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PROBABILISTIC MODELS AND NUMERICAL ALGORITHMS FOR QUASILINEAR PARABOLIC EQUATIONS AND SYSTEMS

YANA BELOPOLSKAYA AND EKATERINA NEMCHENKO

Abstract. We consider two probabilistic approaches to construct stochastic models for quasilinear parabolic equations and systems. These approaches allow to construct stochastic counterparts to quasilinear parabolic equations and systems modeling conservation and balance laws and derive probabilistic representations of the Cauchy problem classical solutions. In addition these approaches allow to construct algorithms for numerical solution of the Cauchy problem based on the derived probabilistic representations.

1. Introduction

Mathematical models of conservation and balance laws in physics, chemistry, biology and other fields often are presented in the form of systems of nonlinear parabolic equations of the form

$$\frac{\partial u_m}{\partial t} + \sum_{i,j=1}^{d}\sum_{q=1}^{d_1} B^i_{mq}(x,u)\nabla_{x_i}u_q = \frac{1}{2}\sum_{i,j=1}^{d} G^{ij}(x,u)\nabla^2_{x_ix_j}u_m + \sum_{q=1}^{d_1} c_{mq}(x,u)u_q, \tag{1.1}$$

called systems with self-diffusion or

$$\frac{\partial u_m}{\partial t} + \sum_{i,j=1}^{d}\sum_{q=1}^{d_1} \nabla_{x_i}[B^i_{mq}(u)u_q] = \frac{1}{2}\sum_{i,j=1}^{d}\sum_{q=1}^{d_1} \nabla^2_{x_ix_j}[G^{ij}_{mq}(x,u)u_q] + \sum_{q=1}^{d_1} c_{mq}(x,u)u_q, \tag{1.2}$$

called systems with cross diffusion [1], [2].

In addition systems of type (1.1) arise when one investigates scalar conservation laws in divergence form

$$\frac{\partial u}{\partial t} = \langle \nabla, (\alpha(u)\nabla u)\rangle, \quad u(0,x) = u_0(x) \in R$$

or balance laws

$$\frac{\partial u}{\partial t} = \langle \nabla, (\alpha(u)\nabla u)\rangle + \gamma(u)u, \quad u(0,x) = u_0(x) \in R \tag{1.3}$$

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which are nonlinear both with respect to u and ∇u. We include this equation into a system of parabolic equations rewriting (1.3) in the form

$$\frac{\partial u}{\partial t} = \alpha(u)\Delta u + \alpha'(u)\langle \nabla u, \nabla u\rangle + \gamma(u)u, \quad u(0,x) = u_0(x), \tag{1.4}$$

and adding to it an equation for the function ∇u. For the resulting semilinear system of parabolic equations we derive a required stochastic system which allows to construct a solution of the Cauchy problem for the original PDE.

Thus, our aim is to construct stochastic counterparts to the Cauchy problem both for (1.1) and (1.2) with initial data $u(0,y) = u_0(y)$ in the form of stochastic equations and obtain probabilistic representations of solutions to the PDE systems. In addition based on the probabilistic representation of a solution to (1.1) we develop an algorithm of constructing of the Cauchy problem numerical solution.

Connections between classical solutions of the Cauchy problem for a system of nonlinear backward Kolmogorov equations and solutions of the corresponding stochastic system were established in [3], [4] under assumptions that coefficients and the Cauchy data of (1.4) are smooth enough and satisfy certain growth conditions. In our previous paper [5] we used this approach to develop a numerical algorithm to construct such solutions and obtained some numerical results. Here we apply this approach to solve numerically the Cauchy problem for a quasilinear parabolic equation.

Numerical schemes based on probabilistic representations of the Cauchy problem solutions for scalar semilinear parabolic equations were developed by Milstein and Tretjakov [6], [7], Talay [8] and others. In our previous paper [5] there were constructed some numerical schemes which allow to obtain approximate numerical solutions of the Cauchy problem for systems of nonlinear backward Kolmogorov equations.

In this paper we consider a class of systems of forward Kolmogorov equations of the form (1.1) which can be reduced to systems of backward ones. This allows to apply the results from [4] and to construct numerical solutions approximating classical solutions of the Cauchy problem based on their probabilistic representations. Since this reduction does not work in the case of systems with cross-diffusion of the form (1.2) we treat them directly as systems of forward Kolmogorov equations similar to [9], [10] and construct probabilistic representations of solutions to the Cauchy problem mild solutions.

The rest of the paper is organized as follows. In section 2 we discuss a stochastic system associated with the Cauchy problem (1.1) and state conditions to ensure existence and uniqueness of classical solutions to (1.1). In addition we construct a probabilistic representation for a classical solution of the Cauchy problem for a scalar quasilinear parabolic equation. In section 3 we discuss a probabilistic interpretation of a mild solution of the Cauchy problem (1.2) for systems of semilinear parabolic equations with cross diffusion. In section 4 we apply the probabilistic approach described in section 2 to develop a numerical scheme which allows to obtain numerical approximation of a classical solution of the Cauchy problem for a nonlinear heat equation and show numerical results.

2. Systems with self-diffusion, probabilistic representation of the Cauchy problem classical solution

Let us reduce the Cauchy problem of the form (1.1) to the Cauchy problem

$$\frac{\partial v_m}{\partial t} - \sum_{q=1}^{d_1} \langle B_{mq}(x,v), \nabla v_q\rangle + \frac{1}{2} Tr[G(x,v)\nabla^2 v_m] + \sum_{q=1}^{d_1} c_{mq}(x,v) v_q = 0, \quad (2.1)$$

$$v_m(T,x) = u_{0m}(x)$$

with respect to a new function $v(T-t,x) = u(t,x)$.

To obtain a stochastic counterpart to (2.1) we fix a probability space $(\Omega, \mathcal{F}, P)$ and the standard Wiener process $w(t) \in R^d$ defined on it. Next we consider a stochastic system of the form

$$d\xi(\tau) = A(\xi(\tau), v(T-\tau, \xi(\tau)))dw(\tau), \quad \xi(t) = x \in R^d, \quad (2.2)$$

$$d\eta(\tau) = c(\xi(\tau), v(T-\tau,\xi(t)))\eta(\tau)d\tau - C(\xi(\tau), v(T-\tau,\xi(\tau)))(\eta(\tau), dw(\tau)), \quad (2.3)$$

$$\eta(t) = h \in R^{d_1}.$$

$$\langle h, v(T-t,x)\rangle = E\langle \eta_{t,h}(T), u_0(\xi_{t,x}(T))\rangle. \quad (2.4)$$

where coefficients of (2.1) and (2.2),(2.3) are connected by relations

$$G_{ij}(x,v) = \sum_{k=1}^{d} A_{ik}(x,v)A_{kj}(x,v), \quad B(x,v) = C(x,v)A(x,v)$$

and $\langle\cdot,\cdot\rangle$ is the scalar product in R^d.

We say that condition **C 2.1** holds if $G(y,u) = A(y,u)A^*(y,u)$, where A^* is a dual matrix to A, and there exist positive constants $K_0, K, L_0, L, \rho_1, \rho, L_\rho$ and a constant ρ_0 such that

$$\|A(y,u) - A(y_1,u_1)\|^2 \le L\|y-y_1\|^2 + L_\rho\|u-u_1\|^2,$$

$$\|A(y,u)\|^2 \le K[1+\|y\|^2+\|u\|^{2p}], \langle c(y,u)h,h\rangle \le [\rho_0+\rho_1\|u\|^p]\|h\|^2, y \in R^d, u,h \in R^{d_1}$$

$$\|[c(y,u) - c(y_1,u_1)]h\|^2 \le L\|y-y_1\|^2 + L_\rho\|u-u_1\|^2]\|h\|^2,$$

$$\|[C(y,u) - C(y_1,u_1)](h,w)\|^2 \le [L\|y-y_1\|^2 + L_\rho\|u-u_1\|^2]\|h\|^2\|w\|^2,$$

$$\|C(y,u)(h,w)\|^2 \le K[1+\|u\|^2]\|h\|^2\|w\|^2, \quad w \in R^d$$

$$\sup_x \|u_0(x)\| \le K_0, \quad \|u_0(x) - u_0(y)\| \le L_0\|x-y\|.$$

Below we often use notations of the type $A^u(y) = A(y, u(y))$.

In [3],[4] there were proved the following assertions.

Theorem 2.1. *Assume that* **C 2.1** *holds. Then there exists a unique solution* $(\xi(t), \eta(t), v(T-t,y))$ *of the system (2.2) –(2.4). In addition* $v(T-t,y)$ *is a bounded Lipschitz continuous function on a certain interval* $[T_1,T]$ *with the length depending on constants in* **C 2.1** *and the initial function* u_0, $\xi(t) \in R^d$ *is a Markov process and* $\eta(t) \in R^{d_1}$ *defines a multiplicative operator functional* $R(s,t) : R^{d_1} \to R^{d_1}$ *of the process* $\xi(t)$, $R(s,t)h = \eta(t), h \in R^{d_1}$.

We denote by $C_b(R^d)$ the space of bounded functions on R^d with the norm $\sup_{R^d}|h(x)| = \|h\|_\infty$ and by $C^k(R^d)$ the space of functions from $C_b(R^d)$ which are k times differentiable.

We say that condition $\mathbf{C2.2}_k$ holds if **C 2.1** holds and in addition all coefficients a^u, A^u, c^u, C^u and the function $u_0(y)$ are functions from the class $C^{k+\beta}(R^d)$ in y, u for $0 < \beta \le 1$.

Theorem 2.2. *Assume that* $\mathbf{C2.2}_2$ *holds. Then the function* $v(T-t,y)$ *defined by (2.4) is a unique classical solution of the Cauchy problem (2.1)*

Proof. Let us give a draft of the proof for the second theorem (see details in [4]). Set

$$\mathcal{L}^v u = \frac{1}{2} Tr A^v(x)\nabla^2 u[A^v]^*(x),$$

$$[\mathcal{M}^v u]_m = -\sum_{q=1}^{d_1} B^v_{mq}(x)\nabla u_q] + \sum_{q=1}^{d_1} c^v_{mq}(x)u_q.$$

Assume that there exists a classical solution $v(T-t,y)$ to (2.1), then there exists a unique solurion to (2.2), (2.3) and applying the Ito formula we derive an expression for a stochastic differential of a process $\gamma(t) = \langle \eta(t), v(T-t,\xi(t))\rangle$. As a result we obtain

$$d\gamma(t) = \langle d\eta(t), v(T-t,\xi(t))\rangle + \langle \eta(t), dv(T-t,\xi(t))\rangle + \langle d\eta(t), dv(T-t,\xi(t))\rangle =$$
$$= \langle \eta(t), \left[\frac{\partial v}{\partial t} + \mathcal{L}^v v + \mathcal{M}^v v\right](T-t,\xi(t))\rangle dt +$$
$$+ \langle \eta(t), [A^v(\xi(t))\nabla v(T-t,\xi(t)) + [C^v]^*(\xi(t))v(T-t,\xi(t))]dw(t)\rangle.$$

Integrating the last equality in time from t up to T and evaluating the expectation we get

$$E[\gamma(T)-\gamma(t)] = E\left[\int_t^T \langle \eta(s), \left[\frac{\partial v}{\partial t} + \mathcal{L}^v v + \mathcal{M}^v v\right](T-s,\xi_{t,y}(s))\rangle ds\right]. \tag{2.5}$$

Then keeping in mind that $E[\gamma(T)] = E\left[\langle \eta_{t,h}(T), u_0(\xi_{t,x}(T))\rangle\right]$ and $\gamma(t) = \langle h, v(T-t,x)\rangle$ we obtain that if $v(t) \in C^2(R^d)$ solves (2.1) it admits a representation (2.4). On the other hand if (2.4) holds and $v(T-t) \in C^2(R^d)$ then applying once again the Ito formula we get

$$E\left[\int_t^T \langle \eta(s), \left[\frac{\partial v}{\partial s} + \mathcal{L}^v v + \mathcal{M}^v v\right](T-s,\xi_{t,x}(s))\rangle ds\right] = E[\gamma(T)] - \gamma(t) = 0.$$

Since this equality holds for arbitrary t and x and $\eta(s) \neq 0$ this means that

$$\frac{\partial v}{\partial t} + \mathcal{L}^v v + \mathcal{M}^v v = 0$$

and one can easily verify that $v(T-0,x) = u_0(x)$.

To prove uniqueness of a classical solution to (2.1) under **C 2.2** we assume that there exist at least two solutions and verify that their probabilistic representations coincide. Thus one can deduce uniqueness of a solution to (2.1) from uniqueness of a solution to (2.2) –(2.4) . □

As a result we obtain that to construct a classical solution to (2.1) it is enough to construct a solution to (2.2) -(2.4) providing the function $v(T-t)$ of the form (2.4) belongs to $C^2(R^d)$.

It should be mentioned that conditions of theorem 2.1 which states existence and uniqueness of a (local in time) solution to (2.2)-(2.4) are more weak then conditions of theorem 2.2 which ensures that the solution of stochastic system gives rise to the classical solution of (2.1).

Now we apply the above approach to construct a solution of a nonlinear heat equation of the form (2.3) assuming that $\alpha(u)>0$. To this end we rewrite (1.4) in the form

$$\frac{\partial u}{\partial t}=\alpha(u)\Delta u+\langle\alpha'(u)u,\nabla u\rangle+\gamma(u)u,\quad u(0,y)=u_0(x)\in R, \tag{2.6}$$

and consider (2.6) as a parabolic equation including two unknown functions u and $v=\nabla u$.

To obtain a closed system we differentiate (2.6) with respect to spatial argument and add to (2.6) an equation for $v=\nabla u$. As a result we get a system including (2.6) and

$$\frac{\partial v}{\partial t}=\alpha(u)\Delta v+2\langle\alpha'(u)v,\nabla v\rangle+\langle\alpha''(u)v,v\rangle v+\gamma'(u)vu+\gamma(u)v,\ v(0,x)=v_0(x)\in R^d. \tag{2.7}$$

To construct a closed stochastic system associated with (2.6), (2.7) we introduce a new couple of functions $f(T-t,x)=u(t,x)$ and $g_i(T-t,x)=\frac{\partial u(t,x)}{\partial x_i}=\frac{\partial f(T-t,x)}{\partial x_i}$ and observe that a system

$$\frac{\partial f}{\partial t}+\alpha(f)\Delta f+\langle\alpha'(f)g,\nabla f\rangle+\gamma(f)f=0,\quad f(T,x)=u_0(y)\in R_+. \tag{2.8}$$

$$\frac{\partial g_i}{\partial t}=\alpha(f)\Delta g_i+2\langle\alpha'(f)g,\nabla g_i\rangle+\alpha''(f)g_i\langle g,g\rangle+\gamma'(f)g_if+\gamma(f)g_i, \tag{2.9}$$

$$g_i(T,x)=\nabla_{x_i}u_{0i}(x)\in R_+,\ i=1,\dots,d,$$

has a structure similar to the structure of the system (2.1). Next we consider a system of stochastic equations similar to (2.2)– (2.4), namely, a system

$$d\xi(\tau)=\alpha'(f(T-\tau,\xi(\tau)))g(T-\tau,\xi(\tau))d\tau+\sqrt{2\alpha(f(T-\tau,\xi(\tau)))}dw(\tau), \tag{2.10}$$

$$\xi(t)=x\in R^d,$$

$$d\eta(\tau)=c(f(T-\tau,\xi(\tau)),g(T-\tau,\xi(t)))\eta(\tau)d\tau+ \tag{2.11}$$

$$+C(f(T-\tau,\xi(\tau)),g(T-\tau,\xi(t)))\eta(\tau)dw(\tau),\quad \eta(t)=h\in R^d,$$

where for $i=1,2,3$, $C^i(f,g)$ and $c(f,g)$ are $(d+1)\times(d+1)$ matrices with components

$$c_{11}(f,g)=\gamma(f),c_{jk}=[\alpha''(f)\langle g,g\rangle+\gamma'(f)f+\gamma(f)]\delta_{jk},\quad j\neq 1\quad\text{or}\quad k\neq 1$$

$$C^i_{1k}=C^i_{k1}=0,\quad\text{and for the remaining indices}\quad C^i_{jk}=2\frac{\alpha'(f)}{\alpha(f)}g.$$

Adding a relation

$$\langle h,V(T-t,x)\rangle=E\langle\eta_{t,h}(T),V_0(\xi_{t,x}(T))\rangle \tag{2.12}$$

where $V(t,x) = (f(t,x), g(t,x))$ and $V(0,x) = (u_0(x), \nabla u_0(x))$ we obtain a closed system (2.10)– (2.12).

Now we can formulate the following assertion.

Theorem 2.3. *Assume that $\alpha \in C^3(R)$, $\gamma \in C^2(R)$ have polynomial growth and $v_0 \in C^2(R^d)$ is bounded. Then there exists an interval $[T_1, T]$ such that for all $s \in [T_1, T]$ there exists a unique solution $(\xi(s), \eta(s), V(T-s,x)) \in R^d \times R^{d_1} \times R^{d_1+1}$ of (2.10)–(2.12). In addition $\xi(s)$ is a Markov process and $V(T-s,x) = (f(T-s,x), g(T-s,x))$ is a bounded Lipschitz continuous function.*

Proof. Since coefficients of (2.10), (2.11) satisfy **C 2.1** we deduce the existence and uniqueness of the solution to (2.10)– (2.12) from theorem 2.1. Besides we obtain that $V(T-t,x)$ is bounded and Lipschitz continuous. □

Theorem 2.4. *If coefficients in (2.10)– (2.12) and u_0 have three bounded continuous derivatives then a solution $(\xi(t), \eta(t), V(T-t,x))$ of (2.10)– (2.12) gives rise to a unique classical solution of the Cauchy problem (2.7) and $u(t,x) = f(T-t,x)$ is a unique classical solution of the Cauchy problem (2.6).*

Proof. Since coefficients of (2.10), (2.11) satisfy $\mathbf{C2.2}_3$ we deduce the existence and uniqueness of the solution to (2.10)– (2.12) from theorem 2.1. Besides we obtain that $V(T-t,x)$ is bounded and twice differentiable. Then applying the Ito formula one can verify the assertions of the theorem. □

3. Systems with cross-diffusion, a probabilistic representation of the Cauchy problem mild solution

As it was mentioned in the introduction the approach described in section 2 does not work for systems of the type (1.2). To develop a probabilistic approach to a solution of the Cauchy problem for such a system we need more weak notions of the solution, namely, the notions of weak and mild solutions of the Cauchy problem. To define these types of solutions we need a number of functional spaces.

We denote by: $C_b(R^d)$ the space of bounded real valued functions on R^d with the norm $\sup_{R^d} |h(x)| = \|h\|_\infty$;

$C_0^\infty(R^d)$ the space of infinitely differentiable real functions with compact supports;

$W^{k,p}(R^d)$ the Sobolev space of order k in $L^p(R^d), \|\cdot\|_p), (1 \le p \le \infty)$ and

$W^{1,1}_{\rm loc}(R^d)$ the space of functions $g : R^d \to R$ such that g and ∇g belong to $L^1_{\rm loc}(R^d) = \{g : \int_{R^d} |h(y)g(y)|dy < \infty, \quad \forall h \in C_0^\infty(R^d)\}$.

In this section we construct a stochastic system which gives rise to a mild solution of the Cauchy problem of the form (1.2). Existence and uniqueness of a solution to this stochastic system will be proved elsewhere.

We restrict ourselves to one of the simplest versions of (1.2). Setting $B^i_{ml} \equiv 0$, $m, l = 1, 2$, $i, j = 1, \dots, d$, $G^{ij}_{ml}(y,u) \equiv \sum_{k=1}^d A^{ik}_m(y) A^{kj}_m(y)\delta_{ml}$, and $[c(y,u)u]_m = c_m(y,u)u_m$, where δ_{ml} is the Kronecker symbol we consider the Cauchy problem

$$\frac{\partial u_m}{\partial t} = \frac{1}{2} \sum_{i,j,k=1}^d \nabla^2_{x_i x_j} [A^{ik}_m(y) A^{kj}_m(y) u_m] + c_m(y,u)u_m, \tag{3.1}$$

$$u_m(0,y) = u_{0m}(y).$$

We say that a function $u(t,\cdot) \in R^2$ is a mild solution of (3.1) if for all $h \in C_0^\infty(R^d;R^2)$, $t \in [0,T]$, $m = 1,2$,

$$\int_{R^d} h_m(y)u_m(t,y)\rangle dy = \int_{R^d} h_m(y)\int_{R^d} u_{0m}(x)p_m(0,x,t,y)dxdy+ \tag{3.2}$$

$$+\int_0^t\int_{R^d}\left(\int^{R^d} h_m(y)p_m(s,z,t,y)dy\right)c_m(z,u(s,z))u_m(s,z)dzds,$$

where $p_m(0,x,t,y)$ is a fundamental solution of a scalar equation

$$\frac{\partial v_m}{\partial t} = \frac{1}{2}\sum_{i,j=1}^d \nabla^2_{y_iy_j}(G_m^{ij}(y)v_m), \quad v_m(0,y) = \delta_y \tag{3.3}$$

and δ_y is the Dirac function at y. One can easily verify that $v_m(t,\cdot)$ belong to $L^1(R^d)$ since

$$\int_{R^d} v_m(t,y)dy = 1, \quad m = 1,2.$$

For a given bounded smooth function $u = (u_1,u_2)$ we consider the Cauchy problem for a linearised system

$$\frac{\partial \mu_m}{\partial t} = \frac{1}{2}\sum_{i,j,k=1}^d \nabla^2_{x_ix_j}[G_m^{ij}(y)\mu_m] + c_m(y,u)\mu_m, \quad \mu_m(0)(dy) = u_{0m}(y)dy. \tag{3.4}$$

We say that a Borel measure $\mu(t,dy) = (\mu_1(t,dy),\mu_2(t,dy))$ is a mild measure -valued solution of the Cauchy problem (3.4) if for all $h \in C_0^\infty(R^d;R^2)$, $t \in [0,T]$

$$\int_{R^d} h_m(y)\mu_m(t,dy) = \int_{R^d} h_m(y)\int_{R^d}\mu_{0m}(dx)P_m(0,x,t,dy)+ \tag{3.5}$$

$$+\int_0^t\int_{R^d}\left(\int_{R^d} h_m(y)P_m(s,z,t,dy)dy\right)c_m(u(s,z))\mu_m(s,dz)ds,$$

where $P_m(0,x,t,dy) = p_m(0,x,t,y)dy$.

To construct a stochastic system associated with (3.1) denoting by $w_m(t)$ independent Wiener processes defined on the given probability space $(\Omega,\mathcal{F},P)$ we consider stochastic equations

$$d\xi^m(\tau) = A_m(\xi^m(\tau))dw_m(\tau),\ \xi^m(0) = \xi_{0m} \sim \mu_{0m},\ 0 \le \tau \le t \tag{3.6}$$

where $\mu_{0m}(dy) = u_{0m}(y)dy = P\{\xi_{0m} \in dy\}$ and closing relations

$$u_m(t,y) = \int_{R^d} p_m(0,x,t,y)\mu_{0m}(dx)+ \tag{3.7}$$

$$+\int_0^t\int_{R^d} p_m(s,z,t,y)c_m(z,u(s,z))u_m(s,z)dzds.$$

Theorem 3.1. *Assume that* $A_m(y), c_m(y,u)$ *satisfy* **C.2.1** *and there exists a solution* $(\xi^m(t), u_m(t,y)), m = 1,2,$ *of the system (3.6)– (3.7) such that* $\xi^m(t)$ *are Markov processes with transition probability densities* $p_m(0,x,t,y)$ *and* $u_m(t,y)$ *are bounded functions.*

A function $u : [0,T] \times R^d \to R^2$, $u \in L^1([0,T], W^{1,1}(R^d;R^2) \cap C_b(R^d;R^2))$ *is a mild solution of (3.1) if for all* $h \in C_0^\infty(R^d)$, $m = 1,2$

$$\int_{R^d} h_m(y)u_m(t,y)dy = E\left[\exp\left\{\int_0^t c_m(\xi^m(s), u(s,\xi^m_{0,\xi_{0m}}(s)))ds\right\} u_{0m}(\xi^m_{0,\xi_{0m}}(t))\right]. \tag{3.8}$$

Proof. Since we can approximate functions $h_m \in C_b(R^d)$ by functions $h_m^n \in C_0^\infty(R^d)$ and $c(u)$ is bounded (this results from **2.1** and boundedness of u) we deduce that the right hand side of (3.8) is a linear functional on $C_b(R^d)$ and by the Riesz theorem there exists a unique measure $\mu_m(t,dy)$ such that

$$\int_{R^d} h_m(y)\mu_m(t,dy) = E\left[\exp\left\{\int_0^t c_m(\xi^m(s), u(s,\xi^m(s)))ds\right\} h_m(\xi^m(t))\right]. \tag{3.9}$$

Since for all $h \in C_0^\infty(R^d, R^2)$ we have

$$E[h_m(\xi^m(t))] = \int_{R^d} u_{0m}(dx)\int_{R^d} h_m(y)p_m(0,x,t,y)dy \tag{3.10}$$

keeping in mind properties of conditional expectations we can rewrite the right hand side of (3.9) in the form

$$E\left[c_m(\xi^m(s), u(s,\xi^m(s)))\exp\left\{\int_0^s c_m(\xi^m(\tau), u(\tau,\xi^m(\tau)))d\tau\right\} h_m(\xi^m(t))\right] = \tag{3.11}$$

$$= E\left[c_m(\xi^m(s), u(s,\xi^m(s)))\exp\left\{\int_0^s c_m(\xi^m(\tau), u(\tau,\xi^m(\tau)))d\tau\right\} E[h_m(\xi^m(t))|\mathcal{F}_s]\right] =$$

$$\int_{R^d} c_m(z,u(s,z))\int_{R^d} h_m(y)p_m(s,z,t,y)dy\,\mu_m(s,dz).$$

To derive the last equality we have used the Riesz theorem choosing

$$c_m(\xi^m(s), u(s,z))\int_{R^d} h_m(y)p^u_m(s,z,t,y)dy$$

for a bounded measurable test function. Substituting (3.10) and (3.11) into (3.9) we obtain that provided $u(t,y)$ is a bounded Lipschitz continuous function, $\mu_m(t,dy)$ is a mild measure-valued solution of (3.4).

Let us prove uniqueness of a measure - valued mild solution of (3.4). Denote by $\mathcal{M}(R^d)$ the space of finite Borel measures on R^d equipped with the total variation norm

$$\|\mu\|_{TV} = \sup_{h\in C_b(R^d), \|h\|_\infty \le 1}\int_{R^d} h(y)\mu(dy)$$

and assume that there exist two measure-valued solutions μ^1 and μ^2 to (3.1). Set $\nu_m = \mu^2_m - \mu^1_m$. Since u is bounded, $c(u)$ is bounded as well and thus $\|\nu_m(t,\cdot)\|_{TV} <$

∞. In addition $\forall h \in C_b(R^d)$ we have

$$\int_{R^d} h_m(y)\nu_m(t,dy) = \int_0^t \int_{R^d} c_m(z,u(s,z)) \int_{R^d} h_m(y)p_m(s,z,t,y)dy\nu_m(s,dz)ds. \tag{3.12}$$

Let us take supremum of both sides of (3.12) over all h_m with $\|h_m\|_\infty \leq 1$ we obtain

$$\|\nu_m(t,\cdot)\|_{TV} \leq \sup_{(s,z)\in[0,T]\times R^d} |c_m(z,u(s,z)) \int_0^t \|\nu_m(\tau,\cdot)\|_{TV} d\tau. \tag{3.13}$$

Then by the Gronwall lemma we deduce $\nu_m(t,\cdot) = 0$. By similar speculations we can prove that the measure $u_m(t,y)dy$ satisfies (3.1) as well and hence $\mu_m(t,dy) = u_m(t,y)dy$. Thus, we deduce that if $u_m(t,y)$ satisfying (3.8) is a mild solution to (3.1). □

4. Numerical scheme

As far as solutions of nonlinear parabolic equations can be rarely solved explicitly an important role is played by the possibility to construct numerical scheme to solve these equations. Applying the standard techniques used for constructing of SDE numerical solutions and successive approximation technique we derive a numerical algorithm to construct approximations for solutions of the Cauchy problem for quasilinear parabolic equations based on their probabilistic representations.

We illustrate this technique constructing a numerical solution for the Cauchy problem of one dimensional nonlinear heat equation

$$\frac{\partial u}{\partial t} = \langle \nabla, u^2 \nabla u\rangle + u^\beta, \quad u_0(x) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{x^2}{\epsilon^2}}, \tag{4.1}$$

where β and ϵ are fixed positive constants. Setting $V_1(T-t,x) = u(t,x), V_2(T-t,x) = \frac{\partial u(t,x)}{\partial x}$ we include the transformed equation (4.1) into a system

$$\frac{\partial V_1}{\partial t} + V_1^2 \frac{\partial^2 V_1}{\partial x^2} + 2V_1V_2\frac{\partial V_1}{\partial x} + V_1^\beta = 0, \quad V_1(T,x) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{x^2}{\epsilon^2}}. \tag{4.2}$$

$$\frac{\partial V_2}{\partial t} + V_1^2 \frac{\partial^2 V_2}{\partial x^2} + 2V_1V_2\frac{\partial V_2}{\partial x} + 4V_1V_2\frac{\partial V_2}{\partial x} + 2V_2^3 + \beta V_1^{\beta-1}V_2 = 0, \tag{4.3}$$

$$V_2(T,x) = \frac{-2x}{\sqrt{2\pi}\epsilon^3} e^{-\frac{x^2}{\epsilon^2}}.$$

A stochastic system associated with (4.1)-(4.2) has the form

$$d\xi(\theta) = a^V(\xi(\theta))d\theta + A^{V_1}(\xi(\theta))dw(\theta), \quad \xi(t) = x, \tag{4.4}$$

$$d\eta(\theta) = [c^V]^*(\xi(\theta))\eta(\theta)d\theta + [C^{V_2}]^*(\xi(\theta))(\eta(\theta), dw(\theta)), \quad \eta(t) = h, \tag{4.5}$$

$$\langle h, V(T-t,x)\rangle = E\langle \eta_{t,h}(T), V_0(\xi_{t,x}(T))\rangle, \tag{4.6}$$

$$a^V(x) = 2V_1(T-t,x)V_2(T-t,x), \quad A^{V_1}(x) = \sqrt{2}V_1(T-t,x),$$

$$c^V(x) = \begin{pmatrix} V_1^{\beta-1}(T-t,x) & 0 \\ 0 & 2V_2^2(T-t,x) + \beta V_1^{\beta-1}(T-t,x) \end{pmatrix},$$

$$C^{V_2}(T-t,x) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}V_2(T-t,x) \end{pmatrix}, \quad V(t,x) = (V_1(t,x), V_2(t,x)),$$

$$h = (1,1), \quad V_0(x) = (V_1(T,x), V_2(T,x)) = \left(u_0(x), \frac{\partial u_0(x)}{\partial x}\right).$$

To solve numerically the system (4.4) –(4.6) we use evolution properties of propagators

$$S(t,T)V_1(T,x) = V_1(T-t,x), \quad G(t,T)\frac{\partial V_2(T,x)}{\partial x} = V_2(T-t,x).$$

Namely, we consider a partition $T_1 = t_0 \le t_1 \le \cdots \le t_n = T$ of the time interval $[T-T_1, T]$ (where a solution to (2.12) – (**??**) is defined) and construct approximations $S^n(s,t)$, $G^n(s,t)$ such that

$$S(t,T)V_1(T,x) = \lim_{n\to\infty} \prod_{k=0}^{n-1} S^n(t_k, t_{k+1})V_1(T,x),$$

$$G(t,T)V_2(T,x) = \lim_{n\to\infty} \prod_{k=0}^{n-1} G^n(t_k, t_{k+1})V_2(T,x),$$

The required approximations $S^n(t_k,t_{k+1}), G^n(t_k,t_{k+1})$ are constructed by the Euler-Maruyama method with an additional approximation of Gaussian random variables by Bernoulli random variables on intervals $[t_k, t_{k+1}]$ similar to [5], [6]. Besides, to obtain a numerical algorithm we introduce a space partition of a space interval $[-L, L]$, $-L = x_0 \le \cdots \le x_i \le \cdots \le x_m = L$.

Set $\bar V(t,x) = V(T-t,x), \Delta = t_{k+1} - t_k = \frac{T_1}{n}$. The resulting algorithm can be presented as follows:

$$\bar V(T,x_i) = (u_0(x_i), \frac{\partial u_0(x_i)}{\partial x}) = (V_1^0(x_i), V_2^0(x_i)),$$

$$\bar V_1^1(t_{n-1},x_i) = \frac{1}{2}[(1+\bar V_1^0(x_i)^{\beta-1}\Delta)(\bar V_1^0(x_i + 2\bar V_1^0(x_i)\bar V_2^0(x_i)\Delta + \tag{4.7}$$
$$+\sqrt{2}\bar V_1^0(x_i)\sqrt{\Delta}) + \bar V_1^0(x_i + 2\bar V_1^0(x_i)\bar V_2^0(x_i)\Delta - \sqrt{2}\bar V_1^0(x_i)\sqrt{\Delta}))],$$

$$\bar V_2^1(t_{n-1},x_i) = \frac{1}{2}[(1 + (2(\bar V_2^0(x_i))^2 + \beta(\bar V_1^0(x_i))^{\beta-1})\Delta + \tag{4.8}$$
$$+2\sqrt{2}\bar V_2^0(x_i)\sqrt{\Delta})\bar V_2^0(x_i + 2\bar V_1^0(x_i)\bar V_2^0(x_i)\Delta + \sqrt{2}\bar V_1^0(x_i)\sqrt{\Delta}) +$$
$$+(1 + (2(\bar V_2^0(x_i))^2 + \beta(\bar V_1^0(x_i))^{\beta-1})\Delta - 2\sqrt{2}\bar V_2^0(x_i)\sqrt{\Delta})\bar V_2^0(x_i +$$
$$+2\bar V_1^0(x_i)\bar V_2^0(x_i)\Delta - \sqrt{2}\bar V_1^0(x_i)\sqrt{\Delta})],$$

for t_k, $k = n-2, ..., 0$

$$\bar V^0(\theta, x) = \bar V^1(t_{k+1}, x) \quad t_k \le \theta \le t_{k+1},$$

$$\bar V_1^1(t_k,x_i) = \frac{1}{2}[(1 + (\bar V_1^0(t_k,x_i))^{\beta-1}\Delta)(\bar V_1^0(t_{k+1}, x_i + 2\bar V_1^0(t_k,x_i)\bar V_2^0(t_k,x_i)\Delta + \tag{4.9}$$
$$+\sqrt{2}\bar V_1^0(t_k,x_i)\sqrt{\Delta}) + \bar V_1^0(t_{k+1}, x_i + 2\bar V_1^0(t_k,x_i)\bar V_2^0(t_k,x_i)\Delta - \sqrt{2}\bar V_1^0(t_k,x_i)\sqrt{\Delta}))],$$

$$\bar V_2^1(t_k,x_i) = \frac{1}{2}[(1 + (2(\bar V_2^0(t_k,x_i))^2 + \beta(\bar V_1^0(t_k,x_i))^{\beta-1})\Delta + \tag{4.10}$$
$$+2\sqrt{2}\bar V_2^0(t_k,x_i)\sqrt{\Delta})\bar V_2^0(t_{k+1}, x_i + 2\bar V_1^0(t_k,x_i)\bar V_2^0(t_k,x_i)\Delta + \sqrt{2}\bar V_1^0(t_k,x_i)\sqrt{\Delta}) +$$
$$+(1 + (2(\bar V_2^0(t_k,x_i))^2 + \beta(\bar V_1^0(t_k,x_i))^{\beta-1})\Delta - 2\sqrt{2}\bar V_2^0(t_k,x_i)\sqrt{\Delta})\bar V_2^0(t_{k+1}, x_i +$$
$$+2\bar V_1^0(t_k,x_i)\bar V_2^0(t_k,x_i)\Delta - \sqrt{2}\bar V_1^0(t_k,x_i)\sqrt{\Delta})].$$

Here $\bar{V}^1(t_k, x)$ is an approximate solution of (4.6) and upper index corresponds to the iteration number, $x_i = x_0 + i\sqrt{\Delta}$, $i = 0, \dots, m$.

To obtain a value of the function $V_l(t_{k+1}, x)$ at a point x which does not coincide with a grid point we use a linear interpolation

$$V_l(t_{k+1}, x) = \frac{x_{i+1} - x}{\sqrt{\Delta}} V_l(t_{k+1}, x_i) + \frac{x - x_i}{\sqrt{\Delta}} V_l(t_{k+1}, x_{i+1}), \tag{4.11}$$

$x_i \le x \le x_{i+1}$, $i = 0, \dots, m$, $k = n-1, \dots, 0$, $l = 1, 2$.

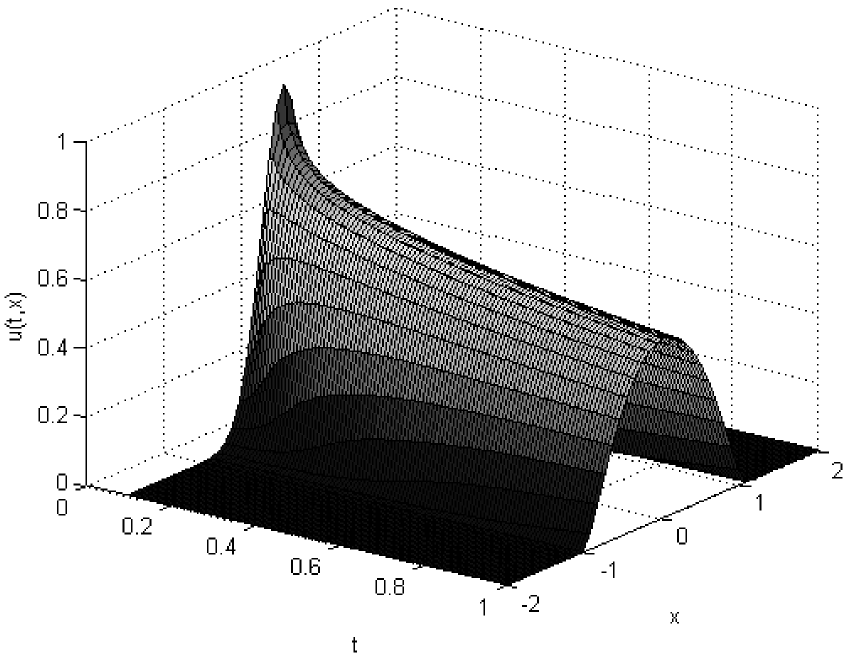

Figure 1. A solution to (4.1) for $\beta = 2$, $\epsilon = 0.4$, $x \in [-1.9922, 1.9922]$.

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Yana Belopolskaya: Saint-Petersburg State University of Architecture and Civil Engineering, Saint-Petersburg, 198005, Russia.

E-mail address: `yana@yb1569.spb.edu`

Ekaterina Nemchenko: Saint-Petersburg State University of Architecture and Civil Engineering, Saint-Petersburg, 198005, Russia.