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COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACE

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Abstract

In this paper, we study complex valued metric spaces and established some fixed point results for a pair of compatible mappings satisfying a rational inequality in complex valued metric spaces. We generalized the result given by Azam et al. [1], Rouzkard and Imdad [17].

Keywords: *Common fixed point; Contractive type mapping; Complex valued metric space.*

Mathematics Subject Classification: *47H10; 54H25.*

1. INTRODUCTION AND PRELIMINARIES.

The Banach contraction principle [3] is considered to be the pioneering result of the fixed point theory, since its simplicity and usefulness, it became a very popular tool in solving many problems in mathematical analysis. Later, a number of articles in this field have been dedicated to the improvement and generalization of the Banach's contraction mapping principle. Inspired from the impact of this natural idea to functional analysis, several researchers have been extended and generalized this principle for different kinds of contractions in various spaces such as 2-metric space, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, Quasi metric spaces, Quasi semi metric spaces, D-metric spaces, and cone metric spaces, one can see [1–12].

Recently, Azam et al. [1] first introduced the complex valued metric spaces which is more general than well-known metric spaces and also gave common fixed point theorems for mappings satisfying generalized contraction condition. In this paper, we study complex valued metric spaces and established some fixed point

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results for a pair of compatible mappings satisfying a rational inequality in complex valued metric spaces.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order "$\precsim$" on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$, $\text{Im}(z_1) \leq \text{Im}(z_2)$.Consequently one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied

(i) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,

(ii) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,

(iii) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,

(iv) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

We will write $z_1 \leq z_2$ if $z_1 \neq z_2$ and one of (ii), (iii), and (iv) is satisfied; also we will write $z_1 < z_2$ if only (iv) is satisfied. Note that

$0 \preccurlyeq z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_1|$ and $z_1 \preccurlyeq z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$

Definition 1.1: Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$, satisfies the following conditions

(i) $0 \preccurlyeq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Definition 1.2: A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists

$$0 \prec r \in \mathbb{C} \text{ such that } B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$$

Definition 1.3: A point $x \in X$ is called a limit point of A whenever for every

$$0 \prec r \in \mathbb{C},\ B(x, r) \cap (A \setminus X) \neq \phi \ i.e. \{B(x, r) - x\} \cap A \neq \phi$$

Definition 1.4: Asubset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

Definition 1.5: A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family $F = \{B(x, r) : x \in X, 0 \prec r\}$is a sub-basis for a Hausdorff topology on X.

Definition 1.6: Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 < c$ there is $n_0 \in \mathbb{N}$such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to \infty$.

Definition 1.7: If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d).

Definition 1.8: If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 1.1: Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.2: Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Here, we give some notions in fixed point theory.

Definition 1.9: Let S and T be self-mappings of a nonempty set X.

(i) A point $x \in$ X is said to be a fixed point of T if T$x = x$.

(ii) A point $x \in$ X is said to be a coincidence point of S and T if Sx = Tx and we shall called w = Sx = Tx, a point of coincidence of S and T.

(iii) A point $x \in$ X is said to be a common fixed point of S and T if x = Sx = Tx.

In 1976, Jungck [12] introduced concept of commuting mappings as follows:

Definition 1.10[12]: Let X be a non-empty set. The mappings S and T are commuting if $TSx = STx$ for all $x \in$ X.

Afterward, Sessa [18] introduced concept of weakly commuting mappings which are more general than commuting mappings as follows:

Definition 1.11[18]: Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx) \text{ for all } x \in X.$$

In 1986, Jungck [13] introduced the more generalized commuting mappings in metric spaces, called compatible mappings, which also are more general than the concept of weakly commuting mappings as follows:

Definition 1.12 [13]: Let S and T be mappings from a metric space (X, d) into itself. The mapping S and T are said to be compatible if

$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in$ X.

Remark 1.1: In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true .

2. MAIN RESULT.

Theorem 2.1: Let (X, d) be a complete complex valued metric space and let the mappings

S, T : X → X are self-mappings satisfying the condition

$$d(Sx, Ty) \le \lambda \frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)} + \mu\, d(x, y) \tag{2.1}$$

for all x, $y \in$ X, where$d(x,Ty)+d(y,Sx) \neq 0$, and λ, μ are nonnegative reals with λ + μ < 1. Also if one of S or T is continuous and the pair (S, T) is compatible. Then S and T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X and we define

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1}, \text{ for } n = 0, 1, 2, 3 \tag{2.2}$$

Then using equations (2.1) and (2.2),

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\le \lambda \frac{d(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Sx_{2n})}{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})}$$

$$+ \mu\, d(x_{2n},x_{2n+1})$$

$$\le \lambda \frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+1})}{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}$$

$$+ \mu\, d(x_{2n},x_{2n+1})$$

i.e., $d(x_{2n+1}, x_{2n+2}) \le (\lambda + \mu)\, d(x_{2n}, x_{2n+1})$

Similarly,

$$d(x_{2n+2}, x_{2n+3}) = d(Sx_{2n+1}, Tx_{2n+2})$$

$$\le \lambda \frac{d(x_{2n+1},Sx_{2n+1})d(x_{2n+1},Tx_{2n+2})+d(x_{2n+2},Tx_{2n+2})d(x_{2n+2},Sx_{2n+1})}{d(x_{2n+1},Tx_{2n+2})+d(x_{2n+2},Sx_{2n+1})}$$

$$+ \mu\, d(x_{2n+1},x_{2n+2})$$

$$\le \lambda \frac{d(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+3})+d(x_{2n+2},x_{2n+3})d(x_{2n+2},x_{2n+2})}{d(x_{2n+1},x_{2n+3})+d(x_{2n+2},x_{2n+2})}$$

$+ \mu\, d(x_{2n+1}, x_{2n+2})$

i.e., $d(x_{2n+2}, x_{2n+3}) \leq (\lambda + \mu)\, d(x_{2n+1}, x_{2n+2})$

Now with $h = \lambda + \mu$, we have

$d(x_{n+1}, x_{n+2}) \leq h\, d(x_n, x_{n+1}) \leq h^2\, d(x_{n-1}, x_n) \leq h^3\, d(x_{n-2}, x_{n-1}) \ldots,$

$d(x_{n+1}, x_{n+2}) \leq h^{n+1}\, d(x_0, x_1).$

So for any $m > n$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots.. + d(x_{m-1}, x_m)$$

$$\leq [h^n + h^{n+1} + \ldots\ldots\ldots + h^{m-1}]\, d(x_0, x_1)$$

$$d(x_n, x_m) \leq \left[\frac{h^n}{1-h}\right] d(x_0, x_1)$$

And so $|d(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right] |d(x_0, x_1)| \to 0$, as $m, n \to \infty$.

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. Consequently, the sequences Sx_{2n} *and* Tx_{2n+1} and its consequences also converges to u.

Let S is continuous. Since S and T are compatible maps,

then, $$\lim_{n\to\infty} d(STx_{2n}, TSx_{2n}) = 0.$$

Also since S is continuous, $STx_{2n} \to Su$.

Consider

$$d(STx_{2n}, Tx_{2n+1}) \leq \lambda \frac{d(Tx_{2n}, STx_{2n})\, d(Tx_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})\, d(x_{2n+1}, STx_{2n})}{d(Tx_{2n}, Tx_{2n+1}) + d(x_{2n+1}, STx_{2n})}$$

$$+ \mu\, d(Tx_{2n}, x_{2n+1})$$

Taking limit as $n \to \infty$, both sides we get

$$d(Su, u) \leq \lambda \frac{d(u, Su)\, d(u, u) + d(u, u)\, d(u, Su)}{d(u, u) + d(u, Su)} + \mu\, d(u, u)$$

Thus it follows that $u = Su$.

Suppose if not, then $d(u, Su) = z > 0$, and we would then have

$z \leq d(u, x_{2n+2}) + d(x_{2n+2}, Su)$

$\leq d(u, x_{2n+2}) + d(Su, Tx_{2n+1})$

$$\leq d(u, x_{2n+2}) + \lambda \frac{d(u, Su)\, d(u, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})\, d(x_{2n+1}, Su)}{d(u, Tx_{2n+1}) + d(x_{2n+1}, Su)} + \mu\, d(u, x_{2n+1})$$

Taking modulus and limit as $n \to \infty$ both sides, we get

$$|z| \leq |d(u, x_{2n+2})| + \lambda \frac{|d(u,Su)||d(u,x_{2n+2})| + |d(x_{2n+1},x_{2n+2})|\,|d(x_{2n+1},Su)|}{|d(u,x_{2n+2})| + |d(x_{2n+1},Su)|}$$

$$+ \mu\, |d(u,x_{2n+1})|$$

Taking limit as $n \to \infty$ both sides, we get

That is $|z| \leq 0$, a contradiction of $d(u, Su) = z > 0$, and hence $u = Su$.

One can easily prove that $u = Tu$, when the continuity of T is assumed.

We now show that *S* and *T* have a unique common fixed point.

For this, assume that u^* in X is another common fixed point of S and T.

Then by (2.1), we have

$$d(u, u^*) = d(Su, Tu^*) \leq \lambda \frac{d(u,Su)d(u,Tu^*) + d(u^*,Tu^*)d(u^*,Su)}{d(u,Tu^*) + d(u^*,Su)} + \mu\, d(u,u^*)$$

$$\leq \mu\, d(u, u^*)$$

$$d(u, u^*) \leq \mu\, d(u, u^*).$$

$\Rightarrow \quad \mu \geq 1$, which is a contradiction

This implies that $u^* = u$. This completes the proof of the theorem.

Theorem 2.2: Let (X, d) be a complete complex valued metric space and let the mappings S, T : X → X are self-mappings satisfying the condition

$$d(Sx, Ty) \leq \lambda \frac{d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)}{d(x,Ty) + d(y,Sx)} + \mu\, d(x, y)$$

for all $x, y \in X$, where $d(x,Ty) + d(y,Sx) \neq 0$,and λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof: The proof of the theorem 2.2 follows immediately from Theorem 2.1.

Theorem 2.3: Let (X, d) be a complete complex valued metric space and let the mappings S, T : X → X are self-mappings satisfying the condition

$$d(Sx, Ty) \leq \lambda \frac{d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)}{d(x,Ty) + d(y,Sx)} \tag{2.3}$$

for all $x, y \in X$, where $d(x,Ty) + d(y,Sx) \neq 0$, and λ is nonnegative real with $\lambda < 1$. Also if one of S or T is continuous and the pair (S, T) is compatible. Then S and T have a unique common fixed point.

Proof: The proof of the theorem 2.3 follows immediately by putting $\mu = 0$ in condition (2.1) of Theorem 2.1.

Corollary 2.1: Let (X, d) be a complete complex valued metric space and let the mapping

T : X → X satisfying the condition

$$d(Tx, Ty) \leq \lambda \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)} + \mu\, d(x, y) \qquad (2.4)$$

for all $x, y \in X$, where $d(x,Ty) + d(y,Tx) \neq 0$, and λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof: The proof of the corollary 2.1 follows immediately by putting S = T in Theorem 2.2.

Corollary 2.2: Let (X, *d*) be a complete complex valued metric space and

T : X → X satisfying the condition

$$d(T^n x, T^n y) \leq \lambda \frac{d(x,T^n x)d(x,T^n y)+d(y,T^n y)d(y,T^n x)}{d(x,T^n y)+d(y,T^n x)} + \mu\, d(x, y) \qquad (2.5)$$

for all $x, y \in X$, where $d(x,T^n y) + d(y,T^n x) \neq 0$, and λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof: By Corollary 2.1, we obtain $v \in X$ such that $T^n v = v$. The result then follows from the fact that

$$\begin{aligned} d(Tv, v) &= d(TT^n v, T^n v) \\ &= d(T^n Tv, T^n v) \\ &\leq \lambda \frac{d(Tv,T^n Tv)d(Tv,T^n v)+d(v,T^n v)d(v,T^n Tv)}{d(Tv,T^n v)+d(v,T^n Tv)} + \mu\, d(Tv, v) \\ &\leq \lambda \frac{d(Tv,T^n Tv)d(Tv,v)+d(v,v)d(v,T^n Tv)}{d(Tv,v)+d(v,T^n Tv)} + \mu\, d(Tv, v) \\ &\leq \mu\, d(Tv, v) \end{aligned}$$

$$|d(Tv, v)| \leq \mu\, |d(Tv, v)|$$

$$\Rightarrow \qquad \mu \geq 1, \text{ which is a contradiction}$$

$$d(Tv, v) \to 0 \Rightarrow Tv = v.$$

This implies that T has a fixed point.

Uniqueness of fixed point follows directly from the inequality (2.5). Our main results in Theorems 2.1, 2.2, 23 and Corollaries 2.1, 2.2 which are generalization and extension of multitude of common fixed point theorems in the recent literature of complex valued metric spaces (see Azam et. al. [1], Rouzkard and Imdad [17]).

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