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Preservation of Arbitrary Fs-Unions and Fs-Intersections by the Inverse of an Fs-function

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Abstract : In this paper we prove that inverse of an Fs-function preserves arbitrary Fs-unions and Fs-intersections
Keywords : Fs-set, Fs-subset , Fs-function, Image of an Fs-subset, Inverse image of an Fs-set.

1. INTRODUCTION

Ever since Zadeh [8] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Recently many researchers put their efforts in order to prove collection of all fuzzy subsets of a given fuzzy set is Boolean algebra under suitable operations [21].Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath[11] introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions. They are successful in their efforts in proving that result with some conditions. In this paper we prove that preservation of arbitrary Fs-unions and Fs-intersections by the inverse of an Fs-function .For smooth reading of the paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first three sections. We denote the largest element of a complete Boolean algebra L_A [1.1] by M_A or 1_A. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [3], Garret Birkhoff[4], Steven Givant • Paul Halmos[3] and Thomas Jech[5].For results in topology one can refer[10].

2. THEORY OF FS-SETS

1. Fs-set : Let U be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple

$$A = (A_1, A, \overline{A} (\mu_{1A_1}, \mu_{2A}), L_A)$$

is said be an FS-set if, and only if

1. $A \subseteq A_1$
2. L_A is a complete Boolean Algebra
3. $\mu_{1A_1} : A_1 \to L_A$, $\mu_{2A} : A \to L_A$, are functions such that $\mu_{1A_1} | A \geq \mu_{2A}$
4. $\overline{A} : A \to L_A$ is defined by

$$\overline{A}x = \mu_{1A_1} x \wedge (\mu_{2A} x)^c, \text{ for each } x \in A$$

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2. Fs-subset : Let $A = (A_1, A, \overline{A}\ (\mu_{1A_1}\ \mu_{2A}), L_A)$ and $B = (B_1, B, \overline{B}\ (\mu_{1B_1}, \mu_{2B}), L_B)$ be a pair of Fs-sets. B is said to be an Fs-subset of A, denoted by $B \subseteq A$, if, and only if

(*a*) $B_1 \subseteq A_1, A \subseteq B$

(*b*) L_B is a complete subalgebra of L_A or $L_B \leq L_A$

(*c*) $\mu_{1B_1} \leq \mu_{1A_1} \mid B_1$, and $\mu_{2B} \mid A \geq \mu_{2A}$

3. Proposition : Let B and A be a pair of Fs-sets such that $B \subseteq A$. Then $\overline{B}x \leq \overline{A}x$ is true for each $x \in A$

4. Definition : For some L_X, such that $L_X \leq L_A$ a four tuple $X = (X_1, X, \overline{X}\ (\mu_{1X_1}, \mu_{2X}), L_X)$ is not an Fs-set if, and only if

(*a*) $X \nsubseteq X_1$ or

(*b*) $\mu_{1X_1} x \ngeq \mu_{2X} x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of B for any $B \subseteq A$.

Definition : An Fs-subset $Y = (Y_1, Y, \overline{Y}\ (\mu_{1Y_1}, \mu_{2Y}), L_Y)$ of A, is said to be an Fs-empty set of second kind if, and only if

(*a*) $Y_1 = Y = A$

(*b*) $L_Y \leq L_A$

(*c*) $\overline{Y} = 0$

4.1. Remark : We denote Fs-empty set of first kind or Fs-empty set of second kind by Φ_A and we prove later (1.15), Φ_A is the least Fs-subset among all Fs-subsets of A.

5. Definition : Let $B_1 = (B_{11}, B_1, \overline{B}_1\ (\mu_{1B_{11}}, \mu_{2B1}), L_{B1})$ and
$B_2 = (B_{12}, B_2, \overline{B}_2\ (\mu_{1B_{12}}, \mu_{2B2}), L_{B2})$ be a pair of Fs-subsets.

1. We say that B_1 and B_2 are (1,5)-equal, if $B_{11} = B_{12}$ and $L_{B_1} = L_{B_2}$
2. We say that B_1 and B_2 are (2,5)-equal, if $B_1 = B_2$ and $L_{B_1} = L_{B_2}$
3. We say that B_1 and B_2 are 3-equal, if B_1 and B_2 are (1,5)-equal and $\mu_{1B_{11}} = \mu_{1B12}$
4. We say that B_1 and B_2 are 4-equal, if B_1 and B_2 are (2,5)-equal and $\mu_{2B_1} = \mu_{2B2}$
5. We say that B_1 and B_2 are Total equal denoted $B_1 = B_2$ (T), if B_1 and B_2 are (2,5)-equal and $\overline{B}_1 = \overline{B}_2$
6. We say that B_1, B_2 are Full-equal, denoted $B_1 = B_2$, if B_1 and B_2 are 3-equal and 4-equal.

6. Proposition : $B_1 = (B_{11}, B_1, \overline{B}_1\ (\mu_{1B_{11}}, \mu_{B1})\ L_{B1})$ and
$B_2 = (B_{12}, B_2, \overline{B}_2\ (\mu_{1B_{12}}, \mu_{B_2}), L_{B2})$

are Full-equal if, only if $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$.

7. Remark : Whenever X and Y are Complete Boolean algebra $\Phi \subseteq X \times Y$ be a relation

(*a*) We say that Φ is $(\vee, \wedge)$-complete relation on X if, and only if $\vee\ \Phi\ (\wedge_{\alpha\in T}\ \alpha) = \wedge_{\alpha\in T}\ (\wedge\ \Phi\ \alpha)$ for any $T \subseteq X$.

(*b*) We say that Φ is $(\vee, \vee)$-complete relation on X if, and only if $\vee\ \Phi\ (\vee_{\alpha\in T}\ \alpha) = \vee_{\alpha\in T}\ (\vee\ \Phi\ \alpha)$ for any $T \subseteq X$.

(*c*) We say that Φ is $(\wedge, \vee)$-complete relation on X if, and only if $\wedge\ \Phi\ (\vee_{\alpha\in T}\ \alpha) = \wedge_{\alpha\in T}\ (\wedge\ \Phi\ \alpha)$ for any $T \subseteq X$.

(*d*) We say that Φ is said to be $\vee$-increasing on X if, and only if, and only if, $\vee\ \Phi\ \alpha \leq \vee\ \Phi\ \beta$ for any $\alpha, \beta \in X$ such that $\alpha \leq \beta$.

8. Proposition : Whenever $\Phi : X \to Y$ is a complete Boolean algebra homomorphism, then

1. Φ^{-1} is join increasing on ΦX
2. Φ^{-1} -complete relation on ΦX
3. Φ^{-1} -complete relation on ΦX

3. FS-FUNCTIONS

1. Definition : A Triplet $(f_1, f, \Phi)$is said to be is an Fs-Function between two given Fs-subsets

$$\begin{aligned} B &= (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \text{ and} \\ &= (C_1, C, \overline{C}\,(\mu_{1C_1}, \mu_{2C}), L_C) \text{ of A, denoted by} \\ (f_1, f, \Phi) : B &= (B_1, B, \overline{B}\,(\mu_{1B_1}, \mu_{2B}), L_B) \to \\ C &= (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C) \text{ if, and only if (using the diagrams).} \end{aligned}$$

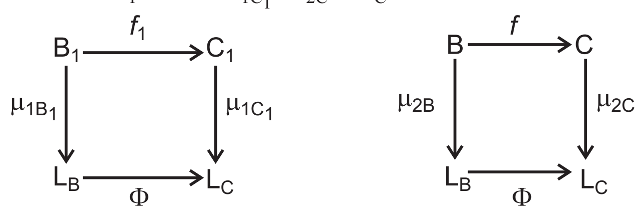

Figure 1: Fs-function $\overline{f}$ B → C

(*a*) $f = f_1|_B^C : B \to C$ be onto

(*b*) $\Phi : L_B \to L_c$ is complete homomorphism

(f_1, f, Φ) is denoted by $\overline{f}$

IMAGES OF FS-SUBSET

2. Definition. Let $D \subseteq B$ and $\overline{f} : B \to C$ be an Fs-function, where

$$\begin{aligned} B &= (B_1, B, \overline{B}\,(\mu_{1B_1}, \mu_{2B}), L_B), \\ C &= (C_1, C, \overline{C}\,(\mu_{1C_1}, \mu_{2C}, L_C), \\ D &= (D_1, D, \overline{D}\,(\mu_{1D_1}, \mu_{2D}), L_D), \\ D &= B \text{ and } f = f_1|_B^C : B \to C \text{ be onto.} \end{aligned}$$

Define $\overline{f}$(D) as follows

$$\overline{f}\,(D) = \varepsilon = (E_1, E, \overline{E}\,(\mu_{1E_1}, \mu_{2E})L_E),$$

where

1. $E_1 = f_1\,(D_1)$
2. $E = f(D)$
3. $\mu_{1E_1} : E_1 \to L_C$ is defined by

$$\mu_{1E_1} y = \begin{cases} \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee\limits_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right)\right], & \text{if } y \in C \\ \mu_{1C_1} y \wedge \left(\bigvee\limits_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right), & \text{if } y \notin C \end{cases}$$

4. $\mu_{2E} : E \to L_C$ is defined by

$$\mu_{2E} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee\limits_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D} x\right)\right]$$

5. $L_E = ([\mu_{1E_1}(E_1)])$ = The complete subalgebra generated by $[\mu_{1E_1}(E_1)]$, where $[\mu_{1E_1}(E_1)]$ = The complete ideal generated by $\mu_{1E_1}(E_1)$

4. INVERSE IMAGE OF FS-SUBSET

1. Definition. Let $D \subseteq B$ and $\bar{f}: B \to C$ be an Fs-function, $\Phi^{-1} \subseteq L_C \times L_B$ be $\vee$-increasing ($\because$ Prop 1.7(d))

$$f = f_1|_B^C : \to \text{be onto.}$$

Let $\varepsilon \subseteq C$,

where $\varepsilon = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$

Define $\bar{f}^{-1}(\varepsilon)$ as follows

$$\bar{f}^{-1}(\varepsilon) = D = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D),$$

where

(a) $D_1 = f_1^{-1}(E_1)$

(b) $D = f^{-1}(E)$

(c) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1}x = \begin{cases} \mu_{1B_1}x & , \text{whenever } \Phi^{-1}\mu_{1E_1}f_1x = \Phi \\ \mu_{2B}\,x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right] & , x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right) & , x \notin B \end{cases}$$

(d) $\mu_{2D} : D \to L_D$ is defined by

$$\mu_{2D}x = \begin{cases} \mu_{2B}x & , \text{whenever } \Phi^{-1}\mu_{2E}fx = \Phi \\ \mu_{2B}\,x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx\right)\right] \end{cases}$$

(e) $L_D = L_B$

1.1 Remark : We definem

(a) Φ_A = Fs-empty set of first kind imply $\bar{f}^{-1}(\Phi_A)$ = Fs-empty set of first kind.

(b) Φ_A = Fs-empty set of second kind imply $\bar{f}^{-1}(\Phi_A)$ = Fs-empty set of second kind.

2. Proposition : $\bar{f}^{-1}(\varepsilon)$ is an Fs-subset of B, if $\Phi^{-1} \subseteq L_C \times L_B$ is $\vee$-increasing

3. Proposition : Let B and C be any pair of Fs-subsets and $\bar{f}: B \to C$ be an Fs-function. Let ε_1 and ε_2 be Fs-subsets C such that $E_1 = E_2$ and $E_1 = E_2 = C$, then $\bar{f}^{-1}(\varepsilon_1) \subseteq \bar{f}^{-1}(\varepsilon_2)$

4. Proposition : Let B and C be any pair of Fs-subsets and $\bar{f}: B \to C$ be an Fs-function. Let ε_1 and ε_2 be Fs-subsets C and $E_1 = E_2 = C$, then $\bar{f}^{-1}(\varepsilon_1 \cup \varepsilon_2)$ and $\bar{f}^{-1}(\varepsilon_1) \cup \bar{f}^{-1}(\varepsilon_2)$ are full-equal.

5. Proposition : Let B and C be any pair of Fs-subsets and $\bar{f}: B \to C$ be an Fs-function. Let ε_1 and ε_2 be Fs-subsets C and $E_1 = E_2 = C$, then $\bar{f}^{-1}(\varepsilon_1 \cap \varepsilon_2)$ and $f^{-1}(\varepsilon_1) \cap \bar{f}^{-1}(\varepsilon_2)$ are full-equal.

5. ARBITRARY FS-UNIONS AND FS-INTERSECTIONS BY THE INVERSE OF AN FS-FUNCTION

1. Proposition : For any Fs-function $\bar{f}: B \to C$, any family of Fs-subsets ε_i, $i \in I$ of C and $E_i = C$, $\bar{f}^{-1}(\cup_{i \in I}\varepsilon_i)$ and $\cup_{i \in I}\bar{f}^{-1}(\varepsilon_i)$ are Full-equal.

Proof : LHS For $I = \Phi$, $\cup_{i \in I} \varepsilon_i = \Phi_A \Rightarrow \bar{f}^{-1}(\cup_{i \in I} \varepsilon_i) = \bar{f}^{-1}(\Phi_A) = \Phi_A$

RHS : $\cup_{i \in I} \bar{f}^{-1}(\varepsilon_i) = \Phi_A$

Hence $\bar{f}^{-1}(\cup_{i \in I} \varepsilon_i)$ and $\cup_{i \in I} \bar{f}^{-1}(\varepsilon_i)$ are Full-equal whenever index set $I = \Phi$.

For $I \neq \Phi$, Let $\cup_{i \in I} \varepsilon_i = F = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$ and $\varepsilon_i \subseteq C$ for each $i \in I$ we have

1. $F_1 = \cup_{i \in I} E_{1_i}$
2. $F = \cap_{i \in I} E_i$
3. $L_F = \vee_{i \in I} L_{E_i}$
4. $\mu_{1F_1} : F_1 \to L_F$ is defined by $\mu_{1F_1} y = (\vee_{i \in I} \mu_{1E_{1i}}) y = \vee_{i \in I_y} \mu_{1E_{1i}} y$, where $I_y = \{i \in I_y \mid y \in E_{1_i}\}$
5. $\mu_{2F} : F \to L_F$ is defined by $\mu_{2F} y = (\wedge_{i \in I} \mu_{2E_i}) y = (\wedge_{i \in I} \mu_{2E_i} y$

   Let $\bar{f}^{-1}(\cup_{i \in I} \varepsilon_i) = \bar{f}^{-1})(F) = G = (G_1, G, \bar{G}(\mu_1 G_1, \mu_{2G}), L_G)$
6. $G_1 = f_1^{-1}(F_1) = f_1^{-1})(\cup_{i \in I} E_{1_i}) = \cup_{i \in I} f_1^{-1}(E_{1i})$
7. $G = f^{-1}(F) = f^{-1})(\cap_{i \in I} E_i) = \cap_{i \in I} f^{-1})(E_1)$
8. $L_G = L_B$
9. $\mu_{1G_1} : G_1 \to L_G$ is defined by

$$\mu_{1G_1} x = \begin{cases} \mu_{1B_1} x & , \text{whenver } \Phi^{-1}\mu_{1F_1} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1F_1} f_1 x\right)\right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1F_1} f_1 x\right) & , x \notin B \end{cases}$$

10. $\mu_{2G} : G \to L_G$ is defined by

$$\mu_{2G} x = \begin{cases} \mu_{2B} x & , \text{whenver } \Phi^{-1}\mu_{2F} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2F} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{2F} f_1 x\right)\right] \end{cases}$$

Let $\bar{f}^{-1}(\varepsilon_i) = D_i = (D_{1i}, D_i, \bar{D}_i(\mu_{1D1i}, \mu_{2Di}), L_{Di})$,

where

11. $D_{1i} = f_1^{-1})(E_{1i})$
12. $D_i = f^{-1})(E_i)$
13. $\mu_{1D1i} : D_{1i} \to L_{D_1}$ is defined by

$$\mu_{1D_{1i}} x = \begin{cases} \mu_{1B_1} x & , \text{whenver } \Phi^{-1}\mu_{1E_{1i}} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E1i} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{1i}} f_1 x\right)\right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{1i}} f_1 x\right) & , x \notin B \end{cases}$$

14. $\mu_{2Di} : D_i \to L_{D_i}$ is defined by

$$\mu_{2D_i} x = \begin{cases} \mu_{2B} x & , \text{whenver } \Phi^{-1}\mu_{2E_i} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_i} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_i} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_i} f_1 x\right)\right] \end{cases}$$

15. $L_{D_i} = L_B$

Let $\cup_{i \in I} \overline{f}^{-1}(\varepsilon_i) = \cup_{i \in I} D_i = H = (H_1, H, \overline{H}(\mu_{1H_1}, \mu_{2H}), L_H)$

Where

16. $E_1 = \cup_{i \in I} D_{1i} = \cup_{i \in I} f_1^{-1}(E_{1i}) = f_1^{-1}(\cup_{i \in I} E_{1i})$
17. $H = \cap_{i \in I} D_i = \cap_{i \in I} f^{-1}(E_i) = f^{-1})(\cap_{i \in I} E_i)$
18. $L_H = \vee_{i \in I} L_{D_i} = L_B$
19. $\mu_{1H_1} : H_1 \to L_H$ is defined by $\mu_{1H_1} x = (\vee_{i \in I} \mu_{1D_{1i}}) x$
20. $\mu_{2H} : H \to L_H$ is defined by $\mu_{2H} x = (\wedge_{i \in I} \mu_{2D_i}) x = \wedge_{i \in I} \mu_{2D_i}$
    Need to show H and G are Full-equal *i.e.* to show
21. $H_1 = G_1, H = G$
22. $L_H = L_G$
23. $\mu_{1H_1} = \mu_{1G_1}, \mu_{2H} = \mu_{2G}$

Proof of (21) follows from (6), (7), (16) and (17)

Proof of (22) follows from (8) and (18)

Proof of (23) :

$$\mu_{1H_1} x = (\vee_{i \in I} \mu_{1D1i}) x = (\vee_{i \in I} \mu_{1D_{1i}}) x \text{ for } x \in B$$

$$= \vee_{i \in I} \left\{ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\}$$

$$\mu_{1G_1} x = \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1F_1} f_1 x \right) \right]$$

$$= \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\vee_{i \in I_y} \mu_{1E_{1i}} f_{1x} \right] \right) \right]$$

$$= \bigvee_{\substack{\Phi\alpha = \mu_{1F_1} y \\ \alpha \in L_B}} \Phi^{-1} \left[\vee_{i \in I_y} \mu_{1E_{1i}} f_1 x \right]$$

$$= \vee_{i \in I_y} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} y \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right] (\because \text{Proposition } 1.7(b))$$

$$\Rightarrow \quad \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} y \\ \alpha \in L_B}} \Phi^{-1} \left[\bigvee_{i \in I_y} \mu_{1E_{1i}} f_1 x \right] \right) \right] = \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{i \in I_y} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right] \right) \right]$$

$$\Rightarrow \quad \mu_{1H_1} x = \mu_{1G_1} x$$

Case : (II) $I_1 = \{i \in I_1 \mid y = f_1 x \in E_{1i}\}$

and $I_2 = \{i \in I_2 \mid y = f_1 x \in E_{1i}\}$

From Case (I)

$$\Rightarrow \quad \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} y \\ \alpha \in L_B}} \Phi^{-1} \left[\bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right] \right) \right] = \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{i \in I_1} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} y \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right] \right) \right]$$

For all $i \in I_2$, LHS : $\mu_{2B} x$, RHS: $\mu_{2B} x$

$\therefore$ LHS = RHS

$$\mu_{1H_1} x = \bigvee_{i \in I} \left\{ \mu_{2B}\ x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\}$$

$$= \bigvee_{i \in I_1} \left\{ \mu_{2B}\ x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\} \vee \left[\bigvee_{i \in I_2} \left\{ \mu_{2B}\ x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\} \right]$$

Since $y \in I_2 \Rightarrow \mu_{1E_{1i}}) f_1 x$ is not defined

$\Rightarrow$ $\Phi^{-1}\ \mu_{1E_{1i}}\ f_1 x = \Phi$

$\Rightarrow$ $\Phi^{-1}\ \mu_{1E_{1i}}\ f_1 x = 0$

$\Rightarrow$ $\vee_{i \in I_2} \Phi^{-1}\ \mu_{1E1i}\ f_1 x = 0$

$$= \bigvee_{i \in I_1} \left\{ \mu_{2B}\ x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\}$$

$$\mu_{1G_1} x = \mu_{2B} x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left\{ \left[\bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right] \vee \left[\bigvee_{i \in I_2} \mu_{1E_{1i}} f_1 x \right] \right\} \right) \right]$$

$$= \mu_{2B} x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right) \right]$$

Case : (III) $x \notin B, y \in \wedge_{i \in I} E_{1i}$

$$I_1 = \{ i \in I_1 \mid y = f_1 x \in E_{1i} \}$$

and

$$I_2 = \{ i \in I_2 \mid y = f_1 x \in E_{1i} \}$$

From Case (I)

$$\Rightarrow \quad \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\bigvee_{i \in I_2} \mu_{1E_{1i}} f_1 x \right] \right) = \mu_{1B_1} x \wedge \left(\bigvee_{i \in I_1} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} y \\ \alpha \in L_B}} \Phi^{-1}\ \mu_{1E_{1i}} f_1 x \right] \right)$$

For all $i \in I_2$, LHS : 0, RHS : 0

$\therefore$ LHS = RHS

$$\mu_{1H_1} x = \bigvee_{i \in I} \left\{ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right\}$$

$$= \bigvee_{i \in I_1} \left\{ \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\} \vee \left[\bigvee_{i \in I_1} \left\{ \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\} \right]$$

Since $y \in I_2$

$\Rightarrow$ $\mu_{1E1i} f_1 x$ is not defined

$\Rightarrow$ $\Phi^{-1}\ \mu_{1E1i} f_1 x = \Phi$

$\Rightarrow$ $\Phi^{-1}\ \mu_{1E1i}\ f_1 x = 0$

$\Rightarrow$ $\vee_{iI_2} \Phi^{-1}\ \mu_{1E1i}\ f_1 x = 0$

$$\mu_{1H_1} x = \bigvee_{i \in I_1} \left\{ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right\}$$

$$\mu_{1G_1} x = \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left\{ \left[\bigvee_{i \in I_1} \mu_{1E_{1i}}\ f_1 x \right] \vee \left[\bigvee_{i \in I_2} \mu_{1E_{1i}}\ f_1 x \right] \right\} \right)$$

$$= \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right)$$

$\Rightarrow$ $\mu_{1H_1} x = \mu_{1G_1} x$

Case (IV) $x \notin B,$

$y = f_1 x \in \wedge_{i \in I} E_{1i}$

$I_1 = \{i \in I_1 \mid y = f_1 x \in E_{1i}\}$

and $I_2 = \{i \in I_2 \mid y = f_1 x \notin E_{1i}\}$

$$\mu_{1H_1} x = \bigvee_{i \in I} \left\{ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right\}$$

$$= \bigvee_{i \in I_1} \left\{ \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\} \vee \left[\bigvee_{i \in I_2} \left\{ \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\} \right]$$

Since $y = f_1 x \in I_2$

$\Rightarrow$ $\mu_{1E1i} f_1 x$ is not defined

$\Rightarrow$ $\Phi^{-1} \mu_{1E1i} f_1 x = \Phi$

$\Rightarrow$ $\Phi^{-1} \mu_{1E1i} f_1 x = 0$

$i \in I_2\ \Phi^{-1} \mu_{1E1i} f_1 \mathrm{x} = 0$

$$= \bigvee_{i \in I_1} \left\{ \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] \right\}$$

$$\mu_{1G1} x = \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left\{ \left[\bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right] \vee \left[\bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right] \right\} \right)$$

$$= \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \bigvee_{i \in I_1} \mu_{1E_{1i}} f_1 x \right)$$

$\Rightarrow$ $\mu_{1H_1} x = \mu_{1G_1} x$

$\mu_{1H_1} x = (\wedge_{i \in I} \mu_{2D_i}) x = \wedge_{i \in I} \mu_{2D_i} x$

$$= \wedge_{i \in I} \left\{ \mu_{2B}\, x \vee \left[\mu_{1B_1}\, x \vee \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_i} fx \\ \alpha \in L_B}} \Phi^{-1} \mu_{2E_i} fx \right) \right] \right\}$$

$$\mu_{2G} x = \mu_{2B} x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2F} fx \\ \alpha \in L_B}} \Phi^{-1} \mu_{2F} fx \right) \right]$$

$$= \mu_{2B} x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2F} fx \\ \alpha \in L_B}} \Phi^{-1} \mu_{2F} fx \left(\wedge_{i \in I} \mu_{2E_i} fx \right) \right) \right]$$

$\vee \Phi^{-1} (\wedge_{i \in I} \mu_{2E_i} fx) = \wedge_{i \in I} (\vee \Phi^{-1} \mu_{2E_i} fx)$ ($\because$ prop 1.7)

$\Rightarrow$ $\mu_{1B_1} x \wedge (\vee \Phi^{-1}) (\wedge_{i \in I} \mu_{2E_i} fx) = \mu_{1B_1} x \wedge (\wedge_{i \in I} (\vee \Phi^{-1} \mu_{2E_i} fx))$

$\Rightarrow$ $\mu_{2B} x \vee [\mu_{1B_1} x \wedge (\vee \Phi^{-1} (\wedge_{i \in I} \mu_{2E_i} fx))] = \mu_{2B} x \vee [\mu_{1B_1} x \wedge (\wedge_{i \in I} (\vee \Phi^{-1} \mu_{2E_i} fx))]$

$\Rightarrow$ $\mu_{2H} x = \mu_{2G} x$

2. Proposition : For any Fs-function $\bar{f}: B \to C$, any family of Fs-subsets ε_i, $i \in I$ of C and $E_i = C$, $\bar{f}^{-1} (\cap_{i \in I} \varepsilon_i)$ and $\cap_{i \in I} \bar{f}^{-1} (\varepsilon_i)$ are Full-equal.

Proof : LHS For $I = \Phi, \cap_{i \in I} \varepsilon_i = C$

$\Rightarrow \quad \bar{f}^{-1}(\cap_{i \in I} \varepsilon_i) = \bar{f}^{-1}(C) = B$

RHS : $\cap_{i \in I} \bar{f}^{-1}(\varepsilon_i) = B$

Hence $\bar{f}^{-1}(\cap_{i \in I} \varepsilon_i)$ and $\cap_{i \in I} \bar{f}^{-1}(\varepsilon_i)$ are Full-equal whenever index set $I = \Phi$.

For $I \neq \Phi$, Let $\cap_{i \in I} \varepsilon_i = F = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$ and $E_i \subseteq C$ for each $i \in I$ we have

1. $F_1 = \cap_{i \in I} E_{1i}$
2. $F = \cup_{i \in I} E_i = C$
3. $L_F = \wedge_{i \in I} L_{Ei}$
4. $\mu_{1F_1} : F_1 \to L_F$ is defined by $\mu_{1F_1} y = (\wedge_{i \in I} \mu_{1E_{1i}})\, y = \wedge_{i \in I} \mu_{1E_{1i}} y$
5. $\mu_{2F} : F \to L_F$ is defined by $\mu_{2F} y = (\vee_{i \in I} \mu_{2E_i})\, y = \vee_{i \in I} \mu_{2E_i} y$

   Let $\bar{f}^{-1}(\cap_{i \in I} \varepsilon_i) = \bar{f}^{-1}(F)$

   $= G = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$

6. $G_1 = f_1^{-1}(F_1) = f_1^{-1}(\cap_{i \in I} E_{1i}) = \cap_{i \in I} f_1^{-1}(E_{1i})$
7. $G = f^{-1})(F) = f^{-1})(\cup_{i \in I} E_i) = \cup_{i \in I} f^{-1}(E_1)$
8. $L_G = L_B$
9. $\mu_{1G_1} : G_1 \to L_G$ is defined by

$$\mu_{1G_1} x = \begin{cases} \mu_{1B_1} x & \text{, whenver } \Phi^{-1}\, \mu_{1F_1} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1F_1} f_1 x \right) \right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1F_1} f_1 x \right) & , x \notin B \end{cases}$$

10. $\mu_{2G} : G \to L_G$ is defined by

$$\mu_{2G} x = \begin{cases} \mu_{2B} x & \text{, whenver } \Phi^{-1}\, \mu_{2F} f x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2F} f x \\ \alpha \in L_B}} \Phi^{-1} \mu_{2F}\, f x \right) \right] \end{cases}$$

   Let $\bar{f}^{-1}(\varepsilon_i) = D_i = (D_{1i}, D_i, \bar{D}_i(\mu_{1D_{1i}}, \mu_{2D_i}), L_{D_i})$,

   where

11. $D_{1i} = f_1^{-1}(E_{1i})$
12. $D_i = f^{-1}(E_i)$
13. $\mu_{1D_{1i}} : D_{1i} \to L_{D_1}$ is defined by

$$\mu_{1D_{1i}} x = \begin{cases} \mu_{1B_1} x & \text{, whenver } \Phi^{-1}\, \mu_{1E_{1i}} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) \right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}} f_1 x \right) & , x \notin B \end{cases}$$

14. $\mu_{1D_i} : D_i \to L_{D_i}$ is defined by

$$\mu_{2D_i} x = \begin{cases} \mu_{2B}x & \text{, whenver } \Phi^{-1}\mu_{2E_i} fx = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_i} fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_i} fx\right)\right] \end{cases}$$

15. $L_{D_i} = L_B$

Let $\cap_{i \in I} \bar{f}^{-1}(\varepsilon_i) = \cap_{i \in I} D_i = H$

$= (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H)$

Where

16. $E_1 = \cap_{i \in I} D_{1i} = \cap_{i \in I} f_1^{-1})(E_{1i}) = f_1^{-1})(\cap_{i \in I} E_{1i})$
17. $H = \cup_{i \in I} D_i = \cup_{i \in I} f^{-1})(E_i) = f^{-1})(\cup_{i \in I} E_i) = f^{-1}(C) = B$
18. $L_H = \wedge_{i \in I} L_{Di} = L_B$
19. $\mu_{1H_1} : H_1 \to L_H$ is defined by $\mu_{1H_1} x = (\wedge_{i \in I} \mu_{1D_{1i}}) x = \wedge_{i \in I} \mu_{1D_{1i}} x$
20. $\mu_{2H} : H \to L_H$ is defined by $\mu_{2H} x = (\vee_{i \in I} \mu_{2D_i}) x = \vee_{i \in I} \mu_{2D_i} x$

Need to show H and G are Full-equal i.e. to show

21. $H_1 = G_1, H = G$
22. $L_H = L_G$
23. $\mu_{1H_1} = \mu_{1G_1}, \mu_{2H} = \mu_{2G}$

Proof of (21) follows from (6), (7), (16) and (17)

Proof of (22) follows from (8) and (18)

Proof of (23) : Case (I): For $x \in B$

$$\begin{aligned}
\mu_{1H_1} x &= (\wedge_{i \in I} \mu_{1D_{1i}}) x = \wedge_{i \in I} \mu_{1D_{1i}} x \\
&= \wedge_{i\in I} \left\{\mu_{2B}\, x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1E_{1i}} f_1 x\right)\right]\right\} \\
&= \mu_{2B}\, x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1F_1}\, f_1 x\right)\right] \\
&= \mu_{2B}\, x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\wedge_{i\in I} \mu_{1E_{1i}} f_1 x\right]\right)\right] \\
&= \bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\wedge_{i\in I} \mu_{1E_{1i}} f_1 x\right] \\
&= \wedge_{i\in I} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1E_{1i}} f_1 x\right] (\because \text{Proposition } 1.7(a))
\end{aligned}$$

$$\Rightarrow \quad \mu_{2B}\, x \vee \left[\mu_{1B_1} x \vee \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\wedge_{i\in I}\, \mu_{1E_{1i}}\, f_1 x\right]\right)\right] = \mu_{2B}\, x \vee \left[\mu_{1B_1} x \wedge \left(\wedge_{i\in I} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1E_{1i}} f_1 x\right]\right)\right]$$

$\Rightarrow \quad \mu_{1H_1} x = \mu_{1G_1} x$

Case : (II) $\quad x \notin B, y = f_1 x \in \wedge_{i \in I} E_{1i}$

$$\begin{aligned}
\mu_{1H_1} x &= (\wedge_{i \in I}\, \mu_{1D_{1i}})\, x \\
&= \wedge_{i \in I}\, \mu_{1D_{1i}}\, x
\end{aligned}$$

$$= \vee_{i \in I} \left\{ \mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{1i}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1E_{1i}} f_1 x \right) \right\}$$

$$\mu_{1G_1} x = \mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1F_1} f_1 x \right)$$

$$= \mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\wedge_{i \in I} \mu_{1E_{1i}} f_1 x \right] \right)$$

$$= \bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\wedge_{i \in I}\, \mu_{1E_{1i}}\, f_1 x \right]$$

$$= \wedge_{i \in I} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_{1i}}\, f_1 x \right] (\because \text{Proposition } 1.7(a))$$

$$\Rightarrow \quad \mu_{2B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1F_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left[\wedge_{i \in I}\, \mu_{1E_{1i}}\, f_1 x \right] \right) = \mu_{1B_1}\, x \wedge \left(\wedge_{i \in I} \left[\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\, \mu_{1E_{1i}} f_1 x \right] \right)$$

$$\Rightarrow \quad \mu_{1H_1} x = \mu_{1G_1} x$$

Now,

$$\mu_{2H} x = (\vee_{i \in I} \mu_{2D_i})\, x = \vee_{i \in I} \mu_{2D_i}\, x$$

$$= \vee_{i \in I} \left\{ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_i} fx \\ \alpha \in L_B}} \Phi^{-1} \mu_{2E_i} fx \right) \right] \right\}$$

$$\mu_{2G} x = \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2F} fx \\ \alpha \in L_B}} \Phi^{-1} \mu_{2E_i} fx \right) \right]$$

$$= \mu_{2B} x \wedge \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2F} fx \\ \alpha \in L_B}} \Phi^{-1} \left(\vee_{i \in I} \mu_{2E_i} fx \right) \right) \right]$$

$$\Rightarrow \quad \Phi^{-1} (\vee_{i \in I}\, \mu_{2E_i} fx) = \vee_{i \in I} (\vee\, \Phi^{-1}\, \mu_{2E_i} fx) \ (\because \text{ prop}1.7(b))$$

$$\Rightarrow \quad \mu_{1B_1} x \wedge (\vee\, \Phi^{-1} (\vee_{i \in I}\, \mu_{2E_i} fx)) = \mu_{1B_1}\, x \wedge (\vee_{i \in I}\, (\vee\, \Phi^{-1})\, \mu_{2E_i} fx))$$

$$\Rightarrow \quad \mu_{2B}\, x \vee [\mu_{1B_1}\, x \wedge (\vee\, \Phi^{-1} (\vee_{i \in I}\, \mu_{2E_i} fx))] = \mu_{2B}\, x \vee [\mu_{1B_1}\, x \wedge (\vee_{i \in I}\, (\vee\, \Phi^{-1}\, \mu_{2E_i}\, fx))]$$

$$\Rightarrow \quad \mu_{2H} x = \mu_{2G}\, x$$

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