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CONDITIONS FOR STATIONARITY AND ERGODICITY OF TWO-FACTOR AFFINE DIFFUSIONS

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Abstract. Sufficient conditions are presented for the existence of a unique stationary distribution and exponential ergodicity of two-factor affine diffusion processes.

1. Introduction

We consider general 2-dimensional two-factor affine diffusion processes

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma_1\sqrt{Y_t}\,\mathrm{d}W_t, \\ \mathrm{d}X_t = (\alpha - \beta Y_t - \gamma X_t)\,\mathrm{d}t + \sigma_2\sqrt{Y_t}\,(\varrho\,\mathrm{d}W_t + \sqrt{1-\varrho^2}\,\mathrm{d}B_t) + \sigma_3\,\mathrm{d}L_t, \end{cases} \tag{1.1}$$

for $t \in [0,\infty)$, where $a \in [0,\infty)$, $b, \alpha, \beta, \gamma \in \mathbb{R}$, $\sigma_1, \sigma_2, \sigma_3 \in [0,\infty)$, $\varrho \in [-1,1]$ and $(W_t, B_t, L_t)_{t\in[0,\infty)}$ is a 3-dimensional standard Wiener process. Affine processes are joint generalizations of continuous state branching processes and Orstein–Uhlenbeck type processes, and they have applications in financial mathematics, see, e.g., in Duffie et al. [7]. The aim of the present paper is to extend the results of Barczy et al. [1] for the processes given in (1.1), where the case of $\beta = 0$, $\varrho = 0$, $\sigma_1 = 1$, $\sigma_2 = 1$, $\sigma_3 = 0$ is covered. We give sufficient conditions for the existence of a unique stationary distribution and exponential ergodicity, see Theorems 3.1 and 4.1, respectively. These results can be used in a forthcoming paper for studying parameter estimation for this model. An important observation is that it is enough to prove the results for the special case of $\varrho = 0$, since there is a non-singular linear transform of a 2-dimensional affine diffusion process which is a special 2-dimensional affine diffusion process with $\varrho = 0$, see Proposition 2.5. Otherwise, the method of the proofs are the same as in Barczy et al. [1].

2. The Affine Two-factor Diffusion Model

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_-$, $\mathbb{R}_{--}$ and $\mathbb{C}$ denote the sets of positive integers, non-negative integers, real numbers, non-negative real numbers, positive real numbers, non-positive real numbers, negative real numbers and complex numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min(x,y)$ and $x \vee y := \max(x,y)$. By $C^2_c(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, we denote the set of twice continuously differentiable real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support. We will

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denote the convergence in distribution and equality in distribution by $\stackrel{\mathcal{D}}{\longrightarrow}$ and $\stackrel{\mathcal{D}}{=}$, respectively.

We start with the definition of a two-factor affine process.

Definition 2.1. A time-homogeneous Markov process $(Y_t, X_t)_{t\in\mathbb{R}_+}$ with state space $\mathbb{R}_+\times\mathbb{R}$ is called a *two-factor affine process* if its (conditional) characteristic function takes the form

$$\begin{aligned}&\mathbb{E}(\mathrm{e}^{\mathrm{i}(u_1Y_t+u_2X_t)}\,|\,(Y_0,X_0)=(y_0,x_0))\\&=\exp\{y_0G_1(t,u_1,u_2)+x_0G_2(t,u_1,u_2)+H(t,u_1,u_2)\},\qquad (u_1,u_2)\in\mathbb{R}^2,\end{aligned}$$

for $(y_0,x_0)\in\mathbb{R}_+\times\mathbb{R}$, $t\in\mathbb{R}_+$, where $G_1(t,u_1,u_2), G_2(t,u_1,u_2), H(t,u_1,u_2)\in\mathbb{C}$.

Let $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P}\big)$ be a filtered probability space satisfying the usual conditions, i.e., $(\Omega,\mathcal{F},\mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $(W_t,B_t,L_t)_{t\in[0,\infty)}$ be a 3-dimensional standard $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-Wiener process.

The next proposition is about the existence and uniqueness of a strong solution of the SDE (1.1).

Proposition 2.2. *Let (η_0,ξ_0) be a random vector independent of the process $(W_t,B_t,L_t)_{t\in\mathbb{R}_+}$ satisfying $\mathbb{P}(\eta_0\in\mathbb{R}_+)=1$. Then for all $a\in\mathbb{R}_+$, $b,\alpha,\beta,\gamma\in\mathbb{R}$, $\sigma_1,\sigma_2,\sigma_3\in\mathbb{R}_+$, $\varrho\in[-1,1]$, there is a (pathwise) unique strong solution $(Y_t,X_t)_{t\in\mathbb{R}_+}$ of the SDE (1.1) such that $\mathbb{P}((Y_0,X_0)=(\eta_0,\xi_0))=1$ and $\mathbb{P}(Y_t\in\mathbb{R}_+$ for all $t\in\mathbb{R}_+)=1$. Further, for all $s,t\in\mathbb{R}_+$ with $s\leqslant t$, we have*

$$Y_t=\mathrm{e}^{-b(t-s)}Y_s+a\int_s^t\mathrm{e}^{-b(t-u)}\,\mathrm{d}u+\sigma_1\int_s^t\mathrm{e}^{-b(t-u)}\sqrt{Y_u}\,\mathrm{d}W_u \tag{2.1}$$

and

$$\begin{aligned}X_t&=\mathrm{e}^{-\gamma(t-s)}X_s+\int_s^t\mathrm{e}^{-\gamma(t-u)}(\alpha-\beta Y_u)\,\mathrm{d}u\\&\qquad+\sigma_2\int_s^t\mathrm{e}^{-\gamma(t-u)}\sqrt{Y_u}\,(\varrho\,\mathrm{d}W_u+\sqrt{1-\varrho^2}\,\mathrm{d}B_u)+\sigma_3\int_s^t\mathrm{e}^{-\gamma(t-u)}\,\mathrm{d}L_u.\end{aligned} \tag{2.2}$$

Moreover, $(Y_t,X_t)_{t\in\mathbb{R}_+}$ is a two-factor affine process with infinitesimal generator

$$\begin{aligned}(\mathcal{A}_{(Y,X)}f)(y,x)&=(a-by)f_1'(y,x)+(\alpha-\beta y-\gamma x)f_2'(y,x)\\&\quad+\frac{1}{2}y\big[\sigma_1^2f_{1,1}''(y,x)+2\varrho\sigma_1\sigma_2f_{1,2}''(y,x)+\sigma_2^2f_{2,2}''(y,x)\big]\\&\quad+\frac{1}{2}\sigma_3^2f_{2,2}''(y,x),\end{aligned} \tag{2.3}$$

where $(y,x)\in\mathbb{R}_+\times\mathbb{R}$, $f\in\mathcal{C}_c^2(\mathbb{R}_+\times\mathbb{R},\mathbb{R})$, and f_i', $i\in\{1,2\}$, and $f_{i,j}''$, $i,j\in\{1,2\}$, denote the first and second order partial derivatives of f with respect to its i-th and i-th and j-th variables.

Conversely, every two-factor affine diffusion process is a (pathwise) unique strong solution of a SDE (1.1) with suitable parameters $a\in\mathbb{R}_+$, $b,\alpha,\beta,\gamma\in\mathbb{R}$, $\sigma_1,\sigma_2,\sigma_3\in\mathbb{R}_+$ and $\varrho\in[-1,1]$.

Proof. Equation (1.1) is a special case of the equation (6.6) in Dawson and Li [6], and Theorem 6.2 in Dawson and Li [6] implies that for any initial value (η_0, ξ_0) with $\mathbb{P}((\eta_0, \xi_0) \in \mathbb{R}_+ \times \mathbb{R}) = 1$ and $\mathbb{E}(\eta_0) < \infty$, $\mathbb{E}(|\xi_0|) < \infty$, there exists a pathwise unique non-negative strong solution satisfying $\mathbb{P}((Y_0, X_0) = (\eta_0, \xi_0)) = 1$ and $\mathbb{P}(Y_t \in \mathbb{R}_+$ for all $t \in \mathbb{R}_+) = 1$.

Applications of the Itô's formula to the processes $(\mathrm{e}^{bt} Y_t)_{t\in\mathbb{R}_+}$ and $(\mathrm{e}^{\gamma t} X_t)_{t\in\mathbb{R}_+}$ give formulas (2.1) and (2.2), respectively.

The form of the infinitesimal generator (2.3) readily follows by (6.5) in Dawson and Li [6]. Further, Theorem 6.2 in Dawson and Li [6] also implies that Y is a continuous state and continuous time branching process with infinitesimal generator given in the Proposition.

The converse follows from Theorems 6.1 and 6.2 in Dawson and Li [6]. □

Next we present a result about the first moment of $(Y_t, X_t)_{t\in\mathbb{R}_+}$ together with its asymptotic behavior as $t \to \infty$. Note that the formula for $\mathbb{E}(Y_t)$, $t \in \mathbb{R}_+$, is well known.

Proposition 2.3. *Let* $(Y_t, X_t)_{t\in\mathbb{R}_+}$ *be the unique strong solution of the SDE* (1.1) *satisfying* $\mathbb{P}(Y_0 \in \mathbb{R}_+) = 1$ *and* $\mathbb{E}(Y_0) < \infty$, $\mathbb{E}(|X_0|) < \infty$. *Then*

$$\begin{bmatrix} \mathbb{E}(Y_t) \\ \mathbb{E}(X_t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-bt} & 0 \\ -\beta \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{(\gamma-b)u} \, \mathrm{d}u & \mathrm{e}^{-\gamma t} \end{bmatrix} \begin{bmatrix} \mathbb{E}(Y_0) \\ \mathbb{E}(X_0) \end{bmatrix} + \begin{bmatrix} \int_0^t \mathrm{e}^{-bu} \, \mathrm{d}u & 0 \\ -\beta \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{\gamma u} \left(\int_0^u \mathrm{e}^{-bv} \, \mathrm{d}v \right) \mathrm{d}u & \int_0^t \mathrm{e}^{-\gamma u} \, \mathrm{d}u \end{bmatrix} \begin{bmatrix} a \\ \alpha \end{bmatrix}, \qquad t \in \mathbb{R}_+.$$

Consequently, as $t \to \infty$, *if* $b \in \mathbb{R}_{++}$, *then* $\mathbb{E}(Y_t) = \frac{a}{b} + \mathrm{O}(\mathrm{e}^{-bt})$ *and*

$$\mathbb{E}(X_t) = \begin{cases} \frac{\alpha}{\gamma} - \frac{a\beta}{b\gamma} + \mathrm{O}(\mathrm{e}^{-(b\wedge\gamma)t}), & \gamma \in \mathbb{R}_{++}, \\ \left(\alpha - \frac{a\beta}{b}\right)t + \mathrm{O}(1), & \gamma = 0, \\ \left(\frac{\beta}{\gamma-b} \mathbb{E}(Y_0) + \mathbb{E}(X_0) - \frac{\alpha}{\gamma} + \frac{a\beta}{b\gamma} - \frac{a\beta}{(\gamma-b)b}\right)\mathrm{e}^{-\gamma t} + \mathrm{O}(1), & \gamma \in \mathbb{R}_{--}; \end{cases}$$

if $b = 0$, *then* $\mathbb{E}(Y_t) = at + \mathrm{O}(1)$, *and*

$$\mathbb{E}(X_t) = \begin{cases} -\frac{a\beta}{\gamma} t + \mathrm{O}(1), & \gamma \in \mathbb{R}_{++}, \\ -\frac{1}{2} a\beta t^2 + \mathrm{O}(t), & \gamma = 0, \\ \left(\frac{\beta}{\gamma} \mathbb{E}(Y_0) + \mathbb{E}(X_0) - \frac{\alpha}{\gamma} - \frac{a\beta}{\gamma^2}\right)\mathrm{e}^{-\gamma t} + \mathrm{O}(t), & \gamma \in \mathbb{R}_{--}; \end{cases}$$

if $b \in \mathbb{R}_{--}$, *then* $\mathbb{E}(Y_t) = \left(\mathbb{E}(Y_0) - \frac{a}{b}\right)\mathrm{e}^{-bt} + \mathrm{O}(1)$, *and*

$$\mathbb{E}(X_t) = \begin{cases} \left(-\frac{\beta}{\gamma-b} \mathbb{E}(Y_0) + \frac{a\beta}{(\gamma-b)b}\right)\mathrm{e}^{-bt} + \mathrm{O}(1), & \gamma \in \mathbb{R}_{++}, \\ \left(\frac{\beta}{b} \mathbb{E}(Y_0) + \mathbb{E}(X_0) - \frac{\beta a}{b^2}\right)\mathrm{e}^{-bt} + \mathrm{O}(t), & \gamma = 0, \\ \left(-\frac{\beta}{\gamma-b} \mathbb{E}(Y_0) + \frac{a\beta}{(\gamma-b)b}\right)\mathrm{e}^{-bt} + \mathrm{O}(\mathrm{e}^{-\gamma t}), & \gamma \in (b, 0), \\ \left(-\beta \mathbb{E}(Y_0) + \frac{a\beta}{b}\right) t\mathrm{e}^{-bt} + \mathrm{O}(\mathrm{e}^{-\gamma t}), & \gamma = b, \\ \left(\frac{\beta}{\gamma-b} \mathbb{E}(Y_0) + \mathbb{E}(X_0) - \frac{\alpha}{\gamma} + \frac{a\beta}{b\gamma} - \frac{a\beta}{b(\gamma-b)}\right)\mathrm{e}^{-\gamma t} + \mathrm{O}(\mathrm{e}^{-bt}), & \gamma \in (-\infty, b). \end{cases}$$

Proof. It is sufficient to prove the statement in the case when $(Y_0, X_0) = (y_0, x_0)$ with an arbitrary $(y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, since then the statement of the proposition follows by the law of total expectation.

The formula for $\mathbb{E}(Y_t)$, $t \in \mathbb{R}_+$, can be found, e.g., in Cox et al. [5, Equation (19)] or Jeanblanc et al. [9, Theorem 6.3.3.1]. Next we observe that

$$\left(\int_0^t \mathrm{e}^{-\gamma(t-u)} \sqrt{Y_u} \, \mathrm{d}(\varrho W_u + \sqrt{1-\varrho^2} B_u)\right)_{t \in \mathbb{R}_+} \tag{2.4}$$

is a square integrable martingale, since

$$\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{-\gamma(t-u)} \sqrt{Y_u} \, \mathrm{d}(\varrho W_u + \sqrt{1-\varrho^2} B_u)\right)^2\right] = \int_0^t \mathrm{e}^{-2\gamma(t-u)} \, \mathbb{E}(Y_u) \, \mathrm{d}u < \infty,$$

where the finiteness of the integral follows from

$$\mathbb{E}(Y_s) = \mathrm{e}^{-bs} y_0 + a \int_0^s \mathrm{e}^{-bu} \, \mathrm{d}u, \qquad s \in \mathbb{R}_+,$$

see, e.g., Cox et al. [5, Equation (19)], Jeanblanc et al. [9, Theorem 6.3.3.1] or Proposition 3.2 in Barczy et al. [1]. In a similar way,

$$\left(\int_0^t \mathrm{e}^{-\gamma(t-u)} \, \mathrm{d}L_u\right)_{t \in \mathbb{R}_+} \tag{2.5}$$

is a square integrable martingale, since

$$\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{-\gamma(t-u)} \, \mathrm{d}L_u\right)^2\right] = \int_0^t \mathrm{e}^{-2\gamma(t-u)} \, \mathrm{d}u < \infty.$$

Taking expectations of both sides of the equation (2.2) and using the martingale property of the processes in (2.4) and (2.5), we have

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathrm{e}^{-\gamma t} x_0 + \int_0^t \mathrm{e}^{-\gamma(t-u)} (\alpha - \beta \, \mathbb{E}(Y_u)) \, \mathrm{d}u \\
&= \mathrm{e}^{-\gamma t} x_0 + \alpha \int_0^t \mathrm{e}^{-\gamma(t-u)} \, \mathrm{d}u - \beta \int_0^t \mathrm{e}^{-\gamma(t-u)} \left(\mathrm{e}^{-bu} y_0 + a \int_0^u \mathrm{e}^{-bv} \, \mathrm{d}v\right) \mathrm{d}u \\
&= \mathrm{e}^{-\gamma t} x_0 - \beta y_0 \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{(\gamma - b)u} \, \mathrm{d}u \\
&\quad + \alpha \int_0^t \mathrm{e}^{-\gamma v} \, \mathrm{d}v - \beta a \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{\gamma u} \left(\int_0^u \mathrm{e}^{-bv} \, \mathrm{d}v\right) \mathrm{d}u, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

The asymptotic behavior of $\mathbb{E}(Y_t)$ as $t \to \infty$ does not depend on γ, which can be derived from

$$\mathbb{E}(Y_t) = \mathrm{e}^{-bt} y_0 + a \int_0^t \mathrm{e}^{-bu} \, \mathrm{d}u = \begin{cases} \frac{a}{b} + \left(y_0 - \frac{a}{b}\right) \mathrm{e}^{-bt}, & b \neq 0, \\ y_0 + at, & b = 0. \end{cases}$$

The asymptotic behavior of $\mathbb{E}(X_t)$ as $t \to \infty$ does depend on b and γ as well. We have

$$\int_0^t \mathrm{e}^{-\gamma v}\,\mathrm{d}v = \begin{cases} \frac{1-\mathrm{e}^{-\gamma t}}{\gamma}, & \gamma \neq 0, \\ t, & \gamma = 0, \end{cases} \qquad \mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{(\gamma-b)u}\,\mathrm{d}u = \begin{cases} \frac{\mathrm{e}^{-bt}-\mathrm{e}^{-\gamma t}}{\gamma-b}, & b \neq \gamma, \\ t\mathrm{e}^{-\gamma t}, & b = \gamma, \end{cases}$$

$$\mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma u}\left(\int_0^u \mathrm{e}^{-bv}\,\mathrm{d}v\right)\mathrm{d}u = \begin{cases} \frac{1}{b}\mathrm{e}^{-\gamma t}\int_0^t \left(\mathrm{e}^{\gamma u} - \mathrm{e}^{(\gamma-b)u}\right)\mathrm{d}u, & b \neq 0, \\ \mathrm{e}^{-\gamma t}\int_0^t u\mathrm{e}^{\gamma u}\,\mathrm{d}u, & b = 0, \end{cases}$$

$$= \begin{cases} \frac{1-\mathrm{e}^{-\gamma t}}{b\gamma} - \frac{\mathrm{e}^{-bt}-\mathrm{e}^{-\gamma t}}{(\gamma-b)b}, & b \neq 0, \ \gamma \neq 0, \ b \neq \gamma, \\ \frac{t}{b} - \frac{1-\mathrm{e}^{-bt}}{b^2} & b \neq 0, \ \gamma = 0, \\ \frac{1-\mathrm{e}^{-\gamma t}}{b\gamma} - \frac{t\mathrm{e}^{-\gamma t}}{b}, & b \neq 0, \ \gamma \neq 0, \ b = \gamma, \\ \frac{t}{\gamma} - \frac{1-\mathrm{e}^{-\gamma t}}{\gamma^2} & b = 0, \ \gamma \neq 0, \\ \frac{t^2}{2}, & b = 0, \ \gamma = 0. \end{cases}$$

Consequently, if $b \neq 0$, $\gamma \neq 0$ and $b \neq \gamma$, then

$$\begin{aligned} \mathbb{E}(X_t) &= \mathrm{e}^{-\gamma t}x_0 - \beta y_0 \frac{\mathrm{e}^{-bt} - \mathrm{e}^{-\gamma t}}{\gamma - b} + \frac{\alpha}{\gamma}(1 - \mathrm{e}^{-\gamma t}) \\ &\quad - \frac{a\beta}{b\gamma}(1 - \mathrm{e}^{-\gamma t}) + \frac{a\beta}{(\gamma - b)b}(\mathrm{e}^{-bt} - \mathrm{e}^{-\gamma t}) \\ &= \frac{\alpha}{\gamma} - \frac{a\beta}{b\gamma} + \left(-\frac{\beta}{\gamma - b}y_0 + \frac{a\beta}{(\gamma - b)b}\right)\mathrm{e}^{-bt} \\ &\quad + \left(\frac{\beta}{\gamma - b}y_0 + x_0 - \frac{\alpha}{\gamma} + \frac{a\beta}{b\gamma} - \frac{a\beta}{(\gamma - b)b}\right)\mathrm{e}^{-\gamma t}. \end{aligned}$$

Moreover, if $b \neq 0$, $\gamma \neq 0$ and $b = \gamma$, then

$$\begin{aligned} \mathbb{E}(X_t) &= \mathrm{e}^{-\gamma t}x_0 - \beta y_0 t\mathrm{e}^{-\gamma t} + \frac{\alpha}{\gamma}(1 - \mathrm{e}^{-\gamma t}) - \frac{a\beta}{b\gamma}(1 - \mathrm{e}^{-\gamma t}) + \frac{a\beta}{b}t\mathrm{e}^{-\gamma t} \\ &= \frac{\alpha}{\gamma} - \frac{a\beta}{b\gamma} + \left(x_0 - \frac{\alpha}{\gamma} + \frac{a\beta}{b\gamma}\right)\mathrm{e}^{-\gamma t} + \left(-\beta y_0 + \frac{a\beta}{b}\right)t\mathrm{e}^{-\gamma t}. \end{aligned}$$

Further, if $b \neq 0$ and $\gamma = 0$, then

$$\begin{aligned} \mathbb{E}(X_t) &= \mathrm{e}^{-\gamma t}x_0 - \beta y_0 \frac{1 - \mathrm{e}^{-bt}}{b} + \alpha t - \frac{a\beta}{b\gamma}t + \frac{a\beta}{b^2}(1 - \mathrm{e}^{-bt}) \\ &= -\frac{\beta}{b}y_0 + \frac{a\beta}{b^2} + \left(\alpha - \frac{a\beta}{b}\right)t + \left(\frac{\beta}{b}y_0 + x_0 - \frac{a\beta}{b^2}\right)\mathrm{e}^{-bt}. \end{aligned}$$

In a similar way, if $b = 0$ and $\gamma \neq 0$, then

$$\begin{aligned} \mathbb{E}(X_t) &= \mathrm{e}^{-\gamma t}x_0 - \beta y_0 \frac{1 - \mathrm{e}^{-\gamma t}}{\gamma} + \frac{\alpha}{\gamma}(1 - \mathrm{e}^{-\gamma t}) - a\beta\left(\frac{t}{\gamma} - \frac{1 - \mathrm{e}^{-\gamma t}}{\gamma^2}\right) \\ &= -\frac{\beta}{\gamma}y_0 + \frac{\alpha}{\gamma} + \frac{a\beta}{\gamma^2} - \frac{a\beta}{\gamma}t + \left(\frac{\beta}{\gamma}y_0 + x_0 - \frac{\alpha}{\gamma} - \frac{a\beta}{\gamma^2}\right)\mathrm{e}^{-\gamma t}, \end{aligned}$$

and if $b = 0$ and $\gamma = 0$, then

$$\mathbb{E}(X_t) = x_0 - \beta y_0 t + \alpha t - a\beta \frac{t^2}{2}.$$

The asymptotic behavior of $\mathbb{E}(X_t)$ as $t \to \infty$ can be derived from the above formulas. □

Based on the asymptotic behaviour of the expectations $(\mathbb{E}(Y_t), \mathbb{E}(X_t))$ as $t \to \infty$, we introduce a classification of affine diffusion processes given by the SDE (1.1).

Definition 2.4. Let $(Y_t, X_t)_{t\in\mathbb{R}_+}$ be the unique strong solution of the SDE (1.1) satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_+) = 1$. We call $(Y_t, X_t)_{t\in\mathbb{R}_+}$ *subcritical, critical* or *supercritical* if $b \wedge \gamma \in \mathbb{R}_{++}$, $b \wedge \gamma = 0$ or $b \wedge \gamma \in \mathbb{R}_{--}$, respectively.

The next proposition describes a non-singular linear transform of a 2-dimensional affine diffusion process which is a special 2-dimensional affine diffusion process with $\varrho = 0$.

Proposition 2.5. *Let us consider the 2-dimensional affine diffusion model* (1.1) *with* $a \in \mathbb{R}_+$, $b, \alpha, \beta, \gamma \in \mathbb{R}$, $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}_+$, $\varrho \in [-1, 1]$, *and with a random initial value* (η_0, ζ_0) *independent of* $(W_t, B_t, L_t)_{t\in\mathbb{R}_+}$ *satisfying* $\mathbb{P}(\eta_0 \in \mathbb{R}_+) = 1$. *Put*

$$c := \begin{cases} 0, & \text{if } \sigma_1 = 0, \\ \frac{\sigma_2 \varrho}{\sigma_1}, & \text{if } \sigma_1 > 0, \end{cases} \qquad Z_t := X_t - cY_t, \qquad t \in \mathbb{R}_+. \tag{2.6}$$

Then the process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ *is a regular affine process with infinitesimal generator*

$$\begin{aligned}(\mathcal{A}_{(Y,Z)} f)(y, z) &= (a - by) f_1'(y, z) + (A - By - \gamma z) f_2'(y, z) \\ &\quad + \frac{1}{2} y \big[\sigma_1^2 f_{1,1}''(y, z) + \Sigma_2^2 f_{2,2}''(y, z)\big] + \frac{1}{2} \sigma_3^2 f_{2,2}''(y, z),\end{aligned} \tag{2.7}$$

for $(y, z) \in \mathbb{R}_+ \times \mathbb{R}$ *and* $f \in \mathcal{C}_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, *where*

$$A := \alpha - ca, \qquad B := \beta - c(b - \gamma), \qquad \Sigma_2 := \begin{cases} \sigma_2, & \text{if } \sigma_1 = 0, \\ \sigma_2\sqrt{1 - \varrho^2}, & \text{if } \sigma_1 > 0. \end{cases}$$

Proof. If $\sigma_1 = 0$, then the statement follows from Proposition 2.2.

If $\sigma_1 > 0$, then, by Itô's formula, $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ is the unique strong solution of the SDE

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma_1\sqrt{Y_t}\,\mathrm{d}W_t, \\ \mathrm{d}Z_t = (A - BY_t - \gamma Z_t)\,\mathrm{d}t + \Sigma_2\sqrt{Y_t}\,\mathrm{d}B_t + \sigma_3\,\mathrm{d}L_t. \end{cases} \qquad t \in \mathbb{R}_+, \tag{2.8}$$

with random initial value $\big(\eta_0, \zeta_0 - c\eta_0\big)$. By Proposition 2.2, $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ is a regular affine process with infinitesimal generator (2.7). □

3. Stationarity

The following result states the existence of a unique stationary distribution of the affine diffusion process given by the SDE (1.1). Let $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) \leqslant 0\}$.

Theorem 3.1. *Let us consider the 2-dimensional affine diffusion model* (1.1) *with* $a \in \mathbb{R}_+$, $b \in \mathbb{R}_{++}$, $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_{++}$, $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}_+$, $\varrho \in [-1, 1]$, *and with a random initial value* (η_0, ζ_0) *independent of* $(W_t, B_t, L_t)_{t\in\mathbb{R}_+}$ *satisfying* $\mathbb{P}(\eta_0 \in \mathbb{R}_+) = 1$. *Then*

(i) $(Y_t, X_t) \xrightarrow{\mathcal{D}} (Y_\infty, X_\infty)$ *as* $t \to \infty$, *and we have*

$$\mathbb{E}\big(\mathrm{e}^{u_1 Y_\infty + \mathrm{i}\lambda_2 X_\infty}\big) = \exp\left\{ a \int_0^\infty \kappa_s(u_1, \lambda_2)\,\mathrm{d}s + \mathrm{i}\frac{\alpha}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2 \right\} \tag{3.1}$$

for $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, *where* $\kappa_t(u_1, \lambda_2)$, $t \in \mathbb{R}_+$, *is the unique solution of the (deterministic) differential equation*

$$\begin{cases} \frac{\partial \kappa_t}{\partial t}(u_1, \lambda_2) = -b\kappa_t(u_1, \lambda_2) - \mathrm{i}\beta \mathrm{e}^{-\gamma t}\lambda_2 + \frac{1}{2}\sigma_1^2 \kappa_t(u_1, \lambda_2)^2 \\ \qquad\qquad\qquad + \mathrm{i}\varrho\sigma_1\sigma_2 \mathrm{e}^{-\gamma t}\lambda_2 \kappa_t(u_1, \lambda_2) - \frac{1}{2}\sigma_2^2 \mathrm{e}^{-2\gamma t}\lambda_2^2, \\ \kappa_0(u_1, \lambda_2) = u_1; \end{cases} \tag{3.2}$$

(ii) *supposing that the random initial value* (η_0, ζ_0) *has the same distribution as* (Y_∞, X_∞) *given in part* (i), $(Y_t, X_t)_{t\in\mathbb{R}_+}$ *is strictly stationary.*

Proof. First we check that it is enough to prove the statement (i) for the special affine diffusion process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ given in Proposition 2.5. Hence we suppose that (i) holds for $(Y_t, Z_t)_{t\in\mathbb{R}_+}$, and we check that then (i) holds for $(Y_t, X_t)_{t\in\mathbb{R}_+}$ as well.

If $\sigma_1 = 0$ then $(Y_t, X_t)_{t\in\mathbb{R}_+} = (Y_t, Z_t)_{t\in\mathbb{R}_+}$, hence (i) trivially holds for $(Y_t, X_t)_{t\in\mathbb{R}_+}$ as well.

If $\sigma_1 > 0$ then $(Y_t, Z_t) \xrightarrow{\mathcal{D}} (Y_\infty, Z_\infty)$ as $t \to \infty$, and we have

$$\mathbb{E}\big(\mathrm{e}^{u_1 Y_\infty + \mathrm{i}\lambda_2 Z_\infty}\big) = \exp\left\{ a \int_0^\infty K_s(u_1, \lambda_2)\,\mathrm{d}s + \mathrm{i}\frac{A}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2 \right\}$$

for $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, where $K_t(u_1, \lambda_2)$, $t \in \mathbb{R}_+$, is the unique solution of the differential equation

$$\begin{cases} \frac{\partial K_t}{\partial t}(u_1, \lambda_2) = -bK_t(u_1, \lambda_2) - \mathrm{i}B\mathrm{e}^{-\gamma t}\lambda_2 \\ \qquad\qquad\qquad + \frac{1}{2}\sigma_1^2 K_t(u_1, \lambda_2)^2 - \frac{1}{2}\Sigma_2^2 \mathrm{e}^{-2\gamma t}\lambda_2^2, \qquad t \in \mathbb{R}_+, \\ K_0(u_1, \lambda_2) = u_1. \end{cases} \tag{3.3}$$

By the continuous mapping theorem, we obtain

$$(Y_t, X_t) = \left(Y_t, Z_t + \frac{\sigma_2\varrho}{\sigma_1}Y_t\right) \xrightarrow{\mathcal{D}} \left(Y_\infty, Z_\infty + \frac{\sigma_2\varrho}{\sigma_1}Y_\infty\right) =: (Y_\infty, X_\infty) \qquad \text{as } t \to \infty.$$

Moreover, for each $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\Big(\mathrm{e}^{u_1 Y_\infty + \mathrm{i}\lambda_2 X_\infty}\Big) &= \mathbb{E}\Big(\exp\Big\{u_1 Y_\infty + \mathrm{i}\lambda_2\Big(Z_\infty + \frac{\sigma_2\varrho}{\sigma_1}Y_\infty\Big)\Big\}\Big) \\
&= \mathbb{E}\Big(\exp\Big\{\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2\Big)Y_\infty + \mathrm{i}\lambda_2 Z_\infty\Big\}\Big) \\
&= \exp\left\{a\int_0^\infty K_s\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big)\,\mathrm{d}s + \mathrm{i}\frac{A}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2\right\} \\
&= \exp\left\{a\int_0^\infty K_s\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big)\,\mathrm{d}s + \mathrm{i}\frac{\alpha}{\gamma}\lambda_2 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\,\frac{a}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2\right\} \\
&= \exp\left\{a\int_0^\infty \kappa_s(u_1, \lambda_2)\,\mathrm{d}s + \mathrm{i}\frac{\alpha}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2\right\},
\end{aligned}$$

where

$$\kappa_t(u_1, \lambda_2) := K_t\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2$$

for $t \in \mathbb{R}_+$ and $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$. Using that $K_t(u_1, \lambda_2)$, $t \in \mathbb{R}_+$, satisfies the differential equation (3.3), we get

$$\begin{aligned}
\frac{\partial \kappa_t}{\partial t}(u_1, \lambda_2) &= \frac{\partial K_t}{\partial t}\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) + \mathrm{i}\gamma\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2 \\
&= -bK_t\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) - \mathrm{i}B\mathrm{e}^{-\gamma t}\lambda_2 + \frac{1}{2}\sigma_1^2 K_t\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big)^2 \\
&\quad - \frac{1}{2}\Sigma_2^2\mathrm{e}^{-2\gamma t}\lambda_2^2 + \mathrm{i}\gamma\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2 \\
&= -b\Big(\kappa_t(u_1, \lambda_2) + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2\Big) - \mathrm{i}\Big(\beta - \frac{\sigma_2\varrho}{\sigma_1}(b - \gamma)\Big)\mathrm{e}^{-\gamma t}\lambda_2 \\
&\quad + \frac{1}{2}\sigma_1^2\Big(\kappa_t(u_1, \lambda_2) + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2\Big)^2 - \frac{1}{2}\sigma_2^2(1 - \varrho^2)\mathrm{e}^{-2\gamma t}\lambda_2^2 + \mathrm{i}\gamma\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2 \\
&= -b\kappa_t(u_1, \lambda_2) - \mathrm{i}\beta\mathrm{e}^{-\gamma t}\lambda_2 + \frac{1}{2}\sigma_1^2\kappa_t(u_1, \lambda_2)^2 \\
&\quad + \mathrm{i}\varrho\sigma_1\sigma_2\mathrm{e}^{-\gamma t}\lambda_2\kappa_t(u_1, \lambda_2) - \frac{1}{2}\sigma_2^2\mathrm{e}^{-2\gamma t}\lambda_2^2,
\end{aligned}$$

and

$$\kappa_0(u_1, \lambda_2) = K_0\Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2 = \Big(u_1 + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2\Big) - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2 = u_1,$$

hence $\kappa_t(u_1, \lambda_2)$, $t \in \mathbb{R}_+$, is a solution of the differential equation (3.2). In a similar way, if $\kappa_t(u_1, \lambda_2)$, $t \in \mathbb{R}_+$, satisfies the differential equation (3.2), then

$$K_t(u_1, \lambda_2) := \kappa_t\Big(u_1 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2, \qquad t \in \mathbb{R}_+,$$

is a solution of the differential equation (3.3) for each $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, since

$$\begin{aligned}
&\frac{\partial K_t}{\partial t}(u_1, \lambda_2) = \frac{\partial \kappa_t}{\partial t}\Big(u_1 - \mathrm{i}\frac{\sigma_2 \varrho}{\sigma_1}\lambda_2, \lambda_2\Big) - \mathrm{i}\gamma\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2 \\
&= -b\kappa_t\Big(u_1 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) - \mathrm{i}\beta\mathrm{e}^{-\gamma t}\lambda_2 + \frac{1}{2}\sigma_1^2\kappa_t\Big(u_1 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big)^2 \\
&\quad + \mathrm{i}\varrho\sigma_1\sigma_2\mathrm{e}^{-\gamma t}\lambda_2\kappa_t\Big(u_1 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) - \frac{1}{2}\sigma_2^2\mathrm{e}^{-2\gamma t}\lambda_2^2 - \mathrm{i}\gamma\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2 \\
&= -b\Big(K_t(u_1, \lambda_2) - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2\Big) - \mathrm{i}\beta\mathrm{e}^{-\gamma t}\lambda_2 + \frac{1}{2}\sigma_1^2\Big(K_t(u_1, \lambda_2) - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2\Big)^2 \\
&\quad + \mathrm{i}\varrho\sigma_1\sigma_2\mathrm{e}^{-\gamma t}\lambda_2\Big(K_t(u_1, \lambda_2) - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2\Big) - \frac{1}{2}\sigma_2^2\mathrm{e}^{-2\gamma t}\lambda_2^2 - \mathrm{i}\gamma\frac{\sigma_2\varrho}{\sigma_1}\mathrm{e}^{-\gamma t}\lambda_2 \\
&= -bK_t(u_1, \lambda_2) - \mathrm{i}B\mathrm{e}^{-\gamma t}\lambda_2 + \frac{1}{2}\sigma_1^2 K_t(u_1, \lambda_2)^2 - \frac{1}{2}\Sigma_2^2\mathrm{e}^{-2\gamma t}\lambda_2^2,
\end{aligned}$$

and

$$K_0(u_1, \lambda_2) = \kappa_0\Big(u_1 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2, \lambda_2\Big) + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2 = \Big(u_1 - \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2\Big) + \mathrm{i}\frac{\sigma_2\varrho}{\sigma_1}\lambda_2 = u_1.$$

Consequently, $\kappa_t(u_1, \lambda_2)$, $t \in \mathbb{R}_+$, is the unique solution of the differential equation (3.2).

(i): We prove this part for the special linear transform described in Proposition 2.5 in three steps.

Step 1. By Theorem 6.1 in Dawson and Li [6] and Proposition 2.2, we have

$$\mathbb{E}\big(\mathrm{e}^{\langle u,(Y_t,Z_t)\rangle} \,\big|\, (Y_0, Z_0) = (y_0, z_0)\big) = \mathrm{e}^{\langle (y_0,z_0),\psi_t(u)\rangle + \phi_t(u)} \tag{3.4}$$

for $u \in \mathbb{C}_- \times (\mathrm{i}\mathbb{R})$, $(y_0, z_0) \in \mathbb{R}_+ \times \mathbb{R}$, $t \in \mathbb{R}_+$, for all $u = (u_1, u_2) \in \mathbb{C}_- \times (\mathrm{i}\mathbb{R})$, we have $\psi_t(u) = (\psi_t^{(1)}(u), \mathrm{e}^{-\gamma t}u_2)$, $t \in \mathbb{R}_+$, where $\psi_t^{(1)}(u)$, $t \in \mathbb{R}_+$, is a solution of the Riccati equation

$$\begin{cases} \frac{\partial \psi_t^{(1)}}{\partial t}(u) = R(\psi_t^{(1)}(u), \mathrm{e}^{-\gamma t}u_2), & t \in \mathbb{R}_+, \\ \psi_0^{(1)}(u) = u_1, \end{cases} \tag{3.5}$$

the function $\mathbb{R}_+ \times (\mathbb{C}_- \times (\mathrm{i}\mathbb{R})) \ni (t, u) \mapsto \psi_t^{(1)}(u)$ is continuous, and

$$\phi_t(u) = \int_0^t F(\psi_s^{(1)}(u), \mathrm{e}^{-\gamma s}u_2)\,\mathrm{d}s, \qquad t \in \mathbb{R}_+,$$

where the (complex valued) functions F and R are given by

$$F(u) = au_1 + Au_2 + \frac{1}{2}\sigma_3^2 u_2^2, \qquad R(u) = -bu_1 - Bu_2 + \frac{1}{2}\sigma_1^2 u_1^2 + \frac{1}{2}\Sigma_2^2 u_2^2$$

for $u = (u_1, u_2) \in \mathbb{C}_- \times (\mathrm{i}\mathbb{R})$. Note that for every $u = (u_1, u_2) \in \mathbb{C}_- \times (\mathrm{i}\mathbb{R})$ and $t \in \mathbb{R}_+$, we have $\psi_t^{(1)}(u) \in \mathbb{C}_-$ and $\phi_t(u) \in \mathbb{C}_-$. Indeed,

$$|\,\mathbb{E}\big(\mathrm{e}^{\langle u,(Y_t,Z_t)\rangle} \,\big|\, (Y_0, Z_0) = (y_0, z_0)\big)| \leqslant \mathbb{E}\big(|\mathrm{e}^{\langle u,(Y_t,Z_t)\rangle}| \,\big|\, (Y_0, Z_0) = (y_0, z_0)\big) \leqslant 1,$$

since $|\mathrm{e}^{\langle u,(Y_t,Z_t)\rangle}| = \mathrm{e}^{\mathrm{Re}(\langle u,(Y_t,Z_t)\rangle)} = \mathrm{e}^{Y_t \mathrm{Re}(u_1)} \leqslant 1$ by $Y_t \geqslant 0$ and $\mathrm{Re}(u_1) \leqslant 0$. Consequently,

$$|\mathrm{e}^{\langle (y_0,z_0),\psi_t(u)\rangle+\phi_t(u)}| = \mathrm{e}^{\mathrm{Re}(\langle (y_0,z_0),\psi_t(u)\rangle)+\mathrm{Re}(\phi_t(u))} = \mathrm{e}^{y_0 \mathrm{Re}(\psi_t^{(1)}(u))+\mathrm{Re}(\phi_t(u))} \leqslant 1,$$

hence $y_0 \mathrm{Re}(\psi_t^{(1)}(u)) + \mathrm{Re}(\phi_t(u)) \leqslant 0$. Putting $y_0 = 0$, we obtain $\mathrm{Re}(\phi_t(u)) \leqslant 0$, thus $\phi_t(u) \in \mathbb{C}_-$. Further, for each $y_0 > 0$, we have $\mathrm{Re}(\psi_t^{(1)}(u)) \leqslant -\mathrm{Re}(\phi_t(u))/y_0$, thus letting $y_0 \to \infty$, we obtain $\mathrm{Re}(\psi_t^{(1)}(u)) \leqslant 0$, and hence $\psi_t^{(1)}(u) \in \mathbb{C}_-$.

Moreover, for all $t \in \mathbb{R}_+$ and $u = (u_1, u_2) \in \mathbb{C}_- \times (\mathrm{i}\mathbb{R})$, we have

$$\begin{aligned}\phi_t(u) &= \int_0^t \Big(a\psi_s^{(1)}(u) + A\mathrm{e}^{-\gamma s}u_2 + \frac{1}{2}\sigma_3^2(\mathrm{e}^{-\gamma s}u_2)^2\Big)\,\mathrm{d}s\\ &= a\int_0^t \psi_s^{(1)}(u)\,\mathrm{d}s + Au_2\frac{1-\mathrm{e}^{-\gamma t}}{\gamma} + \frac{1}{2}\sigma_3^2 u_2^2 \frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}.\end{aligned}$$

In fact, we have

$$\mathbb{E}\big(\mathrm{e}^{u_1Y_t+\mathrm{i}\lambda_2 Z_t} \,\big|\, (Y_0,Z_0)=(y_0,z_0)\big) = \exp\big\{y_0K_t(u_1,\lambda_2) + \mathrm{i}z_0\mathrm{e}^{-\gamma t}\lambda_2 + g_t(u_1,\lambda_2)\big\} \tag{3.6}$$

for $(u_1,\lambda_2) \in \mathbb{C}_- \times \mathbb{R}$ and $(y_0,z_0) \in \mathbb{R}_+ \times \mathbb{R}$, where

$$g_t(u_1,\lambda_2) := a\int_0^t K_s(u_1,\lambda_2)\,\mathrm{d}s + \mathrm{i}A\lambda_2\frac{1-\mathrm{e}^{-\gamma t}}{\gamma} - \frac{1}{2}\sigma_3^2\lambda_2^2\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}, \tag{3.7}$$

and $K_t(u_1,\lambda_2)$, $t \in \mathbb{R}_+$, is the unique solution of the differential equation (3.3). Indeed, by (3.4) with $u_2 = \mathrm{i}\lambda_2$, we have

$$\begin{aligned}&\mathbb{E}\big(\mathrm{e}^{u_1Y_t+\mathrm{i}\lambda_2 Z_t} \,\big|\, (Y_0,Z_0)=(y_0,z_0)\big)\\ &= \exp\Big\{y_0\psi_t^{(1)}(u_1,\mathrm{i}\lambda_2) + \mathrm{i}z_0\mathrm{e}^{-\gamma t}\lambda_2 + a\int_0^t \psi_s^{(1)}(u_1,\mathrm{i}\lambda_2)\,\mathrm{d}s\\ &\qquad + \mathrm{i}A\lambda_2\frac{1-\mathrm{e}^{-\gamma t}}{\gamma} - \frac{1}{2}\sigma_3^2\lambda_2^2\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}\Big\}\end{aligned}$$

for $(y_0,z_0) \in \mathbb{R}_+ \times \mathbb{R}$, where

$$\begin{cases}\frac{\partial\psi_t^{(1)}}{\partial t}(u_1,\mathrm{i}\lambda_2) = -b\psi_t^{(1)}(u_1,\mathrm{i}\lambda_2) - B(\mathrm{e}^{-\gamma t}\mathrm{i}\lambda_2)\\ \qquad\qquad + \frac{1}{2}\sigma_1^2[\psi_t^{(1)}(u_1,\mathrm{i}\lambda_2)]^2 + \frac{1}{2}\Sigma_2^2(\mathrm{e}^{-\gamma t}\mathrm{i}\lambda_2)^2, \qquad t \in \mathbb{R}_+,\\ \psi_0^{(1)}(u_1,\mathrm{i}\lambda_2) = u_1,\end{cases}$$

and hence, for the function $K_t(u_1,\lambda_2) := \psi_t^{(1)}(u_1,\mathrm{i}\lambda_2)$, $t \in \mathbb{R}_+$, we obtain the differential equation (3.3). Recall that $K_t(u_1,\lambda_2) \in \mathbb{C}_-$ for all $(u_1,\lambda_2) \in \mathbb{C}_- \times \mathbb{R}$. The uniqueness of the solution of the differential equation (Cauchy problem) (3.3) follows by general results of Duffie et al. [7, Propositions 6.1, 6.4 and Lemma 9.2].

Step 2. We show that there exists $C_2 \in \mathbb{R}_{++}$ (depending on the parameters b and γ), and for each $(u_1,\lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, there exists $C_1(u_1,\lambda_2) \in \mathbb{R}_+$ (depending on the parameters b, B, γ, σ_1 and Σ_2), such that

$$|K_t(u_1,\lambda_2)| \leqslant C_1(u_1,\lambda_2)\mathrm{e}^{-C_2 t}, \qquad t \in \mathbb{R}_+. \tag{3.8}$$

Let us introduce the functions $v_t(u_1,\lambda_2)$, $t\in\mathbb{R}_+$, and $w_t(u_1,\lambda_2)$, $t\in\mathbb{R}_+$, by

$$v_t(u_1,\lambda_2) := -\operatorname{Re}(K_t(u_1,\lambda_2)), \qquad w_t(u_1,\lambda_2) := \operatorname{Im}(K_t(u_1,\lambda_2)), \qquad t\in\mathbb{R}_+.$$

We observe that, as a consequence of (3.3), the function $(v_t(u_1,\lambda_2), w_t(u_1,\lambda_2))$, $t\in\mathbb{R}_+$, is the unique solution of the system of the Riccati equations

$$\begin{cases} \frac{\partial v_t}{\partial t}(u_1,\lambda_2) = -bv_t(u_1,\lambda_2) - \frac{1}{2}\sigma_1^2\big(v_t(u_1,\lambda_2)^2 - w_t(u_1,\lambda_2)^2\big) + \frac{1}{2}\Sigma_2^2\mathrm{e}^{-2\gamma t}\lambda_2^2,\\ \frac{\partial w_t}{\partial t}(u_1,\lambda_2) = -bw_t(u_1,\lambda_2) - B\mathrm{e}^{-\gamma t}\lambda_2 - \sigma_1^2 v_t(u_1,\lambda_2)w_t(u_1,\lambda_2),\\ v_0(u_1,\lambda_2) = -\operatorname{Re}(u_1),\\ w_0(u_1,\lambda_2) = \operatorname{Im}(u_1). \end{cases} \tag{3.9}$$

Note that $K_t(u_1,\lambda_2)\in\mathbb{C}_-$ implies $v_t(u_1,\lambda_2)\geqslant 0$ for all $(u_1,\lambda_2)\in\mathbb{C}_-\times\mathbb{R}$. Clearly, the function $w_t(u_1,\lambda_2)$, $t\in\mathbb{R}_+$, is the unique solution of the inhomogeneous linear differential equation

$$\begin{cases} \frac{\partial w_t}{\partial t}(u_1,\lambda_2) = -f_t(u_1,\lambda_2)w_t(u_1,\lambda_2) - B\lambda_2\mathrm{e}^{-\gamma t}, \qquad t\in\mathbb{R}_+,\\ w_0(u_1,\lambda_2) = \operatorname{Im}(u_1), \end{cases} \tag{3.10}$$

with $f_t(u_1,\lambda_2) := b + \sigma_1^2 v_t(u_1,\lambda_2)$, $t\in\mathbb{R}_+$. The general solution of the homogeneous linear differential equation

$$\frac{\partial \widetilde{w}_t}{\partial t}(u_1,\lambda_2) = -f_t(u_1,\lambda_2)\widetilde{w}_t(u_1,\lambda_2), \qquad t\in\mathbb{R}_+,$$

takes the form

$$\widetilde{w}_t(u_1,\lambda_2) = C\mathrm{e}^{-\int_0^t f_z(u_1,\lambda_2)\,\mathrm{d}z}, \qquad t\in\mathbb{R}_+,$$

where $C\in\mathbb{R}$. By variation of constants, the function

$$\mathbb{R}_+\ni t\mapsto -B\lambda_2\mathrm{e}^{-\int_0^t f_z(u_1,\lambda_2)\,\mathrm{d}z}\int_0^t \mathrm{e}^{-\gamma s+\int_0^s f_z(u_1,\lambda_2)\,\mathrm{d}z}\,\mathrm{d}s, \qquad t\in\mathbb{R}_+,$$

is a particular solution of the inhomogeneous linear differential equation (3.10). Hence a general solution of the inhomogeneous linear differential equation takes the form

$$w_t(u_1,\lambda_2) = C\mathrm{e}^{-\int_0^t f_z(u_1,\lambda_2)\,\mathrm{d}z} - B\lambda_2\mathrm{e}^{-\int_0^t f_z(u_1,\lambda_2)\,\mathrm{d}z}\int_0^t \mathrm{e}^{-\gamma s+\int_0^s f_z(u_1,\lambda_2)\,\mathrm{d}z}\,\mathrm{d}s$$

for $t\in\mathbb{R}_+$. Taking into account of the initial value $w_0(u_1,\lambda_2)=\operatorname{Im}(u_1)$, we obtain $C=\operatorname{Im}(u_1)$. Consequently,

$$|w_t(u_1,\lambda_2)| \leqslant |\operatorname{Im}(u_1)|\mathrm{e}^{-\int_0^t f_z(u_1,\lambda_2)\,\mathrm{d}z} + |B\lambda_2|\int_0^t \mathrm{e}^{-\gamma s-\int_s^t f_z(u_1,\lambda_2)\,\mathrm{d}z}\,\mathrm{d}s$$

for $t\in\mathbb{R}_+$. Applying $f_t(u_1,\lambda_2)\geqslant b>0$, $t\in\mathbb{R}_+$, we get $\mathrm{e}^{-\int_0^t f_z(u_1,\lambda_2)\,\mathrm{d}z}\leqslant \mathrm{e}^{-bt}$, $t\in\mathbb{R}_+$, and

$$\int_0^t \mathrm{e}^{-\gamma s-\int_s^t f_z(u_1,\lambda_2)\,\mathrm{d}z}\,\mathrm{d}s \leqslant \int_0^t \mathrm{e}^{-\gamma s-(t-s)b}\,\mathrm{d}s = \begin{cases} \frac{\mathrm{e}^{-\gamma t}-\mathrm{e}^{-bt}}{b-\gamma} \leqslant \frac{\mathrm{e}^{-t\min\{\gamma,b\}}}{|b-\gamma|}, & b\neq\gamma,\\ t\mathrm{e}^{-bt}\leqslant \frac{2}{\mathrm{e}b}\mathrm{e}^{-bt/2}, & b=\gamma, \end{cases}$$

since $te^{-bt} \leqslant e^{-bt/2} \sup_{t\in\mathbb{R}_+} te^{-bt/2}$, where $\sup_{t\in\mathbb{R}_+} te^{-bt/2} = 2e^{-1}/b$. Summarizing, we have

$$|w_t(u_1,\lambda_2)| \leqslant C_3(u_1,\lambda_2)e^{-C_2 t}, \qquad t\in\mathbb{R}_+, \tag{3.11}$$

with

$$C_3(u_1,\lambda_2) := |\operatorname{Im}(u_1)| + |B\lambda_2|\left(\frac{1}{|b-\gamma|}\mathbb{1}_{\{b\neq\gamma\}} + \frac{2}{eb}\mathbb{1}_{\{b=\gamma\}}\right) \in \mathbb{R}_+,$$
$$C_2 := \min\{\gamma, b/2\} \in \mathbb{R}_{++}.$$

Using (3.9) and (3.11), we obtain

$$\begin{cases} \frac{\partial v_t}{\partial t}(u_1,\lambda_2) \leqslant -bv_t(u_1,\lambda_2) + C_4(u_1,\lambda_2)e^{-C_2 t}, \qquad t\in\mathbb{R}_+,\\ v_0(u_1,\lambda_2) = -\operatorname{Re}(u_1), \end{cases}$$

with $C_4(u_1,\lambda_2) := (\sigma_1^2 C_3(u_1,\lambda_2)^2 + \Sigma_2^2\lambda_2^2)/2 \in \mathbb{R}_+$. By the help of a version of the comparison theorem (see, e.g., Volkmann [17]), we can derive the inequality $v_t(u_1,\lambda_2) \leqslant \widetilde{v}_t(u_1,\lambda_2)$ for all $t\in\mathbb{R}_+$ and $(u_1,\lambda_2)\in\mathbb{C}_-\times\mathbb{R}$, where $\widetilde{v}_t(u_1,\lambda_2)$, $t\in\mathbb{R}_+$, is the unique solution of the inhomogeneous linear differential equation

$$\begin{cases} \frac{\partial \widetilde{v}_t}{\partial t}(u_1,\lambda_2) = -b\widetilde{v}_t(u_1,\lambda_2) + C_4(u_1,\lambda_2)e^{-C_2 t}, \qquad t\in\mathbb{R}_+,\\ \widetilde{v}_0(u_1,\lambda_2) = -\operatorname{Re}(u_1). \end{cases}$$

This differential equation has the same form as (3.10), hence the solution takes the form

$$\widetilde{v}_t(u_1,\lambda_2) = -\operatorname{Re}(u_1)e^{-bt} + C_4(u_1,\lambda_2)e^{-bt}\int_0^t e^{-C_2 s + bs}\,ds, \qquad t\in\mathbb{R}_+.$$

We have $b - C_2 \geqslant b/2 > 0$ and $b > b/2 \geqslant C_2$, thus

$$\begin{aligned} 0 \leqslant v_t(u_1,\lambda_2) \leqslant \widetilde{v}_t(u_1,\lambda_2) &= -\operatorname{Re}(u_1)e^{-bt} + C_4(u_1,\lambda_2)\frac{e^{-C_2 t} - e^{-bt}}{b - C_2}\\ &\leqslant C_5(u_1,\lambda_2)e^{-C_2 t}, \qquad t\in\mathbb{R}_+, \end{aligned}$$

with $C_5(u_1,\lambda_2) := -\operatorname{Re}(u_1) + 2C_4(u_1,\lambda_2)/b \in \mathbb{R}_+$. Using (3.11), we conclude

$$|K_t(u_1,\lambda_2)| = \sqrt{v_t(u_1,\lambda_2)^2 + w_t(u_1,\lambda_2)^2} \leqslant C_1(u_1,\lambda_2)e^{-C_2 t}, \qquad t\in\mathbb{R}_+,$$

with $C_1(u_1,\lambda_2) := C_5(u_1,\lambda_2) + C_3(u_1,\lambda_2) \in \mathbb{R}_+$, and we obtain (3.8).

Step 3. For each $(u_1,\lambda_2)\in\mathbb{C}_-\times\mathbb{R}$, the function $h_{u_1,\lambda_2}:\mathbb{R}^2\to\mathbb{C}$, given by $h_{u_1,\lambda_2}(y,z) := e^{u_1 y + i\lambda_2 z}$, $(y,z)\in\mathbb{R}^2$, is bounded and continuous, since $|e^{u_1 y + i\lambda_2 z}| = e^{y\operatorname{Re}(u_1)} \leqslant 1$. Hence, by the continuity theorem, by (3.6) and by the portmanteau theorem, to prove (i), it is enough to check that for all $(u_1,\lambda_2)\in\mathbb{C}_-\times\mathbb{R}$ and $(y_0,z_0)\in\mathbb{R}_+\times\mathbb{R}$,

$$\begin{aligned} &\lim_{t\to\infty}[y_0K_t(u_1,\lambda_2) + iz_0e^{-\gamma t}\lambda_2 + g_t(u_1,\lambda_2)]\\ &= a\int_0^\infty K_s(u_1,\lambda_2)\,ds + i\frac{A}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2 =: g_\infty(u_1,\lambda_2), \end{aligned} \tag{3.12}$$

and that the function $\mathbb{C}_- \times \mathbb{R} \ni (u_1, \lambda_2) \mapsto g_\infty(u_1, \lambda_2)$ is continuous. Indeed, using (3.6) and the independence of (η_0, ζ_0) and $(W_t, B_t, L_t)_{t\in\mathbb{R}_+}$, the law of total expectation yields that

$$\mathbb{E}(\mathrm{e}^{u_1 Y_t + \mathrm{i}\lambda_2 Z_t}) = \int_0^\infty \int_{-\infty}^\infty \mathbb{E}\big(\mathrm{e}^{u_1 Y_t + \mathrm{i}\lambda_2 Z_t} \,\big|\, (Y_0, Z_0) = (y_0, z_0)\big)\, \mathbb{P}_{(Y_0,Z_0)}(\mathrm{d}y_0, \mathrm{d}z_0)$$

$$= \int_0^\infty \int_{-\infty}^\infty \exp\left\{y_0 K_t(u_1, \lambda_2) + \mathrm{i}z_0 \mathrm{e}^{-\gamma t}\lambda_2 + g_t(u_1, \lambda_2)\right\} \mathbb{P}_{(Y_0,Z_0)}(\mathrm{d}y_0, \mathrm{d}z_0)$$

for all $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, where $\mathbb{P}_{(Y_0,Z_0)}$ denotes the distribution of (Y_0, Z_0) on $\mathbb{R}_+ \times \mathbb{R}$, and hence (3.12) and the dominated convergence theorem implies that

$$\lim_{t\to\infty} \mathbb{E}(\mathrm{e}^{u_1 Y_t + \mathrm{i}\lambda_2 Z_t}) = \int_0^\infty \int_{-\infty}^\infty \mathrm{e}^{g_\infty(u_1,\lambda_2)}\, \mathbb{P}_{(Y_0,Z_0)}(\mathrm{d}y_0, \mathrm{d}z_0) = \mathrm{e}^{g_\infty(u_1,\lambda_2)}$$

for all $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$. Then, using the continuity of the function $\mathbb{C}_- \times \mathbb{R} \ni (u_1, \lambda_2) \mapsto g_\infty(u_1, \lambda_2)$ (which will be checked later on), the continuity theorem implies $(Y_t, X_t) \stackrel{\mathcal{D}}{\longrightarrow} (Y_\infty, X_\infty)$ as $t \to \infty$, and then, applying the portmanteau theorem for the functions h_{u_1,λ_2}, $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, yields (i).

Next we turn to prove (3.12). By (3.8) and $\gamma > 0$, we have

$$\lim_{t\to\infty} [y_0 K_t(u_1, \lambda_2) + \mathrm{i}z_0 \mathrm{e}^{-\gamma t}\lambda_2] = 0.$$

Recall that

$$g_t(u_1, \lambda_2) = a \int_0^t K_s(u_1, \lambda_2)\, \mathrm{d}s + \mathrm{i}A\lambda_2 \frac{1 - \mathrm{e}^{-\gamma t}}{\gamma} - \frac{1}{2}\sigma_3^2 \lambda_2^2 \frac{1 - \mathrm{e}^{-2\gamma t}}{2\gamma}.$$

Since $\gamma > 0$, we have $\lim_{t\to\infty} \frac{1-\mathrm{e}^{-\gamma t}}{\gamma} = \frac{1}{\gamma}$ and $\lim_{t\to\infty} \frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma} = \frac{1}{2\gamma}$, and by the dominated convergence theorem, we get

$$\lim_{t\to\infty} \int_0^t K_s(u_1, \lambda_2)\, \mathrm{d}s = \int_0^\infty K_s(u_1, \lambda_2)\, \mathrm{d}s.$$

Indeed, by (3.8), $|K_s(u_1, \lambda_2) \mathbb{1}_{[0,t]}(s)| \leqslant |K_s(u_1, \lambda_2)|$ for all $t \in \mathbb{R}_+$ and $s \in [0, t]$, and

$$\int_0^\infty |K_s(u_1, \lambda_2)|\, \mathrm{d}s \leqslant C_1(u_1, \lambda_2) \int_0^\infty \mathrm{e}^{-C_2 s}\, \mathrm{d}s \leqslant \frac{C_1(u_1, \lambda_2)}{C_2} < \infty.$$

The continuity of the function $\mathbb{C}_- \times \mathbb{R} \ni (u_1, \lambda_2) \mapsto g_\infty(u_1, \lambda_2)$ can be checked as follows. It will follow if we prove that for all $s \in \mathbb{R}_+$, the function K_s is continuous. Namely, if $(u_1^{(n)}, \lambda_2^{(n)})$, $n \in \mathbb{N}$, is a sequence in $\mathbb{C}_- \times \mathbb{R}$, such that $\lim_{n\to\infty}(u_1^{(n)}, \lambda_2^{(n)}) = (u_1, \lambda_2)$, where $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, then $\lim_{n\to\infty} K_s(u_1^{(n)}, \lambda_2^{(n)}) = K_s(u_1, \lambda_2)$ for all $s \in \mathbb{R}_+$, and, by (3.8),

$$|K_s(u_1^{(n)}, \lambda_2^{(n)})| \leqslant C_1(u_1^{(n)}, \lambda_2^{(n)}) \mathrm{e}^{-C_2 s}, \qquad n \in \mathbb{N}, \qquad s \in \mathbb{R}_+.$$

Since the sequence $(u_1^{(n)}, \lambda_2^{(n)})$, $n \in \mathbb{N}$, is bounded (since it is convergent), the dominated convergence theorem implies

$$\lim_{n\to\infty} \int_0^\infty K_s(u_1^{(n)}, \lambda_2^{(n)})\, \mathrm{d}s = \int_0^\infty K_s(u_1, \lambda_2)\, \mathrm{d}s,$$

which shows the continuity of g_∞. Finally, the continuity of the function K_s follows from the continuity of the function $\psi_s^{(1)}$.

(ii): First we check that the one-dimensional distributions of $(Y_t, X_t)_{t\in\mathbb{R}_+}$ are translation invariant and have the same distribution as (Y_∞, X_∞) has. Clearly, it is enough to prove that the one-dimensional distributions of $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ are translation invariant and have the same distribution as (Y_∞, Z_∞) has. Using (3.1), (3.6), the tower rule and the independence of (Y_0, Z_0) and (W, B, L), it is enough to check that for all $t \in \mathbb{R}_+$ and $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\left(\exp\left\{K_t(u_1,\lambda_2)Y_\infty + \mathrm{i}\mathrm{e}^{-\gamma t}\lambda_2 Z_\infty + g_t(u_1,\lambda_2)\right\}\right)\\ &= \exp\left\{a\int_0^\infty K_s(u_1,\lambda_2)\,\mathrm{d}s + \mathrm{i}\frac{A}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2\right\}.\end{aligned}$$

By (3.1), (3.7) and using the fact that $K_t(u_1, \lambda_2) \in \mathbb{C}_-$ for all $t \in \mathbb{R}_+$ and $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$ (see Step 1 of the proof of part (i)), we have

$$\begin{aligned}&\mathbb{E}\left(\exp\left\{K_t(u_1,\lambda_2)Y_\infty + \mathrm{i}\mathrm{e}^{-\gamma t}\lambda_2 Z_\infty + g_t(u_1,\lambda_2)\right\}\right)\\ &= \exp\left\{a\int_0^\infty K_s(K_t(u_1,\lambda_2),\mathrm{e}^{-\gamma t}\lambda_2)\,\mathrm{d}s + \mathrm{i}\frac{A}{\gamma}\mathrm{e}^{-\gamma t}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\mathrm{e}^{-2\gamma t}\lambda_2^2 + g_t(u_1,\lambda_2)\right\}\\ &= \exp\left\{a\left(\int_0^\infty K_s(K_t(u_1,\lambda_2),\mathrm{e}^{-\gamma t}\lambda_2)\,\mathrm{d}s + \int_0^t K_s(u_1,\lambda_2)\,\mathrm{d}s\right) + \mathrm{i}\frac{A}{\gamma}\lambda_2 - \frac{\sigma_3^2}{4\gamma}\lambda_2^2\right\}.\end{aligned}$$

Hence it remains to check that

$$\int_0^\infty K_s(u_1,\lambda_2)\,\mathrm{d}s = \int_0^\infty K_s(K_t(u_1,\lambda_2),\mathrm{e}^{-\gamma t}\lambda_2)\,\mathrm{d}s + \int_0^t K_s(u_1,\lambda_2)\,\mathrm{d}s, \quad t\in\mathbb{R}_+,$$

for all $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, i.e.,

$$\int_t^\infty K_s(u_1,\lambda_2)\,\mathrm{d}s = \int_0^\infty K_s(K_t(u_1,\lambda_2),\mathrm{e}^{-\gamma t}\lambda_2)\,\mathrm{d}s, \qquad t\in\mathbb{R}_+,$$

for all $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$. For this it is enough to check that

$$K_s(K_t(u_1,\lambda_2),\mathrm{e}^{-\gamma t}\lambda_2) = K_{s+t}(u_1,\lambda_2), \qquad s,t\in\mathbb{R}_+,$$

for all $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$, or equivalently,

$$K_t(K_s(u_1,\lambda_2),\mathrm{e}^{-\gamma s}\lambda_2) = K_{t+s}(u_1,\lambda_2), \qquad s,t\in\mathbb{R}_+, \tag{3.13}$$

for all $(u_1, \lambda_2) \in \mathbb{C}_- \times \mathbb{R}$. By (3.3), we have

$$\begin{aligned}\frac{\partial K_{s+t}}{\partial t}(u_1,\lambda_2) &= -bK_{s+t}(u_1,\lambda_2) - \mathrm{i}B\mathrm{e}^{-\gamma(s+t)}\lambda_2\\ &\qquad + \frac{1}{2}\sigma_1^2 K_{s+t}(u_1,\lambda_2)^2 - \frac{1}{2}\Sigma_2^2\mathrm{e}^{-2\gamma(s+t)}\lambda_2^2\end{aligned}$$

for $t \in \mathbb{R}_+$ with initial condition $K_{s+0}(u_1, \lambda_2) = K_s(u_1, \lambda_2)$. Note also that, again by (3.3),

$$\begin{aligned}\frac{\partial K_t}{\partial t}(K_s(u_1,\lambda_2),\mathrm{e}^{-\gamma s}\lambda_2) &= -bK_t(K_s(u_1,\lambda_2),\mathrm{e}^{-\gamma s}\lambda_2) - \mathrm{i}B\mathrm{e}^{-\gamma t}(\mathrm{e}^{-\gamma s}\lambda_2)\\ &\qquad + \frac{1}{2}\sigma_1^2 K_t(K_s(u_1,\lambda_2),\mathrm{e}^{-\gamma s}\lambda_2)^2 + \frac{1}{2}\Sigma_2^2\mathrm{e}^{-2\gamma t}(\mathrm{e}^{-\gamma s}\lambda_2)^2\end{aligned}$$

for $t \in \mathbb{R}_+$ with initial condition $K_0(K_s(u_1, \lambda_2), \mathrm{e}^{-\gamma s}\lambda_2) = K_s(u_1, \lambda_2)$. Hence, for all $s \in \mathbb{R}_+$, the left and right sides of (3.13), as functions of $t \in \mathbb{R}_+$, satisfy the differential equation (3.3) with $\mathrm{e}^{-\gamma s}\lambda_2$ instead of λ_2 and with the initial value $K_s(u_1, \lambda_2)$. Since (3.3) has a unique solution for all non-negative initial values, we obtain (3.13).

Finally, the strict stationarity (translation invariance of the finite dimensional distributions) of $(Y_t, X_t)_{t\in\mathbb{R}_+}$ follows by the chain's rule for conditional expectations using also that it is a time homogeneous Markov process. □

4. Exponential Ergodicity

In the subcritical case, the following result states the exponential ergodicity for the process $(Y_t, X_t)_{t\in\mathbb{R}_+}$. As a consequence, according to the discussion after Proposition 2.5 in Bhattacharya [3], one also obtains a strong law of large numbers (4.3) for $(Y_t, X_t)_{t\in\mathbb{R}_+}$.

Theorem 4.1. *Let us consider the 2-dimensional affine diffusion model* (1.1) *with* $a, b \in \mathbb{R}_{++}$, $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_{++}$, $\sigma_1 \in \mathbb{R}_{++}$, $\sigma_2, \sigma_3 \in \mathbb{R}_+$ *and* $\varrho \in [-1, 1]$ *with a random initial value* (η_0, ζ_0) *independent of* $(W_t, B_t, L_t)_{t\in\mathbb{R}_+}$ *satisfying* $\mathbb{P}(\eta_0 \in \mathbb{R}_+) = 1$. *Suppose that* $(1-\varrho^2)\sigma_2^2+\sigma_3^2 > 0$. *Then the process* $(Y_t, X_t)_{t\in\mathbb{R}_+}$ *is exponentially ergodic, namely, there exist* $\delta \in \mathbb{R}_{++}$, $B \in \mathbb{R}_{++}$ *and* $\kappa \in \mathbb{R}_{++}$, *such that*

$$\sup_{|g|\leqslant V+1} \left| \mathbb{E}\left(g(Y_t, X_t) \,|\, (Y_0, X_0) = (y_0, x_0)\right) - \mathbb{E}(g(Y_\infty, X_\infty)) \right| \leqslant B(V(y_0, x_0)+1)\mathrm{e}^{-\delta t} \tag{4.1}$$

for all $t \in \mathbb{R}_+$ *and* $(y_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}$, *where the supremum is running for Borel measurable functions* $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$V(y, x) := y^2 + \kappa x^2, \qquad (y, x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.2}$$

and the distribution of (Y_∞, X_∞) *is given by* (3.1) *and* (3.2). *Moreover, for all Borel measurable functions* $f : \mathbb{R}^2 \to \mathbb{R}$ *with* $\mathbb{E}(|f(Y_\infty, X_\infty)|) < \infty$, *we have*

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{1}{T}\int_0^T f(Y_s, X_s)\,\mathrm{d}s = \mathbb{E}(f(Y_\infty, X_\infty))\right) = 1. \tag{4.3}$$

Proof. First we check that it is enough to prove (4.1) for the special affine diffusion process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ given in Proposition 2.5. Hence we suppose that (4.1) holds for $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ with $\delta \in \mathbb{R}_{++}$, $B \in \mathbb{R}_{++}$ and $\kappa \in \mathbb{R}_{++}$, and we check that then (4.1) holds for $(Y_t, X_t)_{t\in\mathbb{R}_+}$ with $\delta \in \mathbb{R}_{++}$, with some appropriate $\widetilde{B} \in \mathbb{R}_{++}$, and with $\kappa \in \mathbb{R}_{++}$ as well. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a Borel measurable function with $|g(y, x)| \leqslant V(y, x) + 1 = y^2 + \kappa x^2 + 1$, $(y, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then $g(Y_t, X_t) = g(Y_t, Z_t + cY_t) = h(Y_t, Z_t)$, $t \in \mathbb{R}_+$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is given by $h(y, z) := g(y, z+cy)$, $(y, z) \in \mathbb{R}^2$. Clearly, h is also a Borel measurable function. Moreover, (i) of Theorem 3.1 implies $(Y_t, X_t) \xrightarrow{\mathcal{D}} (Y_\infty, X_\infty)$ as $t \to \infty$. Again by (i) of Theorem 3.1, we obtain $(Y_t, X_t) = (Y_t, Z_t + cY_t) \xrightarrow{\mathcal{D}} (Y_\infty, Z_\infty + cY_\infty)$ as $t \to \infty$, consequently, $(Y_\infty, X_\infty) \stackrel{\mathcal{D}}{=} (Y_\infty, Z_\infty+cY_\infty)$, and hence, $\mathbb{E}(g(Y_\infty, X_\infty)) =$

$\mathbb{E}(g(Y_\infty, Z_\infty + cY_\infty)) = \mathbb{E}(h(Y_\infty, Z_\infty))$. We conclude

$$\begin{aligned}
&\Big|\mathbb{E}\left(g(Y_t, X_t) \,|\, (Y_0, X_0) = (y_0, x_0)\right) - \mathbb{E}(g(Y_\infty, X_\infty))\Big| \\
&= \Big|\mathbb{E}\left(g(Y_t, Z_t + cY_t) \,|\, (Y_0, Z_0 + cY_0) = (y_0, x_0)\right) - \mathbb{E}(g(Y_\infty, X_\infty))\Big| \\
&= \Big|\mathbb{E}\left(h(Y_t, Z_t) \,|\, (Y_0, Z_0) = (y_0, z_0)\right) - \mathbb{E}(h(Y_\infty, Z_\infty))\Big|
\end{aligned}$$

with $z_0 := x_0 - cy_0$. By the assumption, (4.1) holds for $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ and Borel measurable functions $\widetilde{g} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with $|\widetilde{g}(y, z)| \leqslant V(y, z) + 1 = y^2 + \kappa z^2 + 1$, $(y, z) \in \mathbb{R}_+ \times \mathbb{R}$. We have

$$\begin{aligned}
|h(y, z)| = |g(y, z + cy)| &\leqslant y^2 + \kappa(z + cy)^2 + 1 \leqslant y^2 + 2\kappa(z^2 + c^2y^2) + 1 \\
&= (1 + 2\kappa c^2)y^2 + 2\kappa z^2 + 1 \leqslant C(y^2 + \kappa z^2 + 1)
\end{aligned}$$

with $C := \max\{1 + 2\kappa c^2, 2\}$, hence we can apply (4.1) for $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ and the Borel measurable function $\frac{1}{C}h(y, z)$, $(y, z) \in \mathbb{R}_+ \times \mathbb{R}$. We obtain

$$\left|\mathbb{E}\Big(\frac{1}{C}h(Y_t, Z_t) \,|\, (Y_0, Z_0) = (y_0, z_0)\Big) - \mathbb{E}\Big(\frac{1}{C}h(Y_\infty, Z_\infty)\Big)\right| \leqslant B(y_0^2 + \kappa z_0^2 + 1)\mathrm{e}^{-\delta t},$$

and hence

$$\begin{aligned}
&\Big|\mathbb{E}\left(g(Y_t, X_t) \,|\, (Y_0, X_0) = (y_0, x_0)\right) - \mathbb{E}(g(Y_\infty, X_\infty))\Big| \\
&\leqslant BC(y_0^2 + \kappa(x_0 - cy_0)^2 + 1)\mathrm{e}^{-\delta t} \leqslant BC(y_0^2 + 2\kappa(x_0^2 + c^2y_0^2) + 1)\mathrm{e}^{-\delta t} \\
&= BC((1 + 2\kappa c^2)y_0^2 + 2\kappa x_0^2 + 1)\mathrm{e}^{-\delta t} \leqslant BC^2(y_0^2 + \kappa x_0^2 + 1)\mathrm{e}^{-\delta t},
\end{aligned}$$

thus (4.1) holds for $(Y_t, X_t)_{t\in\mathbb{R}_+}$ with $\delta \in \mathbb{R}_{++}$, $\widetilde{B} := BC^2 \in \mathbb{R}_{++}$ and with $\kappa \in \mathbb{R}_{++}$.

Next we prove (4.3) for the special affine diffusion process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ given in Proposition 2.5. We use the notations of Meyn and Tweedie [12], [13]. Using Theorem 6.1 (so called Foster-Lyapunov criteria) in Meyn and Tweedie [13], it is enough to check that

(a) $(Y_t, Z_t)_{t\geqslant 0}$ is a right process (defined on page 38 in Sharpe [15]);

(b) all compact sets are petite for some skeleton chain (skeleton chains and petite sets are defined on pages 491, 500 in Meyn and Tweedie [12], and page 550 in Meyn and Tweedie [11], respectively);

(c) there exist $c \in \mathbb{R}_{++}$ and $d \in \mathbb{R}$ such that the inequality

$$(\mathcal{A}_n V)(y, z) \leqslant -cV(y, z) + d, \qquad (y, z) \in O_n$$

holds for all $n \in \mathbb{N}$, where $O_n := \{(y, z) \in \mathbb{R}_+ \times \mathbb{R} : \|(y, z)\| < n\}$ for each $n \in \mathbb{N}$, and $\mathcal{A}_n$ denotes the extended generator of the process $(Y_t^{(n)}, Z_t^{(n)})_{t\in\mathbb{R}_+}$ given by

$$(Y_t^{(n)}, Z_t^{(n)}) := \begin{cases} (Y_t, Z_t), & \text{for } t < T_n, \\ (0, n), & \text{for } t \geqslant T_n, \end{cases}$$

where the stopping time T_n is defined by $T_n := \inf\{t \in \mathbb{R}_+ : (Y_t, Z_t) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus O_n\}$. (Here we note that instead of $(0, n)$ we could have chosen any fixed state in $(\mathbb{R}_+ \times \mathbb{R}) \setminus O_n$, and we could also have defined

$(Y_t^{(n)}, Z_t^{(n)})_{t\in\mathbb{R}_+}$ as the stopped process $(Y_{t\wedge T_n}, Z_{t\wedge T_n})_{t\in\mathbb{R}_+}$, see Meyn and Tweedie [13, page 521].)

To prove (a), it is enough to show that the process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ is a (weak) Feller (see Meyn and Tweedy [12, Section 3.1]), strong Markov process with continuous sample paths, see, e.g., Meyn and Tweedy [12, page 498]. According to Proposition 8.2 (or Theorem 2.7) in Duffie et al. [7], the process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ is a Feller Markov process. Since $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ has continuous sample paths almost surely (especially, it is càdlàg), it is automatically a strong Markov process, see, e.g., Theorem 1 on page 56 in Chung [4].

To prove (b), in view of Proposition 6.2.8 in Meyn and Tweedy [14], it is sufficient to show that the skeleton chain $(Y_n, Z_n)_{n\in\mathbb{Z}_+}$ is irreducible with respect to the Lebesgue measure on $\mathbb{R}_+\times\mathbb{R}$ (see, e.g., Meyn and Tweedy [13, page 520]), and admits the Feller property. The skeleton chain $(Y_n, Z_n)_{n\in\mathbb{Z}_+}$ admits the Feller property, since the process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ is a Feller process. In order to check irreducibility of the skeleton chain $(Y_n, Z_n)_{n\in\mathbb{Z}_+}$ with respect to the Lebesgue measure on $\mathbb{R}_+\times\mathbb{R}$, it is enough to prove that the conditional distribution of (Y_1, Z_1) given (Y_0, Z_0) is absolutely continuous (with respect to the Lebesgue measure on $\mathbb{R}_+\times\mathbb{R}$) with a conditional density function $f_{(Y_1,Z_1)\,|\,(Y_0,Z_0)} : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}_+$ such that $f_{(Y_1,Z_1)\,|\,(Y_0,Z_0)}(y, z\,|\,y_0, z_0) > 0$ for all $(y, z, y_0, z_0) \in \mathbb{R}_{++}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}$. Indeed, the Lebesgue measure on $\mathbb{R}_+\times\mathbb{R}$ is σ-finite, and if B is a Borel set in $\mathbb{R}_+\times\mathbb{R}$ with positive Lebesgue measure, then

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{n=0}^{\infty} \mathbb{1}_B(Y_n, Z_n)\,\middle|\,(Y_0, Z_0) = (y_0, z_0)\right) \geqslant \mathbb{P}((Y_1, Z_1)\in B\,|\,(Y_0, Z_0) = (y_0, z_0))\\
&= \iint_B f_{(Y_1,Z_1)\,|\,(Y_0,Z_0)}(y, z\,|\,y_0, z_0)\,\mathrm{d}y\,\mathrm{d}z > 0
\end{aligned}
$$

for all $(y_0, z_0)\in\mathbb{R}_+\times\mathbb{R}$. The existence of $f_{(Y_1,Z_1)\,|\,(Y_0,Z_0)}$ with the required property can be checked as follows. By Theorem 2.2, we have

$$
\begin{aligned}
Y_1 &= \mathrm{e}^{-b}\left(y_0 + a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u + \sigma_1\int_0^1 \mathrm{e}^{bu}\sqrt{Y_u}\,\mathrm{d}W_u\right),\\
Z_1 &= \mathrm{e}^{-\gamma}\left(z_0 + \int_0^1 \mathrm{e}^{\gamma u}(A - BY_u)\,\mathrm{d}u + \Sigma_2\int_0^1 \mathrm{e}^{\gamma u}\sqrt{Y_u}\,\mathrm{d}B_u + \sigma_3\int_0^1 \mathrm{e}^{\gamma u}\,\mathrm{d}L_u\right),
\end{aligned}
$$

provided that $(Y_0, Z_0) = (y_0, z_0)$, $(y_0, z_0)\in\mathbb{R}_+\times\mathbb{R}$. Recall that a two-dimensional random vector ζ is absolutely continuous if and only if $V\zeta + v$ is absolutely continuous for all invertable matrices $V\in\mathbb{R}^{2\times 2}$ and for all vectors $v\in\mathbb{R}^2$, and if the density function of ζ is positive on a set $S\subset\mathbb{R}^2$, then the density function of $V\zeta + v$ is positive on the set $VS + v$. Hence it is enough to check that the random vector

$$
\left(\sigma_1\int_0^1 \mathrm{e}^{bu}\sqrt{Y_u}\,\mathrm{d}W_u, I\right) \tag{4.4}
$$

with

$$
I := -B\int_0^1 \mathrm{e}^{\gamma u} Y_u\,\mathrm{d}u + \Sigma_2\int_0^1 \mathrm{e}^{\gamma u}\sqrt{Y_u}\,\mathrm{d}B_u + \sigma_3\int_0^1 \mathrm{e}^{\gamma u}\,\mathrm{d}L_u
$$

is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ having a density function being strictly positive on the set

$$\left\{y \in \mathbb{R} : y > -y_0 - a \int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\right\} \times \mathbb{R}.$$

For all $y \leqslant -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u$ and $z \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}\left(\sigma_1 \int_0^1 \mathrm{e}^{bu}\sqrt{Y_u}\,\mathrm{d}W_u < y, I < z\right) &= \mathbb{P}\left(\mathrm{e}^{b}Y_1 - y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u < y, I < z\right)\\ &\leqslant \mathbb{P}(Y_1 < 0) = 0,\end{aligned}$$

since $\mathbb{P}(Y_1 \geqslant 0) = 1$. Note that the conditional distribution of I given $(Y_t)_{t\in[0,1]}$ is a normal distribution with mean $-B\int_0^1 \mathrm{e}^{\gamma u}Y_u\,\mathrm{d}u$ and with variance

$$\Upsilon^2 := \Sigma_2^2 \int_0^1 \mathrm{e}^{2\gamma u}Y_u\,\mathrm{d}u + \sigma_3^2\int_0^1 \mathrm{e}^{2\gamma u}\,\mathrm{d}u$$

due to the fact that $(Y_t)_{t\in[0,1]}$ and $(B_t, L_t)_{t\in\mathbb{R}_+}$ are independent. Indeed, $(Y_t)_{t\in\mathbb{R}_+}$ is adapted to the augmented filtration corresponding to η_0 and $(W_t)_{t\in\mathbb{R}_+}$ (see, e.g., Karatzas and Shreve [10, page 285]), and using the independence of the standard Wiener processes W and B, and Problem 2.7.3 in Karatzas and Shreve [10], one can argue that this augmented filtration is independent of the filtration generated by B. Here we call the attention that the condition $(1-\varrho^2)\sigma_2^2+\sigma_3^2 > 0$ implies $\mathbb{P}(\Upsilon^2 \in \mathbb{R}_{++}) = 1$. Indeed, $\Sigma_2^2 = (1-\varrho^2)\sigma_2^2$, and the assumption $a > 0$ yields $\mathbb{P}(\int_0^1 \mathrm{e}^{2\gamma u}Y_u\,\mathrm{d}u \in \mathbb{R}_{++}) = 1$. Hence, using again the independence of the standard Wiener processes W, B and L, we get for all $y > -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u$ and $z \in \mathbb{R}$, by the law of total expectation,

$$\begin{aligned}&\mathbb{P}\left(\sigma_1 \int_0^1 \mathrm{e}^{bu}\sqrt{Y_u}\,\mathrm{d}W_u < y, I < z\right)\\ &= \mathbb{P}\left(\mathrm{e}^{b}Y_1 - y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u < y, I < z\right)\\ &= \mathbb{E}\left(\mathbb{P}\left(Y_1 < \mathrm{e}^{-b}\left(y + y_0 + a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\right), I < z \,\middle|\, (Y_t)_{t\in[0,1]}\right)\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{\left\{Y_1 < \mathrm{e}^{-b}\left(y+y_0+a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\right)\right\}}\mathbb{1}_{\{I<z\}} \,\middle|\, (Y_t)_{t\in[0,1]}\right)\right)\\ &= \mathbb{E}\left(\mathbb{1}_{\left\{Y_1 < \mathrm{e}^{-b}\left(y+y_0+a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\right)\right\}}\mathbb{P}(I < z \,|\, (Y_t)_{t\in[0,1]})\right)\\ &= \mathbb{E}\left(\mathbb{1}_{\{Y_1 < \mathrm{e}^{-b}\left(y+y_0+a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\right)\}}\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi\Upsilon^2}}\exp\left\{-\frac{(w + B\int_0^1 \mathrm{e}^{\gamma u}Y_u\,\mathrm{d}u)^2}{2\Upsilon^2}\right\}\mathrm{d}w\right)\\ &= \int_0^{\mathrm{e}^{-b}\left(y+y_0+a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\right)}\int_{-\infty}^{z}\mathbb{E}\left(\frac{1}{\sqrt{2\pi\Upsilon^2}}\exp\left\{-\frac{(w + B\int_0^1 \mathrm{e}^{\gamma u}Y_u\,\mathrm{d}u)^2}{2\Upsilon^2}\right\}\,\middle|\, Y_1 = v\right)\\ &\qquad\qquad \times f_{Y_1\,|\,Y_0}(v|y_0)\,\mathrm{d}v\,\mathrm{d}w\end{aligned}$$

$$= \int_{-y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u}^{y} \int_{-\infty}^{z} \mathbb{E}\biggl(\frac{1}{\sqrt{2\pi\Upsilon^2}} \exp\biggl\{-\frac{(w + B\int_0^1 \mathrm{e}^{\gamma u} Y_u\,\mathrm{d}u)^2}{2\Upsilon^2}\biggr\} \,\bigg|\, Y_1 = \mathrm{e}^{-b}\biggl(v + y_0 + a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\biggr)\biggr) \times f_{Y_1\,|\,Y_0}\biggl(\mathrm{e}^{-b}\biggl(v + y_0 + a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\biggr) \,\bigg|\, y_0\biggr) \mathrm{e}^{-b}\,\mathrm{d}v\,\mathrm{d}w,$$

where $f_{Y_1\,|\,Y_0}$ denotes the conditional density function of Y_1 given Y_0. It is known that for each $y_0 \in \mathbb{R}_+$, we have $f_{Y_1\,|\,Y_0}(u\,|\,y_0) > 0$ for Lebesgue a.e. $u \in \mathbb{R}_{++}$, see, e.g., Cox et al. [5, Equation (18)], Jeanblanc et al. [9, Proposition 6.3.2.1] or Ben Alaya and Kebaier [2, the proof of Proposition 2] in case of $y_0 \in \mathbb{R}_{++}$, and Ikeda and Watanabe [8, page 222] in case of $y_0 = 0$.

In what follows we will make use of the following simple observation: if ξ and η are random variables such that $\mathbb{P}(\xi \in \mathbb{R}_{++}) = 1$, $\mathbb{E}(\xi) < \infty$, $\mathbb{P}(\eta \in \mathbb{R}_{++}) = 1$, and η is absolutely continuous with a density function f_η having the property $f_\eta(x) > 0$ Lebesgue a.e. $x \in \mathbb{R}_{++}$, then $\mathbb{E}(\xi\,|\,\eta = y) > 0$ Lebesgue a.e. $y \in \mathbb{R}_{++}$. For a proof, see, e.g., the proof of Theorem 4.1 in the extended arXiv version of Barczy et al. [1].

Now we turn back to the proof that the random vector (4.4) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ with a density function being strictly positive on the set $\bigl\{y \in \mathbb{R} : y > -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\bigr\} \times \mathbb{R}$. Using that $f_{Y_1\,|\,Y_0}\bigl(\mathrm{e}^{-b}\bigl(v + y_0 + a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u\bigr)\,\big|\,y_0\bigr) > 0$ for all $v > -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u$, there exists a measurable function $g : \mathbb{R}^2 \to \mathbb{R}_+$ such that $g(v,w) > 0$ for $v > -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u$, $w \in \mathbb{R}$, and

$$\mathbb{P}\biggl(\sigma_1 \int_0^1 \mathrm{e}^{bu}\sqrt{Y_u}\,\mathrm{d}W_u < y, I < z\biggr) = \begin{cases} \int_{-y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u}^{y} \int_{-\infty}^{\infty} g(v,w)\,\mathrm{d}v\,\mathrm{d}w & \text{if } y > -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u, \ z \in \mathbb{R}, \\ 0 & \text{if } y \leqslant -y_0 - a\int_0^1 \mathrm{e}^{bu}\,\mathrm{d}u, \ z \in \mathbb{R}. \end{cases}$$

Consequently, the random vector (4.4) is absolutely continuous with density function g having the desired property.

To prove (c), first we note that, since the sample paths of $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ are almost surely continuous, for each $n \in \mathbb{N}$, the extended generator has the form

$$(\mathcal{A}_n f)(y,z) = (a - by) f_1'(y,z) + (A - By - \gamma z) f_2'(y,z) + \frac{y}{2}\bigl(\sigma_1^2 f_{1,1}''(y,z) + \Sigma_2^2 f_{2,2}''(y,z)\bigr) + \frac{\sigma_3^2}{2} f_{2,2}''(y,z)$$

for all $(y,z) \in O_n$ and $f \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, see, e.g., page 538 in Meyn and Tweedie [13]. We also note that, by Duffie et al. [7, Theorem 2.7], for functions $f \in \mathcal{C}_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $\mathcal{A}_n f = \mathcal{A} f$ on O_n, where $\mathcal{A}$ denotes the (non-extended) generator of the process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$. For the function V defined in (4.2), we have $V \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and

$$V_1'(y,z) = 2y, \qquad V_2'(y,z) = 2\kappa z, \qquad V_{1,1}''(y,z) = 2, \qquad V_{2,2}''(y,z) = 2\kappa$$

for $(y,z) \in \mathbb{R}_+ \times \mathbb{R}$, and hence for all $n \in \mathbb{N}$ and $c \in \mathbb{R}_{++}$,

$$\begin{aligned}(\mathcal{A}_n V)(y,z) + cV(y,z) &= 2(a-by)y + 2\kappa(A - By - \gamma z)z + y(\sigma_1^2 + \kappa\Sigma_2^2) \\ &\quad + \kappa\sigma_3^2 + cy^2 + c\kappa z^2 \\ &= -(2b-c)y^2 - 2\kappa Byz - \kappa(2\gamma - c)z^2 \\ &\quad + c_1 y + 2\kappa Az + \kappa\sigma_3^2\end{aligned}$$

for all $(y,z) \in O_n$ with

$$c_1 := 2a + \sigma_1^2 + \kappa\Sigma_2^2.$$

Thus

$$(\mathcal{A}_n V)(y,z) + cV(y,z) = -c_2\left(y + \frac{\kappa Bz}{c_2}\right)^2 - c_3 z^2 + c_1\left(y + \frac{\kappa Bz}{c_2}\right) + c_4 z + \kappa\sigma_3^2$$

for all $(y,z) \in O_n$ with

$$c_2 := 2b - c, \qquad c_3 := \kappa\left(2\gamma - c - \frac{\kappa B^2}{c_2}\right), \qquad c_4 := 2\kappa A - c_1\frac{\kappa B}{c_2},$$

whenever $c_2 \neq 0$. Consequently,

$$(\mathcal{A}_n V)(y,z) + cV(y,z) = -c_2\left(y + \frac{\kappa Bz}{c_2} - \frac{c_1}{2c_2}\right)^2 - c_3\left(z - \frac{c_4}{2c_3}\right)^2 + d$$

for all $(y,z) \in O_n$ with

$$d := \frac{c_1^2}{4c_2} + \frac{c_4^2}{4c_3} + \kappa\sigma_3^2,$$

whenever $c_2 \neq 0$ and $c_3 \neq 0$. Let us choose

$$c \in (0, 2\min\{b,\gamma\}), \qquad \kappa \in \left(0, \frac{(2\gamma - c)(2b-c)}{B^2}\right).$$

Then $c_2 > 0$ and $c_3 > 0$, hence

$$(\mathcal{A}_n V)(y,z) \leqslant -cV(y,z) + d, \qquad (y,z) \in O_n, \quad n \in \mathbb{N},$$

and the proof is complete. □

5. Moments of the Stationary Distribution

Theorem 5.1. *Let us consider the 2-dimensional affine diffusion model* (1.1) *with* $a \in \mathbb{R}_+$, $b \in \mathbb{R}_{++}$, $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_{++}$, $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}_+$, $\varrho \in [-1,1]$, *and the random vector* (Y_∞, X_∞) *given by Theorem 3.1. Then all the (mixed) moments of* (Y_∞, X_∞) *of any order are finite, i.e., we have* $\mathbb{E}(Y_\infty^n |X_\infty|^p) < \infty$ *for all* $n, p \in \mathbb{Z}_+$, *and the recursion*

$$\begin{aligned}\mathbb{E}(Y_\infty^n X_\infty^p) = \frac{1}{nb + p\gamma}\bigg[&\left(na + \frac{1}{2}n(n-1)\sigma_1^2\right)\mathbb{E}(Y_\infty^{n-1}X_\infty^p) - p\beta\,\mathbb{E}(Y_\infty^{n+1}X_\infty^{p-1}) \\ &+ p(\alpha + n\varrho\sigma_1\sigma_2)\,\mathbb{E}(Y_\infty^n X_\infty^{p-1}) \\ &+ \frac{1}{2}p(p-1)\sigma_2^2\,\mathbb{E}(Y_\infty^{n+1}X_\infty^{p-2}) \\ &+ \frac{1}{2}p(p-1)\sigma_3^2\,\mathbb{E}(Y_\infty^n X_\infty^{p-2})\bigg],\end{aligned}$$

holds for all $n, p \in \mathbb{Z}_+$ *with* $n + p \geqslant 1$, *where* $\mathbb{E}(Y_\infty^k X_\infty^\ell) := 0$ *for* $k, \ell \in \mathbb{Z}$ *with* $k < 0$ *or* $\ell < 0$. *Especially,*

$$\mathbb{E}(Y_\infty) = \frac{a}{b}, \qquad \mathbb{E}(Y_\infty^2) = \frac{a(2a+\sigma_1^2)}{2b^2}, \qquad \mathbb{E}(Y_\infty^3) = \frac{a(a+\sigma_1^2)(2a+\sigma_1^2)}{2b^3},$$
$$\mathbb{E}(X_\infty) = \frac{b\alpha - a\beta}{b\gamma}, \qquad \mathbb{E}(Y_\infty X_\infty) = \frac{a\,\mathbb{E}(X_\infty) - \beta\,\mathbb{E}(Y_\infty^2) + (\alpha + \varrho\sigma_1\sigma_2)\,\mathbb{E}(Y_\infty)}{b+\gamma},$$
$$\mathbb{E}(X_\infty^2) = \frac{-2\beta\,\mathbb{E}(Y_\infty X_\infty) + 2\alpha\,\mathbb{E}(X_\infty) + \sigma_2^2\,\mathbb{E}(Y_\infty) + \sigma_3^2}{2\gamma},$$
$$\mathbb{E}(Y_\infty^2 X_\infty) = \frac{(2a+\sigma_1^2)\,\mathbb{E}(Y_\infty X_\infty) - \beta\,\mathbb{E}(Y_\infty^3) + (\alpha + 2\varrho\sigma_1\sigma_2)\,\mathbb{E}(Y_\infty^2)}{2b+\gamma},$$
$$\mathbb{E}(Y_\infty X_\infty^2) = \frac{a\,\mathbb{E}(X_\infty^2) - 2\beta\,\mathbb{E}(Y_\infty^2 X_\infty) + 2(\alpha + \varrho\sigma_1\sigma_2)\,\mathbb{E}(Y_\infty X_\infty)}{b+2\gamma} + \frac{\sigma_2^2\,\mathbb{E}(Y_\infty^2) + \sigma_3^2\,\mathbb{E}(Y_\infty)}{b+2\gamma}.$$

If $\sigma_1 > 0$ *then the Laplace transform of* Y_∞ *takes the form*

$$\mathbb{E}(\mathrm{e}^{-\lambda Y_\infty}) = \left(1 + \frac{\sigma_1^2}{2b}\lambda\right)^{-2a/\sigma_1^2}, \qquad \lambda \in \mathbb{R}_+, \tag{5.1}$$

i.e., Y_∞ *has gamma distribution with parameters* $2a/\sigma_1^2$ *and* $2b/\sigma_1^2$, *hence*

$$\mathbb{E}(Y_\infty^\kappa) = \frac{\Gamma\left(\frac{2a}{\sigma_1^2} + \kappa\right)}{\left(\frac{2b}{\sigma_1^2}\right)^\kappa \Gamma\left(\frac{2a}{\sigma_1^2}\right)}, \qquad \kappa \in \left(-\frac{2a}{\sigma_1^2}, \infty\right).$$

If $\sigma_1 > 0$ *and* $(1-\varrho^2)\sigma_2^2 + \sigma_3^2 > 0$ *then the distribution of* (Y_∞, X_∞) *is absolutely continuous.*

Proof. We may and do suppose that all the mixed moments of (Y_0, X_0) are finite and $\mathbb{P}(Y_0 > 0) = 1$, since, due to Theorem 3.1, the distribution of (Y_∞, X_∞) does not depend on the initial value of the model. First we show that

$$\int_0^t \mathbb{E}(Y_u^n X_u^{2p})\,\mathrm{d}u < \infty \qquad \text{for all } t \in \mathbb{R}_+ \text{ and } n, p \in \mathbb{Z}_+. \tag{5.2}$$

One can easily check that it is enough to prove (5.2) for the special affine diffusion process $(Y_t, Z_t)_{t\in\mathbb{R}_+}$ given in Proposition 2.5. Indeed, then

$$\int_0^t \mathbb{E}(Y_u^n X_u^{2p})\,\mathrm{d}u = \int_0^t \mathbb{E}(Y_u^n (Z_u + cY_u)^{2p})\,\mathrm{d}u = \sum_{k=0}^{2p} \binom{2p}{k} c^{2p-k} \int_0^t \mathbb{E}(Y_u^{n+2p-k} Z_u^{2k})\,\mathrm{d}u < \infty.$$

Applying (2.5) and the power means inequality $(a+b+c+d)^{2p} \leqslant 4^{2p-1}(a^{2p} + b^{2p} + c^{2p} + d^{2p})$, $a, b, c, d \in \mathbb{R}$, we obtain

$$\int_0^t \mathbb{E}(Y_u^n Z_u^{2p})\,\mathrm{d}u \leqslant 4^{2p-1} \int_0^t \mathbb{E}\biggl[Y_u^n \biggl(\mathrm{e}^{-2p\gamma u} Z_0^{2p} + \biggl(\int_0^u \mathrm{e}^{-\gamma(u-v)}(A - BY_v)\,\mathrm{d}v\biggr)^{2p} + \biggl(\int_0^u \mathrm{e}^{-\gamma(u-v)}\sqrt{Y_v}\,\mathrm{d}B_v\biggr)^{2p} + \biggl(\int_0^u \mathrm{e}^{-\gamma(u-v)}\,\mathrm{d}L_v\biggr)^{2p}\biggr)\biggr]\mathrm{d}u$$

for all $t \in \mathbb{R}_+$ and $n, p \in \mathbb{Z}_+$. Since for all $u \in [0, t]$, the distribution of $\int_0^u \mathrm{e}^{-\gamma(u-v)}\,\mathrm{d}L_v$ is a normal distribution with mean 0 and with variance $\int_0^u \mathrm{e}^{-2\gamma(u-v)}\mathrm{d}v$, and the conditional distribution of $\int_0^u \mathrm{e}^{-\gamma(u-v)}\sqrt{Y_v}\,\mathrm{d}B_v$ with respect to the σ-algebra generated by $(Y_s)_{s\in[0,t]}$ is a normal distribution with mean 0 and with variance $\int_0^u \mathrm{e}^{-2\gamma(u-v)} Y_v\,\mathrm{d}v$, to prove (5.2), it is enough to show that, for all $t \in \mathbb{R}_+$ and $n, p \in \mathbb{Z}_+$,

$$\int_0^t \mathbb{E}(\mathrm{e}^{-2p\gamma u} Y_u^n Z_0^{2p})\,\mathrm{d}u < \infty, \qquad \int_0^t \mathbb{E}(Y_u^n)\,\mathrm{d}u < \infty,$$
$$\int_0^t \mathbb{E}\left[Y_u^n \left(\int_0^u \mathrm{e}^{-\gamma(u-v)} Y_v\,\mathrm{d}v\right)^{2p}\right]\mathrm{d}u < \infty,$$
$$\int_0^t \mathbb{E}\left[Y_u^n \left(\int_0^u \mathrm{e}^{-2\gamma(u-v)} Y_v\,\mathrm{d}v\right)^{p}\right]\mathrm{d}u < \infty,$$

which can be checked by standard arguments, see, e.g., in the arXiv version of the proof of Theorem 4.2 in Barczy et al. [1].

For all $n, p \in \mathbb{Z}_+$, using the independence of W, B and L, by Itô's formula, we have

$$\begin{aligned}
\mathrm{d}(Y_t^n X_t^p) &= nY_t^{n-1} X_t^p \bigl[(a - bY_t)\,\mathrm{d}t + \sigma_1\sqrt{Y_t}\,\mathrm{d}W_t\bigr] \\
&\quad + pY_t^n X_t^{p-1}\bigl[(\alpha - \beta Y_t - \gamma X_t)\,\mathrm{d}t + \sigma_2\sqrt{Y_t}\,(\varrho\,\mathrm{d}W_t + \sqrt{1-\varrho^2}\,\mathrm{d}B_t) + \sigma_3\,\mathrm{d}L_t\bigr] \\
&\quad + \frac{1}{2}n(n-1)Y_t^{n-2}X_t^p\sigma_1^2 Y_t\,\mathrm{d}t + \frac{1}{2}p(p-1)Y_t^n X_t^{p-2}(\sigma_2^2 Y_t + \sigma_3^2)\,\mathrm{d}t \\
&\quad + npY_t^{n-1}X_t^{p-1}\varrho\sigma_1\sigma_2 Y_t\,\mathrm{d}t
\end{aligned}$$

for $t \in \mathbb{R}_+$. Writing the SDE above in an integrated form and taking the expectation of both sides, we have

$$\begin{aligned}
&\mathbb{E}(Y_t^n X_t^p) - \mathbb{E}(Y_0^n X_0^p) \\
&= \int_0^t \biggl[na\,\mathbb{E}(Y_u^{n-1}X_u^p) - nb\,\mathbb{E}(Y_u^n X_u^p) + p\alpha\,\mathbb{E}(Y_u^n X_u^{p-1}) \\
&\quad - p\beta\,\mathbb{E}(Y_u^{n+1}X_u^{p-1}) - p\gamma\,\mathbb{E}(Y_u^n X_u^p) \\
&\quad + \frac{1}{2}\sigma_1^2 n(n-1)\,\mathbb{E}(Y_u^{n-1}X_u^p) + \frac{1}{2}\sigma_2^2 p(p-1)\,\mathbb{E}(Y_u^{n+1}X_u^{p-2}) \\
&\quad + \frac{1}{2}\sigma_3^2 p(p-1)\,\mathbb{E}(Y_u^n X_u^{p-2}) + \varrho\sigma_1\sigma_2 np\,\mathbb{E}(Y_u^n X_u^{p-1})\biggr]\mathrm{d}u
\end{aligned}$$

for all $t \in \mathbb{R}_+$, where we used that

$$\left(\int_0^t Y_u^{n-1/2} X_u^p \, \mathrm{d}W_u\right)_{t\in\mathbb{R}_+}, \qquad \left(\int_0^t Y_u^{n+1/2} X_u^{p-1} \, \mathrm{d}W_u\right)_{t\in\mathbb{R}_+},$$
$$\left(\int_0^t Y_u^{n+1/2} X_u^{p-1} \, \mathrm{d}B_u\right)_{t\in\mathbb{R}_+}, \qquad \left(\int_0^t Y_u^{n} X_u^{p-1} \, \mathrm{d}L_u\right)_{t\in\mathbb{R}_+}$$

are continuous square integrable martingales due to (5.2), see, e.g., Ikeda and Watanabe [8, page 55]. Introduce the functions $f_{n,p}(t) := \mathbb{E}(Y_t^n X_t^p)$, $t \in \mathbb{R}_+$, for $n, p \in \mathbb{Z}_+$. Then we have

$$\begin{aligned} f'_{n,p}(t) &= -(nb + p\gamma) f_{n,p}(t) - p\beta f_{n+1,p-1}(t) + \left(na + \frac{1}{2}\sigma_1^2 n(n-1)\right) f_{n-1,p}(t) \\ &\quad + p(\alpha + \varrho\sigma_1\sigma_2 n) f_{n,p-1}(t) + \frac{1}{2}\sigma_2^2 p(p-1) f_{n+1,p-2}(t) + \frac{1}{2}\sigma_3^2 p(p-1) f_{n,p-2}(t) \end{aligned}$$

for $t \in \mathbb{R}_+$, where $f_{k,\ell}(t) := 0$ if $k, \ell \in \mathbb{Z}$ with $k < 0$ or $\ell < 0$. Hence for all $M \in \mathbb{N}$, the functions $f_{n,p}$, $n, p \in \mathbb{Z}_+$ with $n + p \leqslant M$ satisfy a homogeneous linear system of differential equations with constant coefficients. The eigenvalues of the coefficient matrix of the above mentioned system of differential equations are $-(kb + \ell\gamma)$, $k, \ell \in \mathbb{Z}_+$ with $k + \ell \leqslant M$ and 0. Thus, for all $n, p \in \mathbb{Z}_+$, the function $f_{n,p}$ is a linear combination of the functions $\mathrm{e}^{-(kb+\ell\gamma)t}$, $t \in \mathbb{R}_+$, $k, \ell \in \mathbb{Z}_+$ with $k + \ell \leqslant n + p$, and the constant function. Consequently, for all $n, p \in \mathbb{Z}_+$, the function $f_{n,p}$ is bounded and the limit $\lim_{t\to\infty} f_{n,p}(t)$ exists and finite. By the moment convergence theorem (see, e.g., Stroock [16, Lemma 2.2.1]), $\lim_{t\to\infty} f_{n,p}(t) = \lim_{t\to\infty} \mathbb{E}(Y_t^n X_t^p) = \mathbb{E}(Y_\infty^n X_\infty^p)$, $n, p \in \mathbb{Z}_+$. Indeed, by Theorem 3.1 and the continuous mapping theorem, $Y_t^n X_t^p \xrightarrow{\mathcal{D}} Y_\infty^n X_\infty^p$ as $t \to \infty$, and the family $\{Y_t^n X_t^p : t \in \mathbb{R}_+\}$ is uniformly integrable. This latter fact follows from the boundedness of the function $f_{2n,2p}$, see, e.g., Stroock [16, condition (2.2.5)]. Hence we conclude that all the mixed moments of (Y_∞, X_∞) are finite.

Next, we calculate these mixed moments. We may and do suppose that the initial value (Y_0, X_0) is independent of $(W_t, B_t, L_t)_{t\in\mathbb{R}_+}$, and its distribution is the same as that of (Y_∞, X_∞), since, due to Theorem 3.1, the distribution of (Y_∞, X_∞) does not depend on the initial value of the model. Then, by Theorem 3.1, the process $(Y_t, X_t)_{t\in\mathbb{R}_+}$ is strictly stationary, and hence, $f_{n,p}(t) = \mathbb{E}(Y_\infty^n X_\infty^p)$ for all $t \in \mathbb{R}_+$ and $n, p \in \mathbb{Z}_+$. The above system of differential equations for the functions $f_{n,p}$, $n, p \in \mathbb{Z}_+$, yields the recursion for $\mathbb{E}(Y_\infty^n X_\infty^p)$, $n, p \in \mathbb{Z}_+$. By this recursion, one can calculate the moments listed in the theorem.

The fact that, in case of $\sigma_1 > 0$, the random variable Y_∞ has gamma distribution with parameters $2a/\sigma_1^2$ and $2b/\sigma_1^2$ follows by Cox et al. [5, Equation (20)].

Finally, we prove that the distribution of (Y_∞, X_∞) is absolutely continuous whenever $\sigma_1 > 0$ and $(1 - \varrho^2)\sigma_2^2 + \sigma_3^2 > 0$. Let us consider a 2-dimensional affine diffusion model (1.1) with random initial value (Y_0, X_0) independent of $(W_t, B_t, L_t)_{t\in[0,\infty)}$ having the same distribution as that of (Y_∞, X_∞). Then, by part (ii) of Theorem 3.1, the process $(Y_t, X_t)_{t\in[0,\infty)}$ is strictly stationary. Hence it is enough to prove that the distribution of (Y_1, X_1) is absolutely continuous. According to the proof of part (b) in the proof of Theorem 3.1, the conditional distribution of (Y_1, X_1) given (Y_0, X_0) is absolutely continuous. This clearly implies that the (unconditional) distribution of (Y_1, X_1) is absolutely continuous, and hence, the distribution of (Y_∞, X_∞) is absolutely continuous. □

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