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A INTEGRAL OPERATOR ON THE $\mathcal{CVH}(\beta)$-CLASS

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Abstract: In this paper we prove some property for a general integral operator on the $\mathcal{CVH}(\beta)$-class of convex functions associated with some hyperbola.

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1. INTRODUCTION

Consider $\mathcal{H}(\mathcal{U})$ be the set of functions which are regular in the unit disc $\mathcal{U} = \{z \in \mathbb{C}, |z| < 1\}$, $\mathcal{A}$ denote the class of the functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \quad z \in \mathcal{U}$$

with the property $f(0) = f'(0) - 1 = 0$ and $\mathcal{S} = \{f \in \mathcal{A} : f \textit{ is univalent in } \mathcal{U}\}$.

We recall here the definition of the class $\mathcal{CVH}(\beta)$ introduced by Acu and Owa in [1].

The function $f \in \mathcal{A}$ is in the class $\mathcal{CVH}(\beta)$ $\beta, > 0$, if

$$\left|\frac{zf''(z)}{f'(z)} - 2\beta\left(\sqrt{2}-1\right) + 1\right| < \operatorname{Re}\left\{\sqrt{2}\,\frac{zf''(z)}{f'(z)}\right\} + 2\beta\left(\sqrt{2}-1\right) + \sqrt{2}, \quad z \in \mathcal{U}. \tag{1}$$

Geometric interpretation: $f \in \mathcal{CVH}(\beta)$ if and only if $\frac{zf''(z)}{f'(z)} + 1$ take all values in the convex domain $\Omega(\beta) = \{w = u + i \cdot v : v^2 < 4\beta u + u^2, u > 0\}$. Note that $\Omega(\beta)$ is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin.

Regarding the class $\mathcal{CVH}(\beta)$ we recall the coefficient estimations obtained in [1]:

Theorem 1.1: If $f(z) = \sum_{j=2}^{\infty} a_j z^j$ belong to the class $\mathcal{CVH}(\beta)$, $\beta > 0$, then

$$|a_2| \le \frac{1+4\beta}{2(1+2\beta)}, \quad |a_3| \le \frac{(1+4\beta)(3+16\beta+24\beta^2}{12(1+2\beta)^3}.$$

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We consider the integral operator defined by

$$F_{\alpha_1, \ldots, \alpha_n}(z) = \int_0^z [f_1'(t)]^{\alpha_1} \cdot \ldots \cdot [f_n'(t)]^{\alpha_n}\, dt \tag{2}$$

where $f_i(z) \in \mathcal{A}$ and $\alpha_i > 0$, for all $i \in \{1, \ldots, n\}$. This operator was introduced by Breaz, Owa and Breaz in [2].

2. MAIN RESULTS

Theorem 2.1: If $f_i \in \mathcal{CVH}(\beta_i)$ for all $i \in \{1, \ldots, n\}$, $\beta_i > 0$ and $\alpha_i > 0$, then the integral operator defined in (2) is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^{n} \alpha_i \left(2 - \sqrt{2}\right) \sum_{i=1}^{n} \alpha_i \beta_i$ and $\mathcal{K}(\delta)$ is the class of convex functions of order δ ($\delta < 1$).

Proof: We have, after the simple calculus:

$$\frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} = \alpha_1 \frac{zf_1''(z)}{f_1'(z)} + \ldots + \alpha_n \frac{zf_n''(z)}{f_n'(z)} = \sum_{i=1}^{n} \alpha_i \frac{zf_i''(z)}{f_i'(z)}.$$

Since $f_i \in \mathcal{CVH}(\beta_i)$, for all $i \in \{1, \ldots, n\}$ we have satisfy for all functions f_i the inequality (1).

Thus, we obtain:

$$\sqrt{2}\, \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} = \sqrt{2} \sum_{i=1}^{n} \alpha_i \frac{zf_i''(z)}{f_i'(z)},$$

$$\sqrt{2}\, \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} = \sum_{i=1}^{n} \left(\sqrt{2}\alpha_i \frac{zf_i''(z)}{f_i'(z)} + 2\alpha_i \beta_i \left(\sqrt{2} - 1\right) + \sqrt{2}\alpha_i - 2\alpha_i \beta_i \left(\sqrt{2} - 1\right) - \sqrt{2}\alpha_i \right),$$

$$\sqrt{2}\, \mathrm{Re} \left(\frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} + 1 \right)$$

$$= \mathrm{Re} \sum_{i=1}^{n} \left(\sqrt{2}\alpha_i \frac{zf_i''(z)}{f_i'(z)} + 2\alpha_i \beta_i \left(\sqrt{2} - 1\right) + \sqrt{2}\alpha_i - 2\alpha_i \beta_i \left(\sqrt{2} - 1\right) - \sqrt{2}\alpha_i \right) + \sqrt{2},$$

$$\sqrt{2}\, \mathrm{Re} \left(\frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} + 1 \right)$$

$$= \sum_{i=1}^{n} \alpha_i \left[\mathrm{Re} \left\{ \sqrt{2} \frac{zf_i''(z)}{f_i'(z)} + 2\beta_i \left(\sqrt{2} - 1\right) + \sqrt{2} \right\} \right] - \sum_{i=1}^{n} \left(2\alpha_i \beta_i \left(\sqrt{2} - 1\right) + \sqrt{2}\alpha_i \right) + \sqrt{2} >$$

$$\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i''(z)}{f_i'(z)}-2\beta_i\left(\sqrt{2}-1\right)+1\right|-\sum_{i=1}^{n}\left(2\alpha_i\beta_i\left(\sqrt{2}-1\right)+\alpha_i\sqrt{2}\right)+\sqrt{2}>$$

$$>-\sqrt{2}\sum_{i=1}^{n}\alpha_i-2\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\alpha_i\beta_i+\sqrt{2}.$$

Thus, we obtain:

$$\operatorname{Re}\left(\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}+1\right)>1-\sum_{i=1}^{n}\alpha_i-\left(2-\sqrt{2}\right)\sum_{i=1}^{n}\alpha_i\beta_i$$

which imply that $F_{\alpha_1,\ldots,\alpha_n}(z)\in K(\delta)$, where $\delta=1-\sum_{i=1}^{n}\alpha_i-\left(2-\sqrt{2}\right)\sum_{i=1}^{n}\alpha_i\beta_i$.

If we consider $\beta_1=\beta_2=\cdots=\beta_n=\beta>0$ in the above theorem we obtain:

Corollary 2.1: If $f_i\in\mathcal{CVH}(\beta)$ for all $i\in\{1,\ldots,n\}$, $\beta>0$ and $\alpha_i>0$, then the integral operator defined in (2) is in the class $\mathcal{K}(\delta)$, where $\delta=1-[1+\beta(2-\sqrt{2})]\sum_{i=1}^{n}\alpha_i$ and $\mathcal{K}(\delta)$ is the class of convex functions of order $\delta\,(\delta<1)$.

If we consider $n=1$ in the Theorem 2.1 we obtain:

Corollary 2.2: If $f_1\in\mathcal{CVH}(\beta_1)$, $\beta_1>0$ and $\alpha_1>0$, then the integral operator defined by $F_{\alpha_1}(z)=\int_0^z[f_1'(t)]^{\alpha_1}dt$ is in the class $\mathcal{K}(\delta)$, where $\delta=1-\alpha_1-(2-\sqrt{2})\,\alpha_1\beta_1$ and $\mathcal{K}(\delta)$ is the class of convex functions of order $\delta\,(\delta<1)$.

If in (2) we consider $\alpha_1=\alpha_2=\cdots=\alpha_n=1$ we obtain the integral operator

$$F_1(z)=\int_0^z f_1'(t)\cdot\ldots\cdot f_n'(t)\,dt. \tag{3}$$

For this integral operator we have:

Theorem 2.2: Let $f_i\in\mathcal{CVH}(\beta_i)$ for all $i\in\{1,\ldots,n\}$, $\beta_i>0$, $f_i(z)=z+\sum_{j=2}^{\infty}a_{i,j}z^i$, $i=\overline{1,n}$.

If we consider the integral operator defined by (3), with $F_1(z)=z+\sum_{j=2}^{n}b_jz^j$, then:

$$|b_2|\le\sum_{i=1}^{n}\frac{1+4\beta_i}{(1+2\beta_i)}$$

and $$|b_3|\le\sum_{i=1}^{n}\frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3}+\frac{1}{3}\sum_{i=1}^{n-1}\left(\frac{1+4\beta_k}{(1+2\beta_k)}\cdot\sum_{i=k+1}^{n}\frac{1+4\beta_i}{(1+2\beta_i)}\right).$$

Proof: From (3) we obtain

$$F_1'(z) = f_1'(z) \cdot f_2'(z) \cdots f_n'(z),$$

namely

$$1 + \sum_{j=2}^{\infty} j b_j z^{j-1} = \left(1 + \sum_{j=2}^{\infty} j a_{1,j} \cdot z^{j-1}\right) \cdot \left(1 + \sum_{j=2}^{\infty} j a_{2,j} \cdot z^{j-1}\right) \cdots \left(1 + \sum_{j=2}^{\infty} j a_{n,j} \cdot z^{j-1}\right).$$

Thus we have:

$$b_2 = \sum_{i=1}^{n} a_{i,2}$$

$$b_3 = \sum_{i=1}^{n} a_{i,3} + \frac{4}{3} \sum_{k=1}^{n} \left(a_{k,2} \sum_{i=k+1}^{n} a_{i,2} \right).$$

But, from Theorem 1.1, we have

$$\left| a_{i,2} \right| \le \frac{1+4\beta_i}{2(1+2\beta_i)}, \quad i = \overline{1,n}$$

$$\left| a_{i,3} \right| \le \frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3}, \quad i = \overline{1,n}.$$

In this conditions we obtain

$$\left| b_2 \right| \le \sum_{i=1}^{n} \left| a_{i,2} \right| \le \frac{1}{2} \sum_{i=1}^{n} \frac{1+4\beta_i}{1+2\beta_i}$$

$$\left| b_3 \right| \le \sum_{i=1}^{n} \left| a_{i,3} \right| + \frac{4}{3} \sum_{k=1}^{n-1} \left(\left| a_{k,2} \right| \sum_{i=k+1}^{n} \left| a_{i,2} \right| \right)$$

$$\le \sum_{i=1}^{n} \frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3} + \frac{1}{3} \sum_{k=1}^{n-1} \left(\frac{1+4\beta_k}{1+2\beta_k} \sum_{i=k+1}^{n} \frac{1+4\beta_i}{1+2\beta_i} \right).$$

For $\beta_1 = \beta_2 = \cdots = \beta_n = \beta > 0$, we obtain

Corollary 2.3: Let $f_i \in \mathcal{CVH}(\beta)$, $\beta > 0$, $f_i(z) = z + \sum_{j=2}^{\infty} a_{i,j} z^i$, $i = \overline{1,n}$. If we consider the integral operator defined by (3), with $F_1(z) = z + \sum_{j=2}^{n} b_j z^j$, then:

$$|b_2| \leq \frac{n(1+4\beta)}{2(1+2\beta)}$$

$$|b_3| \leq \frac{n(1+4\beta)\cdot[2n+1+4(3n+1)\beta+8(2n+1)\beta^2]}{12(1+2\beta)^3}.$$

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