathcal{F}_t)$ adapted strong solution $\{X_t\}$ with rcll paths. Pathwise uniqueness of solutions also holds, i.e. if $\{X^1_t\}$ is another such solution, then $P(X_t = X^1_t, t \geq 0) = 1$.

Proof. We split the proof in the following three steps, depending on assumptions on the random variables κ and ξ.

Step 1: κ, ξ are $\mathcal{F}_0$ measurable with $\mathbb{E}|\kappa|^2 < \infty$ and $\sup_{\omega\in\Omega} \|\xi(\omega)\|_{-p} < \infty$.
Step 2: κ, ξ are $\mathcal{F}_0$ measurable with $\mathbb{E}|\kappa|^2 < \infty$.
Step 3: κ, ξ are $\mathcal{F}_0$ measurable.

Positive constants appearing in our computations may be written as γ and may change its values from line to line.

Step 1: The existence is established by Picard iterations and the uniqueness by Gronwall inequality arguments. This follows the standard approach as in [11, Theorem 5.2.1], where SDEs driven by Brownian motion were considered. In the present case, we get the linear growth of the coefficients directly from the structure of the coefficients, see (3.9) below.

First we prove the uniqueness. Let $\{U^1_t\}$ and $\{U^2_t\}$ be two solutions of (3.3). Define, for $\omega \in \Omega$

$$\begin{aligned}\Theta(t, \omega) &:= \bar{b}(\omega, U^1_{t-}(\omega); \xi(\omega)) - \bar{b}(\omega, U^2_{t-}(\omega); \xi(\omega)),\\ \Xi(t, \omega) &:= \bar{\sigma}(\omega, U^1_{t-}(\omega); \xi(\omega)) - \bar{\sigma}(\omega, U^2_{t-}(\omega); \xi(\omega)),\\ \Psi(t, x, \omega) &:= \bar{F}(\omega, U^1_{t-}(\omega), x; \xi(\omega)) - \bar{F}(\omega, U^2_{t-}(\omega), x; \xi(\omega)).\end{aligned}$$

Note that

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq s\leq t} |U^1_s - U^2_s|^2\right)\\ &= \mathbb{E}\left(\sup_{0\leq s\leq t}\left[\left|\int_0^s \Theta(u)du + \int_0^s \Xi(u)\cdot dB_u + \int_0^s\int_{(0<|x|<1)} \Psi(u, x)\, \widetilde{N}(dudx)\right|^2\right]\right)\\ &\leq \mathbb{E}\left(\sup_{0\leq s\leq t}\left[3\left|\int_0^s \Theta(u)du\right|^2 + 3\left|\int_0^s \Xi(u)\cdot dB_u\right|^2\right.\right.\\ &\qquad\qquad \left.\left. +3\left|\int_0^s\int_{(0<|x|<1)} \Psi(u, x)\, \widetilde{N}(dudx)\right|^2\right]\right)\end{aligned}$$

Using Doob's $\mathcal{L}^2$ maximal inequality and Itô isometry, we have for some positive constant γ,

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le s\le t}|U_s^1-U_s^2|^2\right)\\&\le 3t\ \mathbb{E}\int_0^t|\Theta(s)|^2ds+12\mathbb{E}\int_0^t|\Xi(s)|^2ds+12\mathbb{E}\int_0^t\int_{(0<|x|<1)}|\Psi(s,x)|^2\nu(dx)ds.\end{aligned}$$

Using (3.4), we have

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le s\le t}|U_s^1-U_s^2|^2\right)&\le 3\gamma(t+8)\mathbb{E}\int_0^t|U_{s-}^1-U_{s-}^2|^2ds\\&\le 3\gamma(t+8)\int_0^t\mathbb{E}\left(\sup_{0\le u\le s}|U_u^1-U_u^2|^2\right)ds.\end{aligned}\tag{3.5}$$

We then obtain the uniqueness of the solutions by a Gronwall inequality argument.

To show the existence of a strong solution, we use Picard iteration. Set $U_t^{(0)}=\kappa$ and define

$$\begin{aligned}U_t^{(k+1)}:=\kappa&+\int_0^t\bar b(U_{s-}^{(k)};\xi)ds+\int_0^t\bar\sigma(U_{s-}^{(k)};\xi)\cdot dB_s\\&+\int_0^t\int_{(0<|x|<1)}\bar F(U_{s-}^{(k)},x;\xi)\,\widetilde N(dsdx),\end{aligned}\tag{3.6}$$

for all $k\ge 0$. Fix $M\in\mathbb{N}$. For $k\ge 1$, $t\in[0,M]$ we have

$$\mathbb{E}\left(\sup_{0\le s\le t}|U_s^{(k+1)}-U_s^{(k)}|^2\right)\le 3\gamma(M+8)\int_0^t\mathbb{E}\left(\sup_{0\le u\le s}|U_u^{(k)}-U_u^{(k-1)}|^2\right)ds.\tag{3.7}$$

By (3.4), there exists a constant $C=C(\mathrm{Range}(\xi))$ such that for $z\in\mathbb{R}^d$, $y\in\mathrm{Range}(\xi)$

$$\begin{aligned}&|\bar b(\omega,z;y)-\bar b(\omega,0;y)|^2+|\bar\sigma(\omega,z;y)-\bar\sigma(\omega,0;y)|^2\\&+\int_{(0<|x|<1)}|\bar F(\omega,z,x;y)-\bar F(\omega,0,x;y)|^2\,\nu(dx)\le C\,|z|^2.\end{aligned}\tag{3.8}$$

Using (σ**b**), we have $|\bar b(\omega,0;y)|=|\langle b(\omega),y\rangle|\le\beta\sqrt{d}\|y\|_{-p}$ and $|\bar\sigma(\omega,0;y)|=|\langle\sigma(\omega),y\rangle|\le\beta d\|y\|_{-p}$. From **(F1)**, we have

$$|\bar F(\omega,0,x;y)|=|F(\omega,y,x)|\le C_x\|y\|_{-p}+|F(\omega,0,x)|.$$

Therefore, using (3.8), **(F2)** and **(F3)**, there exists a positive constant $D=D(\mathrm{Range}(\xi))$ such that

$$|\bar b(\omega,z;y)|^2+|\bar\sigma(\omega,z;y)|^2+\int_{(0<|x|<1)}|\bar F(\omega,z,x;y)|^2\,\nu(dx)\le D\,(1+|z|^2).\tag{3.9}$$

As in (3.5), using (3.6), Doob's $\mathcal{L}^2$ maximal inequality and Itô isometry and (3.9) we get

$$\mathbb{E}\left(\sup_{0\le s\le t}|U_s^{(1)}-U_s^{(0)}|^2\right)\le(3t^2+24t)D\ \mathbb{E}(1+|\kappa|^2).\tag{3.10}$$

Therefore by induction from (3.7), there exists a positive constant $\tilde{C}$ s.t.

$$\mathbb{E}\left(\sup_{0\le s\le t}|U_s^{(k+1)}-U_s^{(k)}|^2\right)\le \frac{(\tilde{C}t)^{k+1}}{(k+1)!},\ \forall k\ge 0,\ t\in[0,M]. \tag{3.11}$$

For positive integers $m,\ n$ with $m>n$, we have

$$\begin{aligned}
&\lim_{m,n\to\infty}\mathbb{E}\sup_{0\le t\le M}|U_t^{(m)}-U_t^{(n)}|^2\\
&=\lim_{m,n\to\infty}\mathbb{E}\sup_{0\le t\le M}\left|\sum_{k=n}^{m-1}\left(U_t^{(k+1)}-U_t^{(k)}\right)\right|^2\\
&\le\lim_{m,n\to\infty}\mathbb{E}\left(\sum_{k=n}^{m-1}\sup_{0\le t\le M}\left|U_t^{(k+1)}-U_t^{(k)}\right|\right)^2 \qquad (3.12)\\
&=\lim_{m,n\to\infty}\mathbb{E}\left(\sum_{k=n}^{m-1}\sup_{0\le t\le M}\left|U_t^{(k+1)}-U_t^{(k)}\right|k\frac{1}{k}\right)^2\\
&\le\lim_{n\to\infty}\left(\sum_{k=n}^{\infty}\mathbb{E}\sup_{0\le t\le M}\left|U_t^{(k+1)}-U_t^{(k)}\right|^2k^2\right)\left(\sum_{k=n}^{\infty}k^{-2}\right).
\end{aligned}$$

The second series on the right hand side above converges. By (3.11), the first series is bounded, since $\sum_{k=n}^{\infty}\frac{(\tilde{C}M)^{k+1}}{(k+1)!}k^2\to 0$ as $n\to\infty$. Therefore $\{U_t^{(m)}:m\in\mathbb{N}\}$ is Cauchy and hence converges to some $\{X_t\}_{t\in[0,M]}$ in $\mathcal{L}^2(\lambda\times P)$, where λ denotes the Lebesgue measure on $[0,M]$.

Applying the Chebyshev-Markov inequality in (3.11), we get

$$P\left(\sup_{0\le s\le t}|U_s^{(k+1)}-U_s^{(k)}|\ge\frac{1}{2^{k+1}}\right)\le\frac{(4\tilde{C}t)^{k+1}}{(k+1)!}.$$

By Borel-Cantelli lemma

$$P\left(\limsup_{k\to\infty}\sup_{0\le s\le t}|U_s^{(k+1)}-U_s^{(k)}|\ge\frac{1}{2^{k+1}}\right)=0.$$

Therefore, we conclude that $\{U^{(k)}\}$ is almost surely uniformly convergent on $[0,M]$ to $\{X_t\}$, which is adapted and rcll. Using (3.9) and the fact that a.s. $\{X_t\}$ has at most countably many jumps, we have

$$\begin{aligned}
\mathbb{E}\int_0^M\int_{(0<|x|<1)}|\bar{F}(X_{s-},x;\xi)|^2\nu(dx)ds&\le\mathbb{E}\int_0^M D(1+|X_{s-}|^2)ds\\
&\le D\left[M+\|X\|^2_{\mathcal{L}^2(\lambda\times P)}\right]<\infty.
\end{aligned}$$

Therefore $\{\int_0^t\int_{(0<|x|<1)}\bar{F}(X_{s-},x;\xi)\,\widetilde{N}(dsdx)\}_{t\in[0,M]}$ exists. Similarly, we can show the existence of $\{\int_0^t\bar{\sigma}(X_{s-};\xi)\cdot dB_s\}_{t\in[0,M]}$ and $\{\int_0^t\bar{b}(X_{s-};\xi)ds\}_{t\in[0,M]}$.

By Itô isometry and (3.4), we have the following convergence in $\mathcal{L}^2(P)$, viz.

$$\int_0^t\int_{(0<|x|<1)}\bar{F}(U_{s-}^{(k)},x;\xi)\,\widetilde{N}(dsdx)\xrightarrow{k\to\infty}\int_0^t\int_{(0<|x|<1)}\bar{F}(X_{s-},x;\xi)\,\widetilde{N}(dsdx),$$

for each $t \in [0,M]$. Similarly, we conclude that $\int_0^t \bar{\sigma}(U^{(k)}_{s-};\xi)\cdot dB_s \to \int_0^t \bar{\sigma}(X_{s-};\xi)\cdot dB_s$ and $\int_0^t \bar{b}(U^{(k)}_{s-};\xi)ds \to \int_0^t \bar{b}(X_{s-};\xi)ds$ in $\mathcal{L}^2(P)$ as $k\to\infty$, for each $t\in[0,M]$. Since $\{X_t\}$ is rcll, from (3.6), we have a.s. $\forall t \in [0,M]$,

$$X_t = \kappa + \int_0^t \bar{b}(X_{s-};\xi)ds + \int_0^t \bar{\sigma}(X_{s-};\xi)\cdot dB_s + \int_0^t \int_{(0<|x|<1)} \bar{F}(X_{s-},x;\xi)\,\widetilde{N}(dsdx).$$

Suppose $\{X_t^{(M)}\}$ and $\{X_t^{(M+1)}\}$ denote the solutions up to time M and $M+1$ respectively. Then, by the uniqueness, $\{X_t^{(M+1)}\}_{t\in[0,M]}$ is indistinguishable from $\{X_t^{(M)}\}$ on $[0,M]$. Using this consistency, we obtain the solution of (3.3) on the time interval $[0,\infty)$. This concludes the proof for Step 1.

Step 2: We follow the technique given in [7, Theorem 3.3], where SDEs driven by Brownian motion were considered. For $k\in\mathbb{N}$, define $\chi_k := 1_{\{\|\xi\|_{-p}\le k\}}$ and let $\xi^{(k)} := \chi_k\ \xi$. Let $U^{(k)}$ be the solution of (3.3) with the initial condition $\xi^{(k)}$. Our aim is to show that $\chi_k U^{(k)} = \chi_k U^{(k+1)}$. Let $U_n^{(k)}$ and $U_n^{(k+1)}$ be the approximations of $U^{(k)}$ and $U^{(k+1)}$ obtained in Step 1 above. Now,

$$U_0^{(k)}(t) = \kappa,\ U_0^{(k+1)}(t) = \kappa \text{ and } \chi_k U_0^{(k)}(t) = \chi_k U_0^{(k+1)}(t).$$

Observe that, for $\omega \in \Omega$

$$\chi_k(\omega)\bar{b}(\omega, U_0^{(k)}(s-)(\omega);\xi^{(k)}(\omega)) = \chi_k(\omega)\bar{b}(\omega, U_0^{(k+1)}(s-)(\omega);\xi^{(k+1)}(\omega)).$$

Similar equalities hold for coefficients $\bar{\sigma}$ and $\bar{F}$. Using (3.6) and these equalities, we have a.s. $t\ge 0$,

$$\begin{aligned}
\chi_k U_1^{(k)}(t) &= \chi_k\ \kappa + \int_0^t \chi_k \bar{b}(U_0^{(k)}(s-);\xi^{(k)})ds + \int_0^t \chi_k\bar{\sigma}(U_0^{(k)}(s-);\xi^{(k)})\cdot dB_s\\
&\quad + \int_0^t\int_{(0<|x|<1)} \chi_k \bar{F}(U_0^{(k)}(s-),x;\xi^{(k)})\,\widetilde{N}(dsdx)\\
&= \chi_k\ \kappa + \int_0^t \chi_k \bar{b}(U_0^{(k+1)}(s-);\xi^{(k+1)})ds\\
&\quad + \int_0^t \chi_k\bar{\sigma}(U_0^{(k+1)}(s-);\xi^{(k+1)})\cdot dB_s\\
&\quad + \int_0^t\int_{(0<|x|<1)} \chi_k \bar{F}(U_0^{(k+1)}(s-),x;\xi^{(k+1)})\,\widetilde{N}(dsdx)\\
&= \chi_k U_1^{(k+1)}(t).
\end{aligned}$$

By induction a.s. $t\ge 0, \chi_k U_n^{(k)}(t) = \chi_k U_n^{(k+1)}(t)$.

Letting n go to infinity and using the generalized Lebesgue DCT (see [7, Theorem 3.4]), we have, a.s. $\forall t\in[0,T]$, $\chi_k U^{(k)}(t) = \chi_k U^{(k+1)}(t)$. Note that $P\big(\bigcup_k\{\chi_k = 1\}\big) = 1$. Now define

$$X_t(\omega) := U^{(k)}(t)(\omega), \quad \text{if } \|\xi(\omega)\|_{-p}\le k.$$

Observe that, a.s. $\forall t\in[0,T], \chi_k U^{(k)}(t) = \chi_k X_t$. It is easy to check that $\{X_t\}$ satisfies (3.3).

To prove the uniqueness, let $\{X_t\}$ and $\{Y_t\}$ be two solutions of (3.3). Define

$$\widetilde{F}(\omega, z, x; y) := 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(y)\bar{F}(\omega, z, x; 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(y)y),$$

and $X_t^k := \chi_k X_t$, for $\omega \in \Omega, k \in \mathbb{N}$. Similarly define $\{Y_t^k\}$ for $k \in \mathbb{N}$. Observe that

$$\begin{aligned}\widetilde{F}(\omega, z, x; \xi(\omega)) &= 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(\xi(\omega))\bar{F}\big(\omega, z, x; 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(\xi(\omega))\xi(\omega)\big)\\ &= 1_{\{\tilde{\omega}:\|\xi(\tilde{\omega})\|_{-p}\leq k\}}(\omega)\bar{F}\big(\omega, z, x; 1_{\{\tilde{\omega}:\|\xi(\tilde{\omega})\|_{-p}\leq k\}}(\omega)\xi(\omega)\big),\end{aligned}$$

and

$$\begin{aligned}\chi_k(\omega)\bar{b}(\omega, X_{s-}(\omega); \xi(\omega)) &= \bar{b}(\omega, X_{s-}^k(\omega); \xi^k(\omega)),\\ \chi_k(\omega)\bar{\sigma}(\omega, X_{s-}(\omega); \xi(\omega)) &= \bar{\sigma}(\omega, X_{s-}^k(\omega); \xi^k(\omega)),\\ \chi_k(\omega)\bar{F}(\omega, X_{s-}(\omega), x; \xi(\omega)) &= \chi_k(\omega)\bar{F}(\omega, X_{s-}^k(\omega), x; \chi_k(\omega)\xi^k(\omega))\\ &= \widetilde{F}(\omega, X_{s-}^k(\omega), x; \xi^k(\omega)).\end{aligned}$$

Therefore, a.s.

$$\begin{aligned}X_t^k &= \chi_k X_t\\ &= \chi_k\ \kappa + \int_0^t \bar{b}(X_{s-}^k; \xi^k)ds + \int_0^t \bar{\sigma}(X_{s-}^k; \xi^k)\cdot dB_s\\ &\quad + \int_0^t \int_{(0<|x|<1)} \widetilde{F}(X_{s-}^k, x; \xi^k)\,\widetilde{N}(dsdx).\end{aligned} \tag{3.13}$$

Now, in (3.13) ξ^k is norm bounded. Observe that

$$\begin{aligned}&\int_{(0<|x|<1)} |\widetilde{F}(\omega, z_2, x; y) - \widetilde{F}(\omega, z_1, x; y)|^2\nu(dx)\\ &= 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(y)\\ &\quad\times \int_{(0<|x|<1)} |\bar{F}(\omega, z_1, x; 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(y)y) - \bar{F}(\omega, z_2, x; 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(y)y)|^2\nu(dx)\\ &= 1_{\{\tilde{y}:\|\tilde{y}\|_{-p}\leq k\}}(y)\int_{(0<|x|<1)} |\bar{F}(\omega, z_1, x; y) - \bar{F}(\omega, z_2, x; y)|^2\nu(dx)\\ &\leq C(\{\tilde{y} : \|\tilde{y}\|_{-p} \leq k\})|z_1 - z_2|^2,\end{aligned}$$

for any bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$ and $y \in \mathcal{K}$. Since (3.4) holds for $\bar{b}$, $\bar{\sigma}$ and $\bar{F}$, it also holds for $\bar{b}$, $\bar{\sigma}$ and $\widetilde{F}$. By the uniqueness in Step 1, we conclude that $\{X_t^k\}$ is the unique solution of (3.3) with initial condition $\chi_k\ \kappa$ and in particular,

$$\chi_k(\omega)X_t = X_t^k = Y_t^k = \chi_k(\omega)Y_t.$$

Since k is arbitrary, therefore, a.s. $\forall t \in [0, T]$, $X_t = Y_t$. This completes the proof for Step 2.

Step 3: We follow the argument given in [1, Theorem 6.2.3]. Define $\Omega_M := \{\omega \in \Omega : |\kappa| \leq M\}$ for each $M \in \mathbb{N}$. Then $\Omega = \bigcup_{M\in\mathbb{N}} \Omega_M$ and $\Omega_L \subseteq \Omega_M$ whenever $L \leq M$.

Let $\kappa^M(\omega) := 1_{\{|\kappa|\le M\}}(\omega)\kappa(\omega)$. Note that $\kappa^M \in \mathcal{L}^2$. By Step 2, there exists a unique solution, say $\{X_t^{\kappa^M}\}$, of the reduced equation (3.3) for the initial condition κ^M, i.e. a.s. $t \ge 0$

$$\begin{aligned} X_t^{\kappa^M} &= \kappa^M + \int_0^t \bar{b}(X_{s-}^{\kappa^M};\xi)ds + \int_0^t \bar{\sigma}(X_{s-}^{\kappa^M};\xi)\cdot dB_s \\ &\quad + \int_0^t \int_{(0<|x|<1)} \bar{F}(X_{s-}^{\kappa^M},x;\xi)\,\widetilde{N}(dsdx). \end{aligned}$$

We first show a.s. $1_{\{|\kappa|\le L\}}(\omega)X_t^{\kappa^L}(\omega) = 1_{\{|\kappa|\le L\}}(\omega)X_t^{\kappa^M}(\omega), t\ge 0$ for all $M \ge L$. Define

$$\widetilde{F}(\omega,z,x;y) := 1_{\{|\kappa|\le L\}}(\omega)\bar{F}(\omega,z,x;y).$$

Now, $\{1_{\{|\kappa|\le L\}}X_t^{\kappa^L}\}$ and $\{1_{\{|\kappa|\le L\}}X_t^{\kappa^M}\}$ both satisfy the reduced equation

$$\begin{aligned} dX_t &= \bar{b}(X_{t-};1_{\{|\kappa|\le L\}}\xi)dt + \bar{\sigma}(X_{t-};1_{\{|\kappa|\le L\}}\xi)\cdot dB_t \\ &\quad + \int_{(0<|x|<1)} \widetilde{F}(X_{t-},x;1_{\{|\kappa|\le L\}}\xi)\,\widetilde{N}(dtdx), \\ X_0 &= \kappa^L. \end{aligned} \tag{3.14}$$

It is easy to check that $\bar{b}$, $\bar{\sigma}$, $\widetilde{F}$ satisfy (3.4). Then by the uniqueness in Step 2 for all $M \ge L$ a.s.

$$1_{\{|\kappa|\le L\}}X_t^{\kappa^L} = 1_{\{|\kappa|\le L\}}X_t^{\kappa^M},\ t\ge 0.$$

Since Ω_M increases to Ω, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$, such that $P(\Omega_n) > 1-\epsilon, \forall n > M$. Hence,

$$P\left(\sup_{t\ge 0}|X_t^{\kappa^m} - X_t^{\kappa^n}| > \delta\right) < \epsilon,\ \forall\delta > 0, \forall m,n > M.$$

Therefore the sequence of processes $\{X^{\kappa^n}\}_{n\in\mathbb{N}}$ is uniformly Cauchy in probability and so is uniformly convergent in probability to a process, say $\{X_t\}$. We extract a subsequence for which the convergence holds uniformly and almost surely. This convergence implies that $\{X_t\}$ has rcll paths and solves (3.3).

To prove the uniqueness, we consider the solution $\{X_t\}$ constructed above and compare it with any arbitrary solution $\{X_t'\}_{t\ge 0}$ of (3.3). We claim that for all $M \ge L$, $X_t'(\omega) = X_t^{\kappa^M}(\omega)$ for all $t \ge 0$ and almost all $\omega \in \Omega_L$. Suppose for some $M \ge L$, it doesn't hold. Define

$$X_t''^{\kappa^M}(\omega) := \begin{cases} X_t'(\omega) & \text{for } \omega \in \Omega_L, \\ X_t^{\kappa^M}(\omega), & \text{for } \omega \in \Omega_L^c. \end{cases}$$

Then X''^{κ^M} and X^{κ^M} are two distinct solutions of (3.3) with the same initial condition κ^M, which is a contradiction. This proves our claim. Next by applying a limiting argument we conclude that $P(X_t = X_t', \forall t \ge 0) = 1$. This completes the proof of Step 3 as well as the theorem. □

We now consider the SDE (3.1). The next result follows by the interlacing technique (see [1, Example 1.3.13, pp. 50-51]).

Theorem 3.5. *Suppose all the assumptions of Theorem 3.4 hold. In addition, assume that **(G1)** holds. Then there exists a unique rcll adapted solution to* (3.1).

Proof. We follow the proof of [1, Theorem 6.2.9]. We have already proved the existence and uniqueness of the reduced equation in Theorem 3.4. Now, we use the interlacing technique to complete the proof.

Let $\{\eta_n\}_{n\in\mathbb{N}}$ denote the arrival times for the jumps of the compound Poisson process $\{P_t\}_{t\geq 0}$, where each $P_t = \int_{(|x|\geq 1)} xN(t,dx)$. By Theorem 3.4 there exists a unique solution $\{\widetilde{U}^{(1)}_t\}$ to the reduced equation (3.3). Define

$$U_t := \begin{cases} \widetilde{U}^{(1)}_t; \quad \text{for } 0 \leq t < \eta_1 \\ \widetilde{U}^{(1)}_{\eta_1 -} + \bar{G}(\widetilde{U}^{(1)}_{\eta_1 -}, \triangle P_{\eta_1}; \xi); \quad \text{for } t = \eta_1 \\ U_{\eta_1} + \widetilde{U}^{(2)}_t - \widetilde{U}^{(2)}_{\eta_1}; \quad \text{for } \eta_1 < t < \eta_2 \\ U_{\eta_2 -} + \bar{G}(U_{\eta_2 -}, \triangle P_{\eta_2}; \xi); \quad \text{for } t = \eta_2 \\ \cdots \end{cases}$$

Here $\{\widetilde{U}^{(2)}_t\}$ denotes the unique solution to (3.3) with initial condition U_{η_1}. Then $\{U_t\}$ is an adapted rcll process and solves (3.1).

We show that the uniqueness follows by the interlacing structure. Let $\{\hat{U}_t\}$ be another solution of (3.1). Then by the uniqueness of the reduced equation, a.s.

$$\hat{U}_t = \widetilde{U}_t = U_t; \quad \text{for } 0 \leq t < \eta_1.$$

Since, a.s. $\hat{U}_{\eta_1 -} = \widetilde{U}_{\eta_1 -} = U_{\eta_1 -}$, we have a.s.

$$\hat{U}_{\eta_1} = \hat{U}_{\eta_1 -} + \bar{G}(\hat{U}_{\eta_1 -}, \triangle P_{\eta_1}; \xi) = \widetilde{U}_{\eta_1 -} + \bar{G}(\widetilde{U}_{\eta_1 -}, \triangle P_{\eta_1}; \xi) = U_{\eta_1}.$$

Since $\{\hat{U}_t\}$ has no large jump in the time interval (η_1, η_2) we have, a.s. for $t \in (\eta_1, \eta_2)$

$$\begin{aligned} \hat{U}_t &= \kappa + \int_0^t \bar{b}(\hat{U}_{s-};\xi)ds + \int_0^t \bar{\sigma}(\hat{U}_{s-};\xi)\cdot dB_s \\ &\quad + \int_0^t \int_{(0<|x|<1)} \bar{F}(\hat{U}_{s-}, x;\xi)\, \widetilde{N}(dsdx) + \int_0^t \int_{(|x|\geq 1)} \bar{G}(\hat{U}_{s-}, x;\xi) N(dsdx) \\ &= \hat{U}_{\eta_1} + \int_{\eta_1}^t \bar{b}(\hat{U}_{s-};\xi)ds + \int_{\eta_1}^t \bar{\sigma}(\hat{U}_{s-};\xi)\cdot dB_s \\ &\quad + \int_{\eta_1}^t \int_{(0<|x|<1)} \bar{F}(\hat{U}_{s-}, x;\xi)\, \widetilde{N}(dsdx) \\ &= \hat{U}_{\eta_1} + \int_0^{t-\eta_1} \bar{b}(\hat{U}_{\eta_1+s-};\xi)ds + \int_0^{t-\eta_1} \bar{\sigma}(\hat{U}_{\eta_1+s-};\xi)\cdot dB_{\eta_1+s} \\ &\quad + \int_0^{t-\eta_1} \int_{(0<|x|<1)} \bar{F}(\hat{U}_{\eta_1+s-}, x;\xi)\, \widetilde{N}^{\eta_1}_s(dsdx). \end{aligned} \tag{3.15}$$

We now describe $\{N^{\eta_1}_s\}$, which appeared in the last term of (3.15). For any set $H \subset \mathbb{R}^d$, which is bounded away from 0, i.e. $0 \notin \bar{H}$ and for any stopping time η, define

$$N_t^\eta(H) := (N_{t+\eta}(H) - N_\eta(H))\, 1_{(\eta<\infty)}.$$

By strong Markov property [1, Theorem 2.2.11], we have $\mathbb{E}[e^{i\lambda N_t^\eta(H)}] = \mathbb{E}[e^{i\lambda N_t(H)}]$, $\{N_t^\eta\}$ is independent of $\mathcal{F}_\eta$, has rcll paths and is $(\mathcal{F}_{\eta+t})$ adapted. Furthermore, $\mathbb{E}[N_t^\eta(H)] = t\nu(H) = \mathbb{E}[N_t(H)]$.

Note that the last equality of (3.15) is written in the reduced equation form. Since $\{U_t\}$ also solves the same reduced equation, by Theorem 3.4 a.s. $\hat{U}_t = U_t$ for $\eta_1 < t < \eta_2$. In particular, a.s. $\hat{U}_{\eta_2-} = U_{\eta_2-}$ and hence, a.s.

$$\hat{U}_{\eta_2} = \hat{U}_{\eta_2-} + \bar{G}(\hat{U}_{\eta_2-}, \triangle P_{\eta_2}; \xi) = U_{\eta_2-} + \bar{G}(U_{\eta_2-}, \triangle P_{\eta_2}; \xi) = U_{\eta_2}.$$

Continuing this way, we show that a.s. $U_t = \hat{U}_t, t \geq 0$. This completes the proof. □

3.3. Local Lipschitz coefficients. In the previous subsection, we have established the existence and uniqueness results under 'global Lipschitz' which we now extend for 'local Lipschitz' coefficients.

Let $\widehat{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^d$.

Theorem 3.6. *Let $(\sigma\mathbf{b})$, **(F1)**, **(F2)**, **(F3)** and **(G1)** hold. Suppose the following conditions are satisfied.*

- *(i) κ, ξ are $\mathcal{F}_0$-measurable.*
- *(ii) (Locally Lipschitz in z, locally in y) For every bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$ and positive integer n there exists a constant $C(\mathcal{K}, n) > 0$ s.t. for all $z_1, z_2 \in \mathcal{O}(0, n)$, $y \in \mathcal{K}$ and $\omega \in \Omega$*

$$\begin{aligned} &|\bar{b}(\omega, z_1; y) - \bar{b}(\omega, z_2; y)|^2 + |\bar{\sigma}(\omega, z_1; y) - \bar{\sigma}(\omega, z_2; y)|^2 \\ &+ \int_{(0<|x|<1)} |\bar{F}(\omega, z_1, x; y) - \bar{F}(\omega, z_2, x; y)|^2\, \nu(dx) \leq C(\mathcal{K}, n)\, |z_1 - z_2|^2. \end{aligned} \tag{3.16}$$

Then there exists an $(\mathcal{F}_t)$ stopping time η and an $(\mathcal{F}_t)$ adapted $\widehat{\mathbb{R}^d}$-valued process $\{X_t\}$ with rcll paths such that $\{X_t\}$ solves (3.1) *upto time η and $X_t = \infty$ for $t \geq \eta$. Further η can be identified as follows: $\eta = \lim_m \theta_m$ where $\{\theta_m\}$ are $(\mathcal{F}_t)$ stopping times defined by $\theta_m := \inf\{t \geq 0 : |X_t| \geq m\}$. This is also pathwise unique in this sense: if $(\{X'_t\}, \eta')$ is another such solution, then $P(X_t = X'_t, 0 \leq t < \eta \wedge \eta') = 1$.*

Proof. To prove the existence result, we first obtain a version of the 'global Lipschitz' condition (3.4) for $\bar{b}(\omega, z; y)$, $\bar{\sigma}(\omega, z; y)$, $\bar{F}(\omega, z, x; y)$ from our assumption on 'local Lipschitz' condition (3.16).

Let $n, m \in \mathbb{N}$ and let R be a positive real number. Let $h : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $|h(x) - h(y)| \leq C|x - y|$ for all x, y with $|x|, |y| \leq R$, where C is a positive constant. Define

$$h^R(x) := \begin{cases} h(x), \text{ if } |x| \leq R \\ \frac{2R-|x|}{R} \cdot h(Rx/|x|), \text{ if } R \leq |x| \leq 2R \\ 0, \text{ if } |x| \geq 2R. \end{cases}$$

By [6, Chapter 5, Exercise 3.1], h^R is Lipschitz continuous on $\mathbb{R}^n$. For every fixed y and ω, we construct $\bar{\sigma}^R(\omega, \cdot; y)$ for $\bar{\sigma}(\omega, \cdot; y)$ in the same way viz.,

$$\bar{\sigma}^R(\omega, z; y) := \begin{cases} \bar{\sigma}(\omega, z; y), \text{ for } |z| \leq R; \\ \frac{2R-|z|}{R} \cdot \bar{\sigma}\left(\omega, \frac{Rz}{|z|}; y\right), \text{ for } R \leq |z| \leq 2R, \\ 0, \text{ for } |z| \geq 2R. \end{cases}$$

Similarly define $\bar{b}^R(\omega, \cdot; y)$ and $\bar{F}^R(\omega, \cdot, x; y)$ for every fixed x, y and ω. Then using (3.16) and applying the above exercise, we conclude that $\bar{b}^R(\omega, z; y)$ and $\bar{\sigma}^R(\omega, z; y)$ are globally Lipschitz in z as in (3.4). We now show (3.4) holds for $\bar{F}^R(\omega, z, x; y)$.

By (3.16) and Lemma 3.2, for any $z \in \mathbb{R}^d$ with $|z| \leq R$ and any bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$, we have

$$\begin{aligned} \int_{(0<|x|<1)} \left|\bar{F}(\omega, z, x; y)\right|^2 \nu(dx) &\leq 2\int_{(0<|x|<1)} \left|\bar{F}(\omega, z, x; y) - \bar{F}(\omega, 0, x; y)\right|^2 \nu(dx) \\ &\quad + 2\int_{(0<|x|<1)} \left|\bar{F}(\omega, 0, x; y)\right|^2 \nu(dx) \\ &\leq 2C(\mathcal{K}, R)R^2 + 2\alpha(\mathcal{K}), \ \forall y \in \mathcal{K}. \end{aligned} \tag{3.17}$$

Fix $z_1, z_2 \in \mathbb{R}^d$, with $|z_1| \leq R$ and $R \leq |z_2| \leq 2R$. Then

$$\begin{aligned} &\int_{(0<|x|<1)} |\bar{F}^R(\omega, z_1, x; y) - \bar{F}^R(\omega, z_2, x; y)|^2 \nu(dx) \\ &= \int_{(0<|x|<1)} \left|\bar{F}(\omega, z_1, x; y) - \frac{2R - |z_2|}{R} \cdot \bar{F}\left(\omega, \frac{Rz_2}{|z_2|}, x; y\right)\right|^2 \nu(dx) \\ &\leq 2\int_{(0<|x|<1)} \left|\bar{F}(\omega, z_1, x; y) - \bar{F}\left(\omega, \frac{Rz_2}{|z_2|}, x; y\right)\right|^2 \nu(dx) \\ &\quad + 2\frac{||z_2| - R|^2}{R^2} \int_{(0<|x|<1)} \left|\bar{F}\left(\omega, \frac{Rz_2}{|z_2|}, x; y\right)\right|^2 \nu(dx) \\ &\leq 2C(\mathcal{K}, R)\left|z_1 - \frac{Rz_2}{|z_2|}\right|^2 + 2\frac{||z_2| - R|^2}{R^2}\left[2C(\mathcal{K}, R)R^2 + 2\alpha(\mathcal{K})\right] \\ &= |z_1 - z_2|^2 \left[6C(\mathcal{K}, R) + \frac{4}{R^2}\alpha(\mathcal{K})\right]. \end{aligned}$$

In the above calculation, we have used (3.17) and two inequalities, viz.

$$\left|z_1 - \frac{Rz_2}{|z_2|}\right|^2 \leq |z_1 - z_2|^2, \quad ||z_2| - R|^2 \leq |z_1 - z_2|^2.$$

We indicate a proof of the first inequality. We have

$$\begin{aligned} \left|z_1 - \frac{Rz_2}{|z_2|}\right|^2 \leq |z_1 - z_2|^2 &\iff |z_1|^2 + R^2 - 2\frac{R}{|z_2|}(z_1)^t z_2 \leq |z_1|^2 + |z_2|^2 - 2(z_1)^t z_2 \\ &\iff 2(|z_2| - R)\ (z_1)^t z_2 \leq |z_2|(|z_2|^2 - R^2) \\ &\iff 2(z_1)^t z_2 \leq (|z_2| + R)|z_2|. \end{aligned}$$

Of course, the last statement holds true, since $2(z_1)^t z_2 \leq 2|z_1||z_2| \leq (|z_2|+R)|z_2|$. The inequality $||z_2|-R|^2 \leq |z_1-z_2|^2$ follows from the observation that $z_1 \in \overline{\mathcal{O}(0,R)}$ and that the distance between $\overline{\mathcal{O}(0,R)}$ and z_2 is $||z_2|-R|$, which is achieved by the point $\frac{Rz_2}{|z_2|}$.

Similar arguments show that (3.4) holds for $\bar{F}^R$ for all $z_1, z_2 \in \mathbb{R}^d$. This shows that the 'global Lipschitz' regularity (3.4) holds for $\bar{b}^R$, $\bar{\sigma}^R$ and $\bar{F}^R$. Since $\bar{b}^R(\omega, 0; y) = \bar{b}(\omega, 0; y), \bar{\sigma}^R(\omega, 0; y) = \bar{\sigma}(\omega, 0; y)$ and $\bar{F}^R(\omega, 0, x; y) = \bar{F}(\omega, 0, x; y)$ for all $|x| < 1, y \in \mathcal{S}_{-p}$, the growth condition (3.9) can be established for $\bar{b}^R$, $\bar{\sigma}^R$ and $\bar{F}^R$ as done in Step 1 of Theorem 3.4. Then arguing as in Theorem 3.4 (Steps 1, 2 and 3) and Theorem 3.5, for $R \in \mathbb{N}$, we have the existence of a unique process $\{X_t^R\}$ satisfying a.s. for every $t \geq 0$

$$\begin{aligned} X_t^R = \kappa &+ \int_0^t \bar{b}^R(X_{s-}^R; \xi) ds + \int_0^t \bar{\sigma}^R(X_{s-}^R; \xi) \cdot dB_s \\ &+ \int_0^t \int_{(0<|x|<1)} \bar{F}^R(X_{s-}^R, x; \xi)\, \widetilde{N}(dsdx) + \int_0^t \int_{(|x|\geq 1)} \bar{G}(X_{s-}^R, x; \xi)\, N(dsdx). \end{aligned} \tag{3.18}$$

Let $\pi_i, i = 1, 2, \cdots$ denote the arrival times for the jumps of the compound Poisson process $\{P_t\}_{t\geq 0}$, where each $P_t = \int_{(|x|\geq 1)} xN(t, dx)$. Let $m, n \in \mathbb{N}$ and $m < n$. Consider the stopping times

$$\theta_{m,i}^n := \inf\{t \geq 0 : \ |X_t^m| \text{ Or } |X_t^n| \geq m\} \wedge \pi_i.$$

Take $i = 1$. Then $\{X_t^m\}$ and $\{X_t^n\}$ both satisfy the same reduced equation

$$\begin{aligned} dX_t &= \bar{b}^m(X_{t-}; \xi) dt + \bar{\sigma}^m(X_{t-}; \xi) \cdot dB_t \\ &\qquad + \int_{(0<|x|<1)} \bar{F}^m(X_{t-}, x; \xi)\, \widetilde{N}(dtdx), \quad t < \theta_{m,1}^n, \\ X_0 &= \kappa; \end{aligned} \tag{3.19}$$

First assume ξ is norm bounded and consider the stopped processes $\{X_{t\wedge\theta_{m,1}^n}^m\}$ and $\{X_{t\wedge\theta_{m,1}^n}^n\}$. Then arguing as in the uniqueness proof of Step 1 in Theorem 3.4, we conclude a.s. $X_t^m = X_t^n, t < \theta_{m,1}^n$. In particular, a.s. $X_{t-}^m = X_{t-}^n$ for $t = \theta_{m,1}^n$. Further, for almost all ω such that $\pi_1(\omega) = \theta_{m,1}^n(\omega)$, we have

$$X_t^m(\omega) = X_{t-}^m(\omega) + \bar{G}(X_{t-}^m(\omega), \triangle N_t, ; \xi) = X_t^n(\omega), \quad t = \pi_1(\omega).$$

We extend this result for $\mathcal{F}_0$ measurable ξ by arguing as in Step 2 in Theorem 3.4.

Take $i = 2$. Note that the contribution of the term involving $\bar{G}$ in $X_{t\wedge\theta_{m,2}^n}^m$ and $X_{t\wedge\theta_{m,2}^n}^n$ for the large jump at $t = \pi_1$ are the same. Arguing as in the case $i = 1$, we conclude a.s $X_t^m = X_t^n, t < \theta_{m,2}^n$.

Repeating the arguments, we have a.s. for all i, m, n with $m < n, X_t^m = X_t^n, t < \theta_{m,i}^n$. Since a.s. $\pi_i \uparrow \infty$ as $i \to \infty$, a.s. for all m, n with $m < n$ we have $X_t^m = X_t^n, t < \theta_m^n$, where

$$\theta_m^n := \inf\{t \geq 0 : \ |X_t^m| \text{ Or } |X_t^n| \geq m\}.$$

In particular, $\theta_m^n = \inf\{t \geq 0 : \ |X_t^m| \geq m\} = \inf\{t \geq 0 : \ |X_t^n| \geq m\}$. As such, θ_m^n is independent of $n(> m)$. Define $\theta_m := \inf\{t \geq 0 : \ |X_t^m| \geq m\}$ and set

$$X_t := \begin{cases} X_t^m & \text{for } t \leq \theta_m, \\ \infty, & \text{for } t \geq \eta, \end{cases}$$

so that $(\{X_t\}, \eta)$ is a solution of (3.1) for $t < \eta := \lim_{m\uparrow\infty} \theta_m$.

To prove the uniqueness, we consider the solution $(\{X_t\}, \eta)$ constructed above and compare it with any arbitrary solution $(\{X_t'\}, \eta')$ of (3.1). In the proof of existence of solutions, we had compared $\{X_t^m\}$ and $\{X_t^n\}$. We follow the same approach and define

$$\theta^R := \inf\{t \geq 0 : |X_t| \text{ Or } |X_t'| \geq R\} \wedge \eta \wedge \eta', \forall R \in \mathbb{N}.$$

We then conclude a.s. $X_t = X_t', t < \theta^R, \forall R \in \mathbb{N}$. Letting R go to infinity concludes the proof. □

Remark 3.7. In Theorem 3.6, it is easy to see that a.s. $\eta > 0$, where η is as in the proof of existence. Since the processes $\{X_t^m\}$ are right continuous at 0, it follows that a.s. $\theta_m > 0$ for all m. Moreover, θ_m are increasing in m. As such a.s. $\eta = \sup_m \theta_m > 0$.

Remark 3.8. The 'local Lipschitz' condition (3.16) follows from regularity assumptions on σ, b and F, provided other hypotheses are satisfied (see [5, Proposition 3.7]). As mentioned in Section 1, the class of SDEs (3.1) considered above are related to a class of stochastic PDEs taking values in $\mathcal{S}'$. The existence and uniqueness problems for these stochastic PDEs are studied in [5].

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References

1. Applebaum, D.: *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.
2. Bhar, S.: Correction to: An Itō Formula in the Space of Tempered Distributions. *J. Theoret. Probab.* **30** (2017), 1786–1787.
3. Bhar, S.: An Itō Formula in the Space of Tempered Distributions. *J. Theoret. Probab.* **30** (2017), 510–528.
4. Bhar, S.: Stationary solutions of stochastic partial differential equations in the space of tempered distributions. *Commun. Stoch. Anal.* **11** (2017), 169–193.
5. Bhar, S., Rajeev, B., and Sarkar, B.: Stochastic PDEs in $\mathcal{S}'$ for SDEs driven by Lévy noise. *arXiv:1801.06772v2 [math.PR] (Preprint)* (2018).
6. Durrett, R.: *Stochastic calculus*. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996. A practical introduction.

7. Gawarecki, L. and Mandrekar, V.: *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*. Probability and its Applications (New York). Springer, Heidelberg, 2011.
8. Ikeda, N. and Watanabe, S.: *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.
9. Itô, K.: *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.
10. Karatzas, I. and Shreve, S. E.: *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
11. Øksendal, B.: *Stochastic differential equations*. Universitext. Springer-Verlag, Berlin, sixth edition, 2003. An introduction with applications.
12. Protter, P. E.: *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.
13. Rajeev, B.: Translation invariant diffusion in the space of tempered distributions. *Indian J. Pure Appl. Math.* **44** (2013), 231–258.
14. Rajeev, B. and Thangavelu, S.: Probabilistic representations of solutions to the heat equation. *Proc. Indian Acad. Sci. Math. Sci.* **113** (2003), 321–332.
15. Rajeev, B. and Thangavelu, S.: Probabilistic representations of solutions of the forward equations. *Potential Anal.* **28** (2008), 139–162.

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