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Theory of Inverse Images of Fs-Subsets under an Fs-Function – Some Observations

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Abstract : In this paper we introduce the concept of inverse image of an Fs-subset under an Fs-function and prove some results.
Keywords : Fs-set, Fs-subset, Fs-empty set , Fs-function, Image of an Fs-subset, Inverse image of an Fs-set.

1. INTRODUCTION

Ever since Zadeh [8] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Recently many researchers put their efforts in order to prove collection of all fuzzy subsets of a given fuzzy set is Boolean algebra under suitable operations [21].Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath[11] introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions. They are successful in their efforts in proving that result with some conditions. In this paper we introduce the concept of inverse image of an Fs-subset under an Fs-function and prove some results. For smooth reading of the paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra L_A[1.1] by M_A or 1_A. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [3], Garret Birkhoff [4], Steven Givant • Paul Halmos [2] and Thomas Jech [5]. For results in topology one can refer [10].

2. THEORY OF FS-SETS

1. Fs-set : Let U be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple

$$A = (A_1, A, \overline{A} (\mu_{1A_1}, \mu_{2A}), L_A)$$

is a complete Boolean Algebra

(*a*) $A \subseteq A_1$

(*b*) L_A is a complete Boolean Algebra

(*c*) $\mu_{1A_1} : A_1 \to L_A$, $\mu_{2A} : A \to L_A$, are functions such that $\mu_{1A_1} \mid A \geq \mu_{2A}$

2. Fs-subset : Let $A = (A_1, A, \overline{A} (\mu_{1A_1}\ x \wedge (\mu_{2A}), L_A)$ and $B = (B_1, B, \overline{B} (\mu_{1B1}, \mu_{2B}), L_B)$ be a pair of Fs-sets. B is said to be an Fs-subset of A, denoted by $B \subseteq A$, if, and only if

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(*a*) $B_1 \subseteq A_1, A \subseteq B$

(*b*) L_B is a complete subalgebra of L_A or $L_B \leq L_A$

(*c*) $\mu_{1B_1} \leq \mu_{1A_1} \mid B_1$, and $\mu_{2B} \mid A \geq \mu_{2A}$

3. Proposition : Let B and A be a pair of Fs-sets such that $B \subseteq A$. Then $\overline{B}x \leq \overline{A}x$ is true for each $x \in A$

4. Definition : For some L_X, such that $L_X \leq L_A$ a four tuple $X = (X_1, X, \overline{X}(\mu_{1X_1}, \mu_{2X}), L_X)$ is not an Fs-set if, and only if

(*a*) $X \nsubseteq X_1$ or

(*b*) $\mu_{1X_1} x \ngeq \mu_{2X} x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of B for any $B \subseteq A$.

Definition : An Fs-subset $Y = (Y1, Y, \overline{Y}(\mu_{1Y_1}, \mu_{2Y}), L_Y)$ of A, is said to be an Fs-empty set of second kind if, and only if

(*a*) $Y_1 = Y = A$

(*b*) $L_Y \leq L_A$

(*c*) $\overline{Y} = 0$

4.1. Remark : We denote Fs-empty set of first kind or Fs-empty set of second kind by Φ_A and we prove later (1.15), Φ_A is the least Fs-subset among all Fs-subsets of A.

5. Definition : Let $B_1 = (B_{11}, B_1, \overline{B}_1(\mu_{1B_{11}}, \mu_{2B1}), L_{B1})$ and

$B_2 = (B_{12}, B_2, \overline{B}_2(\mu_{1B12}), \mu_{2B2}), L_{B2}$ be a pair of Fs-subsets.

1. We say that B_1 and B_2 are (1,5)-equal, if $B_{11} = B_{12}$ and $L_{B_1} = L_{B_2}$
2. We say that B_1 and B_2 are (2,5)-equal, if $B_1 = B_2$ and $L_{B1} = L_{B_2}$
3. We say that B_1 and B_2 are 3-equal, if B_1 and B_2 are (1,5)-equal and $\mu_{1B_{11}} = \mu_{1B12}$
4. We say that B_1 and B_2 are 4-equal, if B_1 and B_2 are (2,5)-equal and $\mu_{2B_1} = \mu_{2B2}$
5. We say that B_1 and B_2 are Total equal denoted $B_1 = B_2$ (T), if B_1 and B_2 are (2,5)-equal and $\overline{B}_1 = \overline{B}_2$
6. We say that B_1, B_2 are Full-equal, denoted $B_1 = B_2$, if B_1 and B_2 are 3-equal and 4-equal.

6. Proposition : $B_1 = (B_{11}, B_1, \overline{B}_1(\mu_{1B_{11}}), \mu_{B1}, L_{B1}$ and

$B_2 = (B_{12}, B_2, \overline{B}_2(\mu_{1B12}, \mu_{B_2}), L_{B2})$ are Full-equal if, only if $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$.

7. Remark : Whenever X and Y are Complete Boolean algebra $\Phi \subseteq X \times Y$ be a relation

(*a*) We say that Φ is $(\vee, \wedge)$-complete relation on X if, and only if $\vee \Phi(\wedge_{\alpha\in T} \alpha) = \wedge_{\alpha\in T}(\wedge \Phi \alpha)$ for any $T \subseteq X$.

(*b*) We say that Φ is $(\vee, \wedge)$-complete relation on X if, and only if $\vee \Phi(\wedge_{\alpha\in T} \alpha) = \wedge_{\alpha\in T}(\vee \Phi \alpha)$ for any $T \subseteq X$.

(*c*) We say that Φ is $(\vee, \wedge)$-complete relation on X if, and only if $\wedge \Phi(\wedge_{\alpha\in T} \alpha) = \wedge_{\alpha\in T}(\wedge \Phi \alpha)$ for any $T \subseteq X$.

(*d*) We say that Φ is said to be $\vee$-increasing on X if, and only if, and only if, $\vee \ \Phi \ \alpha \ \leq \vee \beta$ for any $\alpha, \beta \in X$ such that $\alpha \leq \beta$.

8. Proposition : Whenever $\Phi : X \to Y$ is a complete Boolean algebra homomorphism, then

1. is join increasing on
2. is -complete relation on
3. is -complete relation on

3. FS-FUNCTIONS

1. Definition : A Triplet (f_1, f, Φ)is said to be is an Fs-Function between two given Fs-subsets

$$B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \text{ and}$$
$$= (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C) \text{ of A, denoted by}$$
$$(f_1, f, \Phi) : B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \rightarrow C$$
$$= (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C) \text{ if, and only if (using the diagrams).}$$

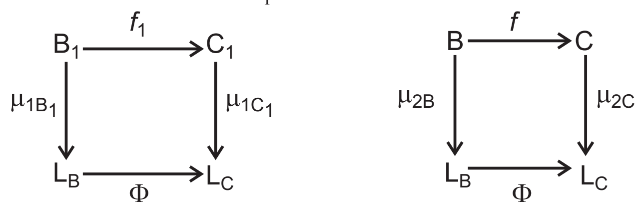

Figure 1: Fs-function $\overline{f}$ B → C

(*a*) $f = f_1|_B^C : B \rightarrow C$ be into

(*b*) $\Phi : L_B \rightarrow L_c$ is complete homomorphism

(f_1, f, Φ) is denoted by $\overline{f}$

2. Proposition : (*i*) $\mu_{1C_1}|_C \circ f_1|_B \geq \mu_{2C} \circ f$

(*ii*) $\Phi \circ \mu_{1B_1}|_B \geq \Phi \circ \mu_{2B}$

3. Def : Increasing Fs-function

$\overline{f}$ is said to be an increasing Fs- function, and denoted by $\overline{f}_i$ if, and only if(using fig-1)

(*a*) $\mu_{1C_1}|C \circ f_1|_B \geq \Phi \circ \mu_{1B_1}$

(*b*) $\mu_{2C} \circ f \leq \Phi \circ \mu_{2B}$

4. Proposition : $\Phi \circ (\mu_{2B}\, x)^c = [(\Phi \circ \mu_{2B})x]^c$

5. Proposition: $\Phi \circ \overline{B} \leq \overline{C} \circ f$, provided $\overline{f}$ is an increasing Fs-function

6. Def : Decreasing Fs-function

$\overline{f}$ is said to be decreasing Fs-function denoted as $\overline{f}_d$ and if and only if

(*a*) $\mu_{1C_1}|\, C \circ f_1|_B \leq \Phi \circ \mu_{1B_1}$

(*b*) $\mu_{2C} \circ f \geq \Phi \circ \mu_{2B}$

7. Proposition : $\Phi \circ \overline{B} \geq \overline{C} \circ f$, provided $\overline{f}$ is a decreasing Fs-function

8. Def : Preserving Fs- function

$\overline{f}$ is said to be preserving Fs-function and denoted as $\overline{f}_p$ if, and only if

(*a*) $\mu_{1C_1}|_C \circ f_1|\, B = \Phi \circ \mu_{1B1}$

(*b*) $\mu_{2C} \circ f = \Phi \circ \mu_{2B}$

9. Proposition : $\Phi \circ \overline{B} = \overline{C} \circ f$, provided $\overline{f}$ is Fs- preserving function

IMAGES OF FS-SUBSET

10. Definition. Let $D \subseteq B$ and $\overline{f} : B \rightarrow C$ be an Fs-function, where

$$B = (B_1, B, \overline{B}(\mu_{1B1}, \mu_{2B}), L_B),$$
$$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}, L_C),$$
$$D = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D),$$
$$D = D \text{ and } f = f_1|_B^C : B \rightarrow C \text{ be onto.}$$

Define $\overline{f}$(D) as follows

$$\overline{f}\,(\mathrm{D}) \;=\; \varepsilon = (\mathrm{E}_1, \mathrm{E}, \overline{\mathrm{E}}\,(\mu_{1\mathrm{E}_1}, \mu_{2\mathrm{E}})\mathrm{L}_\mathrm{E}), \text{ where}$$

1. $\mathrm{E}_1 = f_1\,(\mathrm{D}_1)$
2. $\mathrm{E} = f(\mathrm{D})$
3. $\mu_{1\mathrm{E}_1} : \mathrm{E}_1 \to \mathrm{L}_\mathrm{C}$ is defined by

$$u_{1\mathrm{E}_1} y \;=\; \begin{cases} \mu_{2\mathrm{C}} y \vee \left[\mu_{1\mathrm{C}_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in \mathrm{D}_1}} \Phi\mu_{1\mathrm{D}_1} x\right)\right], & \text{if } y \in \mathrm{C} \\ \mu_{1\mathrm{C}_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in \mathrm{D}_1}} \Phi\mu_{1\mathrm{D}_1} x\right), & \text{if } y \notin \mathrm{C} \end{cases}$$

4. $\mu_{2\mathrm{E}} : \mathrm{E} \to \mathrm{L}_\mathrm{C}$ is defined by

$$u_{2\mathrm{E}} y \;=\; \mu_{2\mathrm{C}} y \vee \left[\mu_{1\mathrm{C}_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in \mathrm{D}}} \Phi\mu_{2\mathrm{D}} x\right)\right]$$

5. $\mathrm{L}_\mathrm{E} = ([\mu(_{1\mathrm{E}_1}\,(\mathrm{E}_1)]) =$ The complete subalgebra generated by $[\mu_{1\mathrm{E}_1}\,(\mathrm{E}_1)]$, where $[\mu_{1\mathrm{E}_1}\,(\mathrm{E}_1)] =$ The complete ideal generated by $\mu_{1\mathrm{E}_1}(\mathrm{E}_1)$

4. INVERSE IMAGE OF FS-SUBSET

1. Definition. Let $\mathrm{D} \subseteq \mathrm{B}$ and $\overline{f}: \mathrm{B} \to \mathrm{C}$ be an Fs-function, $\phi^{-1} \subseteq \mathrm{L}_\mathrm{C} \times \mathrm{L}_\mathrm{B}$ be $\vee$-increasing ($\because$ Prop 1.7(d))

$$f \;=\; f_1\big|_\mathrm{B}^\mathrm{C} : \to \text{ be onto.}$$

Let $\varepsilon \subseteq \mathrm{C}$,

where $\varepsilon = (\mathrm{E}_1, \mathrm{E}, \overline{\mathrm{E}}\,(\mu_{1\mathrm{E}_1}, \mu_{2\mathrm{E}}), \mathrm{L}_\mathrm{E})$

Define $\overline{f}^{\,-1}\,(\mathrm{E})$ as follows

$$\begin{aligned} \overline{f}^{\,-1}(\varepsilon) &= \mathrm{D} \\ &= (\mathrm{D}_1, \mathrm{D}, \overline{\mathrm{D}}\,(\mu_{1\mathrm{D}_1}, \mu_{2\mathrm{D}}, \mathrm{L}_\mathrm{D}), \end{aligned}$$

where

(a) $\mathrm{D}_1 = f_1^{-1}\,(\mathrm{E}_1)$

(b) $\mathrm{D} = f^{-1}\,(\mathrm{E})$

(c) $\mu_{1\mathrm{D}_1} : \mathrm{D}_1 \to \mathrm{L}_\mathrm{D}$ is defined by

$$\mu_{1\mathrm{D}_1} x \;=\; \begin{cases} \mu_{1\mathrm{B}_1} x & \text{, whenever } \Phi^{-1}\mu_{1\mathrm{E}_1} f_1 x = \Phi \\ \mu_{2\mathrm{B}}\; x \vee \left[\mu_{1\mathrm{B}_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1\mathrm{E}_1} f_1 x \\ \alpha \in \mathrm{L}_\mathrm{B}}} \Phi^{-1}\mu_{1\mathrm{E}_1} f_1 x\right)\right] & , x \in \mathrm{B} \\ \mu_{1\mathrm{B}_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1\mathrm{E}_1} f_1 x \\ \alpha \in \mathrm{L}_\mathrm{B}}} \Phi^{-1}\mu_{1\mathrm{E}_1} f_1 x\right) & , x \notin \mathrm{B} \end{cases}$$

(d) $\mu_{2\mathrm{D}} : \mathrm{D} \to \mathrm{L}_\mathrm{D}$ is defined by

$$\mu_{2\mathrm{D}} x \;=\; \begin{cases} \mu_{2\mathrm{B}} x & \text{, whenever } \Phi^{-1}\,\mu_{2\mathrm{E}} f x = \Phi \\ \mu_{2\mathrm{B}}\; x \vee \left[\mu_{1\mathrm{B}_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2\mathrm{E}} f x \\ \alpha \in \mathrm{L}_\mathrm{B}}} \Phi^{-1}\mu_{2\mathrm{E}} f x\right)\right] \end{cases}$$

(e) $\mathrm{L}_\mathrm{D} = \mathrm{L}_\mathrm{B}$

2. Proposition : $\overline{f}^{-1}(\varepsilon)$ is an Fs-subset of B, if $\Phi^{-1} \subseteq L_C \times L_B$ is $\vee$-increasing ($\because$Prop 1.7(*d*))

Proof : Let $\overline{f}^{-1}(\varepsilon) = D = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$,

where

(*a*) $D_1 = f_1^{-1}(E_1)$

(*b*) $D = f^{-1}(E)$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1}x = \begin{cases} \mu_{1B_1}x & , \text{whenever } \Phi^{-1}\mu_{1E_1}f_1x = \Phi \\ \mu_{2B}\, x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee\limits_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right] & , x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee\limits_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right) & , x \notin B \end{cases}$$

(*d*) $\mu_{2D} : D_1 \to L_D$ is defined by

$$\mu_{2D}x = \begin{cases} \mu_{2B}x, \text{ whenever } \Phi^{-1}\mu_{2E}fx = \Phi \\ \mu_{2B}\, x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee\limits_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx\right)\right] \end{cases}$$

(*e*) $L_D = L_B$

$$E \subseteq C \Rightarrow E_1 \subseteq C_1 \Rightarrow f_1^{-1}(E_1) = D_1 \subseteq f_1^{-1}(C_1) = B_1$$

so that, $D_1 \subseteq B_1$

$$E \subseteq C \Rightarrow E \supseteq C \Rightarrow f^{-1}(E) = D \supseteq f^{-1}(C) = B$$

so that $D \supseteq B$

Case(I) For $x \in B$,

$\mu_{2E}fx \leq \mu_{1E_1}f_1x$ and Φ^{-1} is $\vee$-increasing ($\because$ Prop 1.7(*d*))

$$\Rightarrow \bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx \leq \bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x$$

$$\Rightarrow \mu_{2B}\, x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx\right)\right] \leq \mu_{2B}\, x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right]$$

$$\Rightarrow \mu_{2D}x \leq \mu_{1D_1}x$$

Case(*ii*). For $x \notin B$,

$\mu_{2E}\, fx \leq \mu_{1E_1}\, f_1x$ and Φ^{-1} is $\vee$-increasing ($\because$ Prop 1.7(*d*))

$$\Rightarrow \bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx \leq \bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x$$

$$\Rightarrow \mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx\right) \leq \mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)$$

$$\Rightarrow \mu_{2D}x \leq \mu_{1D_1}x$$

Hence $\overline{f}^{-1}(\varepsilon)$ is an Fs-subset of B.

3. Proposition : Let B and C be any pair of Fs-subsets and $\overline{f}$: B → C be an Fs-function. Let ε_1 and ε_2 be Fs-subsets C such that $\varepsilon_1 \subseteq \varepsilon_2$ and $\varepsilon_1 = \varepsilon_2 = C$, then $\overline{f}^{-1}(\varepsilon_1) \subseteq \overline{f}^{-1}(\varepsilon_2)$

Proof : Let

$$B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B),$$
$$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C),$$
$$\varepsilon_1 = (E_{11}, E_1, \overline{E}_1(\mu_{1E_{11}}, \mu_{2E_1}), L_{E_1})$$

and

$$\varepsilon_2 = (E_{12}, E_2, \overline{E}_2(\mu_{1E_{12}}, \mu_{2E_2}), L_{E_2})$$
$$\varepsilon_1 \subseteq \varepsilon2 \text{ imply}$$

(*a*) $E_{11} \subseteq E_{12}, E_1 \supseteq E_2$

(*b*) $L_{E_1} \leq L_{E_2}$

(*c*) $\mu_{1E_{11}} y \leq \mu_{1E12} y, \forall y \in E_{11}$ and $\mu_{2E_1} y \geq \mu_{2E_2} y, \forall y \in E_2$

Let,

$$\overline{f}^{-1}(\varepsilon_1) = D_1 = (D_{11}, D_1, \overline{D}_1(\mu_{1D_{11}}, \mu_{2D_1}), L_{D_1}),$$

and

$$\overline{f}^{-1}(\varepsilon_2) = D_2 = (D_{12}, D_2, \overline{D}_2(\mu_{1D12}, \mu_{2D_2}), L_{D_2})$$

where

1. $D_{11} = f_1^{-1}(E_{11})$
2. $D_1 = f^{-1}(E_1)$
3. $\mu_{1D11} : D_{11} \to L_{D_1}$ is defined by

$$\mu_{1D11}x = \begin{cases} \mu_{1B_1}x & \text{, whenever } \Phi^{-1}\mu_{1E_{11}} f_1 x = \Phi \\ \mu_{2B}\, x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{11}} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x\right)\right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{11}} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x\right) & , x \notin B \end{cases}$$

4. $\mu_{2D_1} : D_1 \to L_{D_1}$ is defined by

$$\mu_{2D_1}x = \begin{cases} \mu_{2B}x, \quad \text{whenver } \Phi^{-1}\mu_{2E_1} fx = \Phi \\ \mu_{2B}\, x \wedge \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_1} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_1} fx\right)\right] \end{cases}$$

5. $L_{D1} = L_B$
6. $D_{12} = f_1^{-1}(E_{12})$
7. $D_2 = f^{-1}(E_2)$
8. $\mu_{1D_{12}} : D_{12} \to L_{D_2}$ is defined by

$$\mu_{1D12}x = \begin{cases} \mu_{1B_1}x & \text{, whenever } \Phi^{-1}\mu_{1E_{12}} f_1 x = \Phi \\ \mu_{2B}\, x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{12}} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{12}} f_1 x\right)\right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{12}} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{12}} f_1 x\right) & , x \notin B \end{cases}$$

9. $\mu_{2D_2} : D_{12} \rightarrow L_{D_2}$ is defined by

$$\mu_{2D_2}x = \begin{cases} \mu_{2B}x, & \text{whenver } \Phi^{-1}\mu_{2E_2} fx = \Phi \\ \mu_{2B}\ x \wedge \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_2}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_2} fx \right)\right] & \end{cases}$$

10. $LD_2 = L_B$

Need to show that, $\bar{f}^{-1}(\varepsilon^1) \subseteq \bar{f}^{-1}(\varepsilon_2) \Rightarrow D_1 \subseteq D_2$

It is enough to show that,

11. $D_{11} \subseteq D_{12}$, $D_1 \supseteq D_2$
12. $L_{D_1} \leq L_{D_2}$
13. $\mu_{1D_{11}}\ x \leq \mu_{1D12}x$, $\forall x \in D_{11}$ and $\mu_{2D_1}x \geq \mu_{2D_2}x, \forall x \in D_2$

Proof of (11) (*a*) implies

$$\left.\begin{array}{l} f_1^{-1}(E_{11}) \subseteq f_1^{-1}(E_{12}) \text{ or } D_{11} \subseteq D_{12} \text{ and} \\ f^{-1}(E_1) \supseteq f^{-1}(E_2) \text{ or } D_1 \supseteq D_2) \end{array}\right\} \quad \ldots(1)$$

Proof of (12) follows from (5) and (10) ....(2)

Proof of (13) From (*c*)

$$\mu_{1E_{11}} f_1 x \leq \mu_{1E12}\ f_1\ x$$

$$\Rightarrow \bigvee_{\substack{\Phi\alpha = \mu_{1E_{11}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x \leq \bigvee_{\substack{\Phi\alpha = \mu_{1E_{12}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x$$

also

Case (1) For $x \in B$,

$$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{11}}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x \right) \leq \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{12}}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{12}} f_1 x \right)$$

$$\Rightarrow \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{11}}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x \right)\right] \leq \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{12}}y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{12}} f_1 x \right)\right]$$

Case (1) For $x \notin B$,

$$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{11}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{11}} f_1 x \right) \leq \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_{12}} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_{12}} f_1 x \right)$$

$$\Rightarrow \mu_{1D_{11}} \leq \mu_{1D_{12}}x$$

From Case 1 and Case 2 for each $x \in D_{11}$ we can conclude that,

$$\mu_{1D_{11}}x \leq \mu_{1D12}\ x \text{ for } x \in B \text{ or } x \notin B \quad \ldots(3)$$

Again From (*c*) we have $\mu_{2E_1} fx \geq \mu_{2E_2}\ fx$

$$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_2} fx \right) \geq \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_2} fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_2} fx \right)$$

$$\Rightarrow \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_1} fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_1} fx \right)\right] \geq \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_2} fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E_2} fx \right)\right]$$

$$\Rightarrow \qquad \mu_{2D_1} x \ \geq \ \mu_{2D_2} x, \ \forall x \in D_2 \qquad \text{...(4)}$$

From (1),(2),(3) and (3) we can conclude that,

$$D_1 \subseteq D_2 \ i.e. \ \bar{f}^{-1}(\varepsilon_1) \subseteq \bar{f}^{-1}(\varepsilon_2).$$

4. Proposition : Let $\bar{f}$: B → C be an Fs-function. Let E ⊆ C be such that E = C and assume

(*a*) $\mu_{2E} y \leq \Phi \circ \mu_{2B} x$

(*b*) $\Phi \circ \mu_{1B_1} x \leq \mu_{1E_1} y$ for each $x \in B$ such that $y = f_1 x$

then $\bar{f}\,\bar{f}^{-1}(\varepsilon) \subseteq \varepsilon.$

Proof : Observe that, $\mu_{2C} y \leq \mu_{2E}\, y \leq \Phi \circ \mu_{2B}\, x \leq \Phi \circ \mu_{1B_1} x \leq \mu_{1E_1} y \leq \mu_{1C_1} y$

Let $B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}, L_B),$

$C = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}, L_C)$ and

$E = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}, L_E)$

Let, $\bar{f}^{-1}(\varepsilon) = D = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}, L_D),$h

where

(*a*) $D_1 = f_1^{-1}(E_1)$

(*b*) $D = f_1^{-1}(E)$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1} x = \begin{cases} \mu_{1B_1} x & , \text{whenever } \Phi^{-1}\mu_{1E_1} f_1 x = \Phi \\ \mu_{2B}\ x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x \right) \right] & , x \in B \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x \right) & , x \notin B \end{cases}$$

(*d*) $\mu_{2D} : D \to L_D$ is defined by

$$\mu_{2D} x = \begin{cases} \mu_{2B} x, \quad \text{whenver } \Phi^{-1}\mu_{2E} f x = \Phi \\ \mu_{2B}\ x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} f x \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E} f_l x \right) \right] \end{cases}$$

(*e*) $L_D = L_B$

Again suppose $\bar{f}\,\bar{f}^{-1}(\varepsilon) = \bar{f}(D) = F = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F),$

where

(*f*) $F_1 = f_1(D_1) = f_1(f_1^{-1})(E_1) = E_1$

(*g*) $F = f(D) = f(f^{-1})(E)) = f(f^{-1})(C)) = C = E$

(*h*) $\mu_{1F_1} : F_1 \to L_C$ is defined by

$$\mu_{1F_1} y = \begin{cases} \mu_{2C}\ y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x, \right) \right], \text{if } y \in C \\ \mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right), \text{if } y \notin C \end{cases}$$

(*i*) $\mu_{2F_1} : F \to L_C$ is defined by

$$\mu_{2F}\, y = \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D}\, x\right)\right]$$

(*j*) $L_F = ([\mu_{1F_1}(F_1)])$ = The complete subalgebra generated by $[\mu_{1F_1}(F_1)]$, where $[\mu_{1F_1})(F_1)]$ = The complete ideal generated by $\mu_{1F_1}(F_1)$

Need to show that, $F \subseteq \varepsilon$ *i.e.* it is enough to show that

(*k*) $F_1 \subseteq E_1$, $E \subseteq F$

(*l*) L_F is a complete subalgebra of L_E or $L_F \leq L_E$

(*m*) $\mu_{1F_1} \leq \mu_{1E_1} \mid F_1$, and $\mu_{2F} \mid E \geq \mu_{2E}$

(*k*) is follows from (*f*) and (*g*)

(*l*) is follows from (*e*) and (*j*)

Proof of (*m*) : For $y \in C$,

$$\mu_{1F_1}\, y = \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1}\, x\right)\right]$$

$$= \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\left(\mu_{2B}\, x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}\, f_1 x\right)\right]\right)\right)\right]$$

$$= \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\,\mu_{2B}\, x \vee \left[\Phi\,\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1}\, f_1 x\right)\right]\right)\right)\right]$$

Case 1 : (*a*) : $\Phi^{-1}\,\mu_{1E1} f_1 x = \Phi$,

$$\mu_{1F_1}\, y = \mu_{2C} y \vee \left[\mu_{1C_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}{}_{2B} x\right)\right]$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E1} f_1 x \\ \alpha \in L_B}} \Phi\mu_{2B} x\right)$$

$$\leq \bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\mu_{2B} x \leq \mu_{1E_1} f_1 x \ (\because \Phi \circ \mu_{2B}\, x \leq \Phi \circ \mu_{1B_1}\, x \leq \mu_{1E_1}\, y)$$

Case : (*b*) $\Phi^{-1}\,\mu_{1E_1} f_1\, x \neq \Phi$

$$\mu_{1F_1}\, y = \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\,\mu_{2B}\, x \vee \left[\Phi\,\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1}\, f_1 x\right)\right]\right)\right)\right]$$

$$= \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\,\mu_{2B}\, x \vee \left[\Phi\,\mu_{1B_1} x \wedge \mu_{1E_1} f_1 x\right]\right)\right)\right]$$

$$= \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \Phi\mu_{1B1} x\right)\right)\right]$$

$$= \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1B_1} x\right)\right]$$

$$= \mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{f = f_1 x \\ x \in D_1}} \Phi\mu_{1B_1} x \right) \leq \bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1B_1} x \leq \mu_{1E_1} f_1 x$$

For $y = f_1 x \notin C$,

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x\, \Phi \\ x \in D_1}} \left(\mu_{2B}\ x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_1} f_1 x \right) \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1} \mu_{1E_1} f_1 x \right) \right] \right) \right)$$

Case 2 : (a) $\Phi^{-1} \mu_{1E_1}\ y = \Phi$,

$$\mu_{1F1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\mu_{2B} x \right)$$

$$\leq \bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\mu_{2B} x \leq \mu_{1E_1} f_1 x\ (\because \Phi \circ \mu_{2B}\ x \leq \Phi \circ \mu_{1B_1}\ x \leq \mu_{1E_1}\ y)$$

Case 2 : (b) $\Phi^{-1} \mu_{1E_1}\ y = \Phi$,

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1} \mu_{1E_1} f_1 x \right) \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y \quad f\, x \\ \in}} \left(\Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \wedge \mu_{1E_1}\ f_1 x \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \Phi\mu_{1B_1} x \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1B_1} x \right) \leq \bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1B_1} x \leq \mu_{1E_1} f_1 x$$

$$\mu_{2F} y = \mu_{2C} y \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \Phi\mu_{2D} x \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \Phi \left\{ \mu_{2B}\ x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ x \in L_B}} \Phi^{-1} \mu_{2E} fx \right) \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ x \in L_B}} \Phi\Phi^{-1} \mu_{2E} fx \right) \right] \right\} \right) \right]$$

Case 3 : (a) $\Phi^{-1} \mu_{2E}\ fx = \Phi$,

$$\mu_{2F} y = \mu_{2C}\ y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \Phi\mu_{2B} x \right) \right]$$

$$= \mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \Phi\mu_{2B} x \right) \geq \mu_{1C_1}\, y \wedge \mu_{2E} y = \mu_{2E} fx$$

Case 3 : (*b*) $\Phi^{-1} \mu_{2E}\, fx \neq \Phi$,

$$\mu_{2F} y = \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \Phi\mu_{2B} x \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{2E} fx \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \mu_{2E} fx \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2B}\, x \right) \right] \geq \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2B}\, x \right) \geq \mu_{1C_1} y \wedge \mu_{2E} fx = \mu_{2E} fx$$

Hence $\quad F \subseteq \varepsilon$ *i.e.*

$$\bar{f}\, \bar{f}^{-1}(\varepsilon) \subseteq \varepsilon.$$

5. Proposition : If $\bar{f} : B \to C$ be a decreasing Fs-function, $\varepsilon \subseteq C$, $E = C$, then $\bar{f}\, \bar{f}^{-1}(\varepsilon) \subseteq \varepsilon$.

Proof : We have, $\Phi \circ \mu_{2B}\; x \leq \mu_{2C}\, y \leq \mu_{2E}\, y \leq \mu_{1E_1}\, y \leq \mu_{1C_1}\, y \leq \Phi \circ \mu_{1B_1}\, x$ for $x \in B$

Let $\quad B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B),$

$\quad C = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and

$\quad E = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$

Let $\quad \bar{f}^{-1}(\varepsilon) = D = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D),$

Where

(*a*) $D_1 = f_1^{-1}(E_1)$

(*b*) $D = f^{-1}(E)$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1} x = \begin{cases} \mu_{2B} x & , \text{whenever } \Phi^{-1} \mu_{1E_1}\; f_1 x = \Phi \\ \mu_{2B} x \vee \left[\left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_1} f_1 x \right) \right] & , x \in V \\ \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_1} f_1 x \right) & , x \notin V \end{cases}$$

(*d*) $\mu_{2D} : D \to L_D$ is defined by

$$\mu_{2D} x = \begin{cases} \mu_{2B} x \text{ , whenever } \Phi^{-1} \mu_{2E}\; fx = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \Phi^{-1} \mu_{2E} fx \right) \right] \end{cases}$$

(*e*) $L_D = L_B$

Again suppose $\bar{f}\, \bar{f}^{-1}(\varepsilon) = \bar{f}(D) = F = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}, L_F),$

where

(*f*) $F_1 = f_1(D_1) = f_1(f_1^{-1})(E_1)) \subseteq E_1$ ($\because f_1$ is onto) (*g*) $F = f(D) = f(f^{-1})(E)) = f(f^{-1})(C)) = C = E$

(*h*) $\mu_{1F_1} : F_1 \to L_C$ is defined by

$$\mu_{1F_1} y = \begin{cases} \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right) \right] & , y \in C \\ \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right) & , y \notin C \end{cases}$$

(*i*) $\mu_{2F} : F \to L_C$ is defined by

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{2D} x \right) \right]$$

(*j*) $L_F = ([\mu_{1F_1}) (F_1)]) =$ The complete subalgebra generated by $[\mu_{1F_1}(F_1)]$, where $[\mu_{1F_1}(F_1)] =$ The complete ideal generated by $\mu_{1F_1}(F_1)$

Need to show that, $F \subseteq \varepsilon$ *i.e.* it is sufficient to show that,

(*k*) $F_1 \subseteq E_1$, $F \supseteq E$

(*l*) $L_F \leq L_E$

(*m*) $\mu_{1F_1} y \leq \mu_{1E_1} y$, $\mu_{2F} y \geq \mu_{2E} y$

(*k*) is follows from (*f*) and (*g*)

(*l*) is follows from (*e*) and (*j*)

Proof of (*m*) : For $x \in B$,

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\left(\mu_{2B}\ x \vee \left[\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x \right) \right] \right) \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x \right) \right] \right) \right) \right]$$

Case :1 (*a*) : $\Phi^{-1} \mu_{1E_1} f_1 x = \Phi$, $\mu_{1F_1} f_1 x = \mu_{1E_1} f_1 x$

Case :1 (*b*) : $\Phi^{-1} \mu_{1E_1} f_1\, x \neq \Phi$

$$\mu_{1F_1} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x \right) \right] \right) \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \wedge \mu_{1E_1} f_1 x \right] \right) \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \mu_{1E_1} f_1 x \right) \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1E_1} f_1 x \right) \right]$$

$$= \mu_{2C} y \vee (\mu_{1C_1} y \wedge \mu_{1E_1} f_1 x)$$

$$= \mu_{2C} y \vee \mu_{1E_1} f_1 x$$

$$= \mu_{1E_1} f_1 x, \text{ whenever } x \in B$$

For $\quad y \notin C,$

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \left(\mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_1} f_1 x \right) \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1} \mu_{1E_1} f_1 x \right) \right] \right) \right)$$

Case : 2 (*a*) : $\Phi^{-1} \mu_{1E_1} f_1 x = \Phi$, $\mu_{1F_1} f_1 x = \mu_{1E_1} f_1 x$

Case :2 (*b*) : $\Phi^{-1} \mu_{1E_1} f_1 x \neq \Phi$

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi\Phi^{-1} \mu_{1E_1} f_1 x \right) \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{1E_1} f_1 x \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \mu_{1E_1} f_1 x \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1E_1} f_1 x \right)$$

$$= \mu_{1C_1} y \wedge \mu_{1E_1} f_1 x \wedge = \mu_{1E_1} f_1 x$$

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D} x \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi \left\{ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} y \\ \alpha \in L_B}} \Phi^{-1} \mu_{2E} f x \right) \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} y \\ \alpha \in L_B}} \Phi\Phi^{-1} \mu_{2E} f x \right) \right] \right\} \right) \right]$$

Case : 3 (*a*) : $\Phi^{-1} \mu_{2E} fx = \Phi$, $\mu_{2F} fx = \mu_{2E} fx$

Case : 3 (*b*) : $\Phi^{-1} \mu_{2E} fx \neq \Phi$

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{2E} fx \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \mu_{2E} fx \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \mu_{2E} \; fx \right) \right]$$

$$= \mu_{2C}y \vee [\mu_{1C_1}y \wedge \mu_{2E}fx]$$
$$= \mu_{2C}y \vee \mu_{2E}fx = \mu_{2E}fx$$

Hence $F \subseteq \varepsilon$ *i.e.*

$$\bar{f}\,\bar{f}^{-1}(\varepsilon) \subseteq \varepsilon.$$

5.1. Corollary : If $\bar{f} : B \to C$ be a decreasing Fs-function, f_1 is onto, $\varepsilon \subseteq C$, $E = C$, then $\bar{f}\,\bar{f}^{-1}(\varepsilon)$ and ε are full-equal.

Proof : We have, $\Phi \circ \mu_{2B}\ x \le \mu_{2C}\, y \le \mu_{2E}\, y \le \mu_{1E_1}\, y \le \mu_{1C_1}\, y \le \Phi \circ \mu_{1B_1}\, x$ for $x \in B$

Let $B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$,

$C = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and

$E = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$

Let, $\bar{f}^{-1}(\varepsilon) = D = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$,

Where

(*a*) $D_1 = f_1^{-1}(E_1)$

(*b*) $D = f^{-1})(E)$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1}x = \begin{cases} \mu_{1B_1}x & \text{, whenever } \Phi^{-1}\mu_{1E_1} f_1x = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right] & , x \in V \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right) & , x \notin V \end{cases}$$

(*d*) $\mu_{2D} : D \to L_D$ is defined by

$$\mu_{2D}x = \begin{cases} \mu_{2B}x & \text{, whenever } \Phi^{-1}\mu_{2E}\ fx = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E_E}fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}\ fx\right)\right] & \end{cases}$$

(*e*) $L_D = L_B$

Again suppose $\bar{f}\,\bar{f}^{-1}(E) = \bar{f}(D) = F = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$,

where

(*f*) $F_1 = f_1(D_1) = f_1(f_1^{-1}(E_1) = E_1$ ($\because f_1$ is onto)

(g) $F = f(D) = f(f^{-1})(E)) = f(f^{-1})(C)) = C = E$

(*h*) $\mu_{1F_1} : F_1 \to L_C$ is defined by

$$\mu_{1F_1}y = \begin{cases} \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x \in D_1}} \Phi\mu_{1D_1}\ x\right)\right] & , y \in C \\ \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x \in D_1}} \Phi\mu_{1D_1}x\right) & , y \in C \end{cases}$$

(*i*) $\mu_{2F} : F \to L_C$ is defined by

$$\mu_{2F}\ y = \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x \in D}} \Phi\mu_{2D}\ x\right)\right]$$

(*j*) $L_F = ([\mu_{1F_1}(F_1)])$ = The complete subalgebra generated by $[\mu_{1F_1}(F_1)]$, where $[\mu_{1F1}(F_1)]$ = The complete ideal generated by $\mu_{1F_1}(F_1)$

Need to show that, F and E are full-equal *i.e.* it is sufficient to show that,

(*k*) $F_1 = E_1$

(*l*) $F = E$

(*m*) $L_F = L_E$

(*n*) $\mu_{1F1} f_1 x = \mu_{1E_1} f_1 x$

(*o*) $\mu_{2F}\, fx = \mu_{2E}\, fx$

(*k*) and (*l*) is follows from (*f*) and (*g*)

(*m*) is follows from (*e*) and (*j*)

Proof of (*n*) : For $x \in B$,

$$\mu_{1F_1} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1}\, x\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\left(\mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_l x\right)\right]\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_l x\right)\right]\right)\right)\right]$$

Case : 1 (*a*) : $\Phi^{-1}\mu_{1E_1} f_1 x = \Phi$, $\mu_{1F_1} y = \mu_{1E_1} f_1 x$

Case : 1 (*b*) : $\Phi^{-1}\mu_{1E_1} y \neq \Phi$

$$\mu_{1F_1} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_l x\right)\right]\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\, x \vee \left[\Phi\mu_{1B_1}\, x \wedge \mu_{1E_1} f_1 x\right]\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\, x \vee \mu_{1E_1}\, f_1 x\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1E_1} f_1 x\right)\right]$$

$$= \mu_{2C}\, y \vee (\mu_{1C_1}\, y \wedge \mu_{1E_1}\, y)$$

$$= \mu_{2C}\, y \vee \mu_{1E_1} f_1 x$$

$$= \mu_{1E_1} f_1 x, \text{ whenever } x \in B$$

For $y \notin C$,

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\left(\mu_{2B} x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} y \right) \right] \right) \right)$$

Case : 2 (*a*) : $\Phi^{-1}\mu_{1E_1} y = \Phi$, $\mu_{1F_1} y = \mu_{1E_1} y$

Case :2 (*b*) : $\Phi^{-1}\mu_{1E_1} y \neq \Phi$

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x \right) \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \vee \mu_{1E_1} f_1 x \right] \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \mu_{1E_1} f_1 x \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1E_1} f_1 x \right)$$

$$= \mu_{1C_1} y \wedge \mu_{1E_1} f_1 x$$

$$= \mu_{1E_1} f_1 x$$

$$\mu_{1F_1} y = \mu_{2C} y \wedge \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D} x \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi \left\{ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E} f_1 x \right) \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{2E} f_1 x \right) \right] \right\} \right) \right]$$

Case : 3 (*a*) : $\Phi^{-1}\mu_{2E} fx = \Phi$, $\mu_{2F} fx = \mu_{2E} fx$

Case : 3 (*b*): $\Phi^{-1}\mu_{2E} fx \neq \Phi$

$$\mu_{1F_1} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{2E} fx \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B} x \vee \mu_{2E} fx \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \mu_{2E} fx \right) \right]$$

$$= \mu_{2C} y \vee [\mu_{1C_1} y \wedge \mu_{2E} fx]$$

$$= \mu_{2C} y \vee \mu_{2E} fx = \mu_{2E} fx$$

Hence F = ε *i.e.*

$\bar{f}\,\bar{f}^{-1}(\varepsilon)$ and E are full-equal.

6. Proposition : If $\bar{f} : B \to C$ be a preserving Fs-function, $\varepsilon \subseteq C$, $\varepsilon = C$, then $\bar{f}\,\bar{f}^{-1}(\varepsilon) \subseteq \varepsilon$.

Proof : We have, $\Phi \circ \mu_{2B} x = \mu_{2C} y = \mu_{2E} y = \mu_{1E_1} y = \mu_{1C_1} y = \Phi\mu_{1B_1} x$ for $x \in B$

Let $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$,

$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and

$E = (E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E}), L_E)$

Let $\overline{f}^{-1}(\varepsilon) = D = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$,

Where

(*a*) $D_1 = f_1^{-1}(E_1)$

(*b*) $D = f^{-1}(E)$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1} x = \begin{cases} \mu_{1B_1} x & , \text{whenever } \Phi^{-1}\mu_{1E_1} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee\limits_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x \right)\right] & , x \in V \\ \mu_{1B_1} x \wedge \left(\bigvee\limits_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x \right) & , x \notin V \end{cases}$$

(*d*) $\mu_{2D} : D \to L_D$ is defined by

$$\mu_{2D} x = \begin{cases} \mu_{2B} x \vee, \text{ whenever } \Phi^{-1}\mu_{2E} fx = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee\limits_{fx} \Phi^{-1}\mu_{2E} fx \right)\right] \end{cases}$$

(*e*) $L_D = L_B$

Again suppose $\overline{f}\,\overline{f}^{-1}(\varepsilon) = \overline{f}(D) = F = (F_1, F, \overline{F}(\mu_{1F_1}, \mu_{2F}), L_F)$,

where

(*f*) $F_1 = f_1(D_1) = f_1(f_1^{-1})(E_1)) \subseteq E_1$ ($\because f_1$ is onto)

(*g*) $F = f(D) = f(f^{-1})(E)) = f(f^{-1})(C)) = C = E$

(*h*) $\mu_{1F_1} : F_1 \to L_C$ is defined by

$$\mu_{1F_1} y = \begin{cases} \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee\limits_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right)\right], & y \in C \\ \mu_{1C_1} y \wedge \left(\bigvee\limits_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \right) & , y \notin C \end{cases}$$

(*i*) $\mu_{2F} : F \to L_C$ is defined by

$$\mu_{2F}\, y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee\limits_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D}\, x \right)\right]$$

(*j*) $LF = ([\mu_{1F_1}(F_1)])$ = The complete subalgebra generated by $[\mu_{1F_1}(F_1)]$, where $[\mu_{1F_1})(F_1)]$ = The complete ideal generated by $\mu_{1F_1}(F_1)$

Need to show that, $F \subseteq \varepsilon$ *i.e.* it is sufficient to show that,

(*k*) $F_1 \subseteq E_1$, $F \supseteq E$

(*l*) $L_F \leq L_E$

(*m*) $\mu_{1F_1} f_1 x \leq \mu_{1E_1} f_1 x$, $\mu_{2F} fx \geq \mu_{2E} fx$

(*k*) is follows from (*f*) and (*g*)

(*l*) is follows from (*e*) and (*j*)

Proof of (*m*) : For $x \in B$,

$$\mu_{1F_1}y = \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D}} \Phi\mu_{1D_1}\, x\right)\right]$$

$$= \mu_{2C}y \vee \left[\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \Phi\left(\mu_{2B}x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_1}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right]\right)\right)\right]$$

$$= \mu_{2C}y \vee \left[\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}x \vee \left[\Phi\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_1}y\\ \alpha\in L_B}} \Phi\Phi^{-1}\mu_{1E_1}f_1x\right)\right]\right)\right)\right]$$

Case : 1 (*a*) : $\Phi^{-1}\,\mu_{1E_1}f_1x = \Phi,\ \mu_{1F_1}f_1x = \mu_{1E_1}f_1x$

Case :1 (*b*) : $\Phi^{-1}\,\mu_{1E}f_1x \neq \Phi$

$$\mu_{1F_1}y = \mu_{2C}y \vee \left[\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}x \vee \left[\Phi\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_1}y\\ \alpha\in L_B}} \Phi\Phi^{-1}\mu_{1E_1}f_1x\right)\right]\right)\right)\right]$$

$$= \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D}} \left(\Phi\mu_{2B_1}\, x \vee \left[\Phi\mu_{1B_1}\, x \wedge \mu_{1E_1}\, f_1x\right]\right)\right)\right]$$

$$= \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}\, x \vee \mu_{1E_1}f_1x\right)\right)\right]$$

$$= \mu_{2C}\, y \vee (\mu_{1C_1}\, y \wedge \mu_{1E_1}f_1x)$$

$$= \mu_{2C}\, y \vee \mu_{1E_1}f_1x$$

$$= \mu_{1E_1}f_1x, \text{ whenever } x \in B$$

For $y \notin C$,

$$\mu_{1F_1}y = \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \Phi\mu_{1D_1}\, x\right)$$

$$= \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \Phi\left(\mu_{2B}x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_1}f_1x\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right]\right)\right)$$

$$= \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}x \vee \left[\Phi\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_1}f_1x\\ \alpha\in L_B}} \Phi\Phi^{-1}\mu_{1E_1}f_1x\right)\right]\right)\right)$$

Case: 2 (*a*) : $\Phi^{-1}\,\mu_{1E_1}f_1x = \Phi,\ \mu_{1F_1}f_1x = \mu_{1E_1}f_1x$

Case :2 (*b*) : $\Phi^{-1}\,\mu_{1E_1}f_1x \neq \Phi$

$$\mu_{1F_1}y = \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}x \vee \left[\Phi\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_1}f_1x\\ \alpha\in L_B}} \Phi\Phi^{-1}\mu_{1E_1}f_1x\right)\right]\right)\right)$$

$$= \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}x \vee \left[\Phi\mu_{1B_1}x \wedge \mu_{1E_1}f_1x\right]\right)\right)$$

$$= \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x\\ x\in D_1}} \left(\Phi\mu_{2B}x \vee \mu_{1E_1}f_1x\right)\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1E_1} y \right)$$

$$= \mu_{1C_1} y \wedge \mu_{1E_1} y$$

$$= \mu_{1E_1} y$$

$$\mu_{2F}\, y = \mu_{2C} y \vee \left[m_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ x \in L_B}} \Phi\mu_{2D} x \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D_1}} \Phi \left\{ \mu_{2B} x \vee \left[\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}\, fx \right) \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D_1}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{2E} f_1 x \right) \right] \right\} \right) \right]$$

Case : 3 (*a*) : $\Phi^{-1}\, \mu_{2E} fx = \Phi$, $\mu_{2F} fx = \mu_{2E} fx$

Case :3 (*b*) : $\Phi^{-1}\, \mu_{2E} fx \neq \Phi$

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} fx \\ \alpha \in L_B}} \left\{ \Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{2E} fx \right] \right\} \right) \right.$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} y \\ \alpha \in L_B}} \left\{ \Phi\mu_{2B} x \vee \mu_{2E} fx \right\} \right) \right.$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \mu_{2E} fx \right) \right]$$

$$= \mu_{2C}\, y \vee [\mu_{1C_1}\, y \wedge \mu_{2E} fx]$$

$$= \mu_{2C}\, y \vee \mu_{2E} fx = \mu_{2E} fx$$

Hence $\quad F \subseteq \varepsilon$ *i.e.*

$$\bar{f}\bar{f}^{-1}(\varepsilon) \subseteq \varepsilon.$$

6.1. Corollary : If $\bar{f}: B \to C$ be a preserving Fs-function, f_1 is onto, $\varepsilon \subseteq C$, $E = C$, then $\bar{f}\,\bar{f}^{-1}(\varepsilon)$ and ε are full-equal.

Proof : We have, $\Phi \circ \mu_{2B}\, x = \mu_{2C}\, y = \mu_{2E}\, y = \mu_{1E_1}\, y = \mu_{1C_1} y = \Phi \circ\ \mu_{1B_1}\, x$ for $x \in B$

Let
$$B = (B_1, B, \bar{B}\, (\mu_{1B_1}, \mu_{2B}), L_B),$$
$$C = (C_1, C, \bar{C}\, (\mu_{1C_1}, \mu_{2C}), L_C) \text{ and}$$
$$E = (E_1, E, \bar{E}\, (\mu_{1E_1}, \mu_{2E}), L_E)$$

Let,
$$\bar{f}^{-1}(\varepsilon) = D = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D),$$

where

(*a*) $D_1 = f_1^{-1})\, (E_1)$

(*b*) $D = f^{-1})\, (E)$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$$\mu_{1D_1}x = \begin{cases} \mu_{1B_1}x & \text{, whenever } \Phi^{-1}\,\mu_{1E_1}\,f_1x = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right)\right] & , x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1}f_1x \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_1}f_1x\right) & , x \notin B \end{cases}$$

(d) $\mu_{2D} : D \to L_D$ is defined by

$$\mu_{2D}x = \begin{cases} \mu_{1B_1}x & \text{, whenever } \Phi^{-1}\,\mu_{2E}\,f_1x = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}f_1x \\ \alpha\in L_B}} \Phi^{-1}\mu_{2E}f_1x\right)\right] \end{cases}$$

(e) $L_D = L_B$

Again suppose $\bar{f}\,\bar{f}^{-1}(\varepsilon) = \bar{f}(D) = F = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$,

where

(f) $F_1 = f_1(D_1) = f_1(f_1^{-1})(E_1)) = E_1$ ($\because f_1$ is onto)

(g) $F = f(D) = f(f^{-1})(E)) = f(f^{-1})(C)) = C = E$

(h) $\mu_{1F_1} : F_1 \to L_C$ is defined by

$$\mu_{1F_1}y = \begin{cases} \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x\in D_1}} \Phi\mu_{1D_1}x\right)\right] & , y \in C \\ \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x\in D_1}} \Phi\mu_{1D_1}x\right) & , y \notin C \end{cases}$$

(i) $\mu_{2F} : F \to L_C$ is defined by

$$\mu_{2F}\,y = \begin{cases} \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x\in D_1}} \Phi\mu_{1D_1}x\right)\right] & , y \in C \\ \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = f_1x \\ x\in D_1}} \Phi\mu_{1D_1}x\right) & , y \notin C \end{cases}$$

$$= \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y = fx \\ x\in D}} \Phi\mu_{2D}x\right)\right]$$

(j) $L_F = ([\mu_{1F_1}(F_1)])$ = The complete subalgebra generated by $[\mu_{1F_1}(F_1)]$, where $[\mu_{1F_1}(F_1)]$ = The complete ideal generated by $\mu_{1F_1}(F_1)$

Need to show that, F and ε are full-equal *i.e*. it is sufficient to show that,

(k) $F_1 = E_1$

(l) $F = E$

(m) $L_F = L_E$

(n) $\mu_{1F_1}y = \mu_{1E_1}y$

(o) $\mu_{2F}y = \mu_{2E}y$

(k) and (l) is follows from (f) and (g)

(m) is follows from (e) and (j)

Proof of (*n*) : For $x \in \mathrm{B}$,

$$\mu_{1F_1} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\left(\mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)\right]$$

Case : 1 (*a*) : $\Phi^{-1}\mu_{1E_1} f_1 x = \Phi$, $\mu_{1F_1} f_1 x = \mu_{1E_1} f_1 x$

Case : 1 (*b*) : $\Phi^{-1}\mu_{1E_1} f_1 x \neq \Phi$

$$\mu_{1F_1} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{1E_1} f_l x\right]\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \mu_{1E_1} f_1 x\right)\right)\right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1E_1} f_1 x\right)\right]$$

$$= \mu_{2C} y \vee (\mu_{1C_1} y \wedge \mu_{1E_1} f_1 x)$$

$$= \mu_{2C} y \vee \mu_{1E_1} f_1 x$$

$$= \mu_{1E_1} f_1 x, \text{ whenever } x \in \mathrm{B}$$

For $y \notin \mathrm{C}$,

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\left(\mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)$$

Case : 2 (*a*) : $\Phi^{-1}\mu_{1E_1} y = \Phi$, $\mu_{1F_1} f_1 x = \mu_{1E_1} f_1 x$

Case : 2 (*b*) : $\Phi^{-1}\mu_{1E_1} f_1 x \neq \Phi$

$$\mu_{1F_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{1E_1} f_1 x\right)\right]\right)\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B} x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{1E_1} f_1 x\right]\right)\right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \mu_{1E_1} f_1 x \right) \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \left(\Phi\mu_{2B}\ x \vee \Phi\mu_{1E_1} f_1 x \right) \right)$$

$$= \mu_{1C_1} y \wedge \mu_{1E_1} f_1 x$$

$$= \mu_{1E_1} f_1 x$$

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \ \wedge \ \left(\bigvee_{\substack{y = fx \\ x \in D}} \Phi\mu_{2D} x \right) \right]$$

$$= \mu_{2C} y \left[\mu_{1C_1} y \ \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \Phi \left\{ \mu_{2B}\ x \vee \left[\mu_{1B_1}\ x \ \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} y \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E} fx \right) \right] \right\} \right) \right]$$

$$= \mu_{2C} y \left[\mu_{1C_1} y \ \wedge \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1}\ x \ \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} y \\ \alpha \in L_B}} \Phi\Phi^{-1}\mu_{2E} fx \right) \right] \right\} \right) \right]$$

Case : 3 (*a*) : $\Phi^{-1}\ \mu_{2E}\ fx = \Phi$, $\mu_{2F} fx = \mu_{2E}\ fx$

Case : 3 (*b*) : $\Phi^{-1}\ \mu_{2E} fx \neq \Phi$

$$\mu_{2F} y = \mu_{2C} y \vee \left[\mu_{1C_1} y \ \wedge \ \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B}\ x \vee \left[\Phi\mu_{1B_1} x \wedge \mu_{2E} fx \right] \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \ \wedge \ \left(\bigvee_{\substack{y = fx \\ x \in D}} \left\{ \Phi\mu_{2B}\ x \vee \mu_{2E} fx \right\} \right) \right]$$

$$= \mu_{2C} y \vee \left[\mu_{1C_1} y \ \wedge \ \left(\bigvee_{\substack{y = fx \\ x \in D}} \mu_{2E} fx \right) \right]$$

$$= \mu_{2C} y \vee [\mu_{1C_1} y \wedge \mu_{2E} fx]$$

$$= \mu_{2C}\ y \vee \mu_{2E} fx$$

$$= \mu_{2E} fx$$

Hence $\quad$ F = ε *i.e.*

$\bar{f}\,\bar{f}^{-1}$ (ε) and E are full-equal.

7. Proposition : If $\bar{f}$: B → C be a increasing Fs-function, D ⊆ B, D = B , then

$$\bar{f}^{-1}\bar{f}(D) \supseteq D.$$

Proof : Let D ⊆ B and $\bar{f}$: B → C be an Fs-function,

where

$$B = (B_1, B, \bar{B}\ (\mu_{1B_1}, \mu_{2B}), L_B),$$
$$C = (C_1, C, \bar{C}\ (\mu_{1C_1}, \mu_{2C}), L_C),$$
$$D = (D_1, D, \bar{D}\ (\mu_{1D1}, \mu_{2D}), L_D),$$
$$D = B \text{ and } f = f_1 |^C_B B \to C \text{ be onto.}$$

Define $\bar{f}$(D) as follows $\quad \bar{f}(D) = \varepsilon = (E_1, E, \bar{E}\ (\mu_{1E_1}, \mu_{2E})\ L_E),$

where

1. $E_1 = f_1 (D_1)$
2. $E = f (D)$
3. $\mu_{1E_1} : E_1 \to L_C$ is defined by

$$\mu_{1E_1} y = \begin{cases} \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right] & , \text{if } y \in C \\ \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) & , \text{if } y \notin C \end{cases}$$

4. $\mu_{2E} : E \to L_C$ is defined by

$$\mu_{2E}\, y = \mu_{2C}\, y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi \mu_{2D}\, x \right) \right]$$

5. $L_E = ([\mu_{1E_1} (E_1)])$ = The complete subalgebra generated by $[\mu_{1E_1}(E_1)]$, where $[\mu_{1E_1} (E_1)]$ = The complete ideal generated by $\mu_{1E_1}(E_1)$

Again suppose $\bar{f}^{-1} \bar{f}(D) = \bar{f}^{-1} (\varepsilon) = H = (H_1, H, \bar{H} (\mu_{1H_1}, \mu_{2H}), L_H)$, where

(*a*) $H_1 = f_1^{-1} (E_1) = f_1^{-1} (f_1 (D_1)) \supseteq D_1$ ($\because f_1$ is onto)

(*b*) $H = f^{-1} (E) = f^{-1} (f (D)) = f^{-1}) (f (B)) = B = D$

(*c*) $\mu_{1H_1} : H_1 \to L_H$ is defined by

$$\mu_{1H_1} x = \begin{cases} \mu_{1B_1} x & , \text{whenver } \Phi^{-} \ \mu_{1E_1} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E} \\ \in}} \quad {}^{-} \quad {}_{1E}\, f\, x \right) \right] & , x \in \\ {}_{1B}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E} \\ \in}} \quad {}^{-} \quad {}_{1E}\, f\, x \right) & , x \notin \end{cases}$$

(*d*) $\mu_{2H} : H \to L_H$ is defined by

$$\mu_{2H}\, x = \begin{cases} \mu_{2B} x & , \text{whenver } \Phi^{-1}\, \mu_{2E_1} f_1 x = \Phi \\ \mu_{2B} x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E} y \\ \alpha \in L_B}} \Phi^{-1} \mu_{2E_1} f_1 x \right) \right] \end{cases}$$

(*e*) $L_H = L_B$

Need to show that, $H \supseteq D$ *i.e.* sufficient to show that,

(*f*) $H_1 \supseteq D_1$, $H \subseteq D$

(g) $L_H \geq LD$

(*h*) $\mu_{1H_1} \mid D_1 \geq \mu_{1D_1}$, $\mu_{2H} \leq \mu_{2D} \mid H$

(*f*) follows from (*a*) and (*b*)

(*g*) follows from (*e*)

Sufficient to show (*h*)

For $x \in B$

$$\mu_{1H_1}\, x = \mu_{2B}\, x \vee \left[\mu_{1B_1} \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} y \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_1} f_1 x \right) \right]$$

$$= \mu_{2B}\, x \vee \left[\mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left[\mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right] \right) \right) \right]$$

$$\geq \mu_{1B_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left[\mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right] \right) \right)$$

$$\geq \mu_{1D_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left[\mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right] \right) \right)$$

$$= \mu_{1D_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left(\mu_{2C}\, y \vee \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right) \right) \right)$$

$$\left\{ \because \begin{array}{ll} (i) & \mu_{1C_1} y \geq \Phi \circ \mu_{1B_1} x \geq \Phi \circ \mu_{1D_1} x \geq \Phi \circ \mu_{2D} x \\ (ii) & \mu_{2C} y \leq \Phi \circ \mu_{2B} x \leq \Phi \circ \mu_{2D} x \leq \Phi \circ \mu_{1D_1} x \end{array} \right.$$

$$= \mu_{1D_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right)$$

$$= \mu_{1D_1}\, x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \Phi \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1D_1} x \right) \right)$$

$$\geq \mu_{1D_1}\, x \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1D_1} x \right)$$

$$= \mu_{1D_1}\, x$$

$$\left\{ \because \Phi^{-1}\, \Phi A \supseteq A \Rightarrow \vee \Phi^{-1}\, \Phi A \geq VA,\ A = \bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1D_1} x \right.$$

So that, $\mu_{1H_1} x \geq \mu_{1D_1} x$

For $x \notin B$

$$\mu_{1H_1} x = \mu_{1H_1} x \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \mu_{1E_1} f_1 x \right)$$

$$= \mu_{1B_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left[\mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right] \right) \right)$$

$$\geq \mu_{1D_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left[\mu_{1C_1}\ y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right] \right) \right)$$

$$= \mu_{1D_1} x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1} \left(\mu_{2C}\, y \vee \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi \mu_{1D_1} x \right) \right) \right)$$

$$\left\{ \because \begin{array}{ll} (i) & \mu_{1C_1} y \geq \Phi \circ \mu_{1B_1} x \geq \Phi \circ \mu_{1D_1} x \geq \Phi \circ \mu_{2D} x \\ (ii) & \mu_{2C} y \leq \Phi \circ \mu_{2B} x \leq \Phi \circ \mu_{2D} x \leq \Phi \circ \mu_{1D_1} x \end{array} \right.$$

$$= \mu_{1D_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \Phi\mu_{1D_1}x\right)\right)$$

$$= \mu_{1D_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} f_1 x \\ \alpha \in L_B}} \Phi^{-1}\Phi\left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1D_1}x\right)\right)$$

$$\geq \mu_{1D_1}x \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1D_1}x\right)$$

$$= \mu_{1D_1}x$$

$$\left\{\because \Phi^{-1}\Phi A \supseteq A \Rightarrow \vee \Phi^{-1}\Phi A \geq VA, A = \bigvee_{\substack{y = f_1 x \\ x \in D_1}} \mu_{1D_1}x\right.$$

So that, $\mu_{1H_1}x \geq \mu_{1D_1}x$

$$\mu_{2H}x = \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{1E_1} \\ \alpha \in L_B}} \Phi^{-1}\mu_{2E}fx\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\left(\mu_{2C}\, y \vee \left[\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D}x\right)\right]\right)\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\left(\mu_{1C_1}\, y \wedge \left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D}x\right)\right)\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \Phi\mu_{2D}x\right)\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\Phi\left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \mu_{2D}x\right)\right)\right]$$

$$= \mu_{2B}x \vee \left[\bigvee_{\substack{\Phi\alpha = \mu_{2E}fx \\ \alpha \in L_B}} \Phi^{-1}\Phi\left(\bigvee_{\substack{y = f_1 x \\ x \in D}} \mu_{2D}x\right)\right]$$

$$= \mu_{2B}x \vee \mu_{2D}x$$

$$= \mu_{2D}x$$

$$= \mu_{2D}x \left\{\because \bigvee_{\substack{\Phi\alpha = \mu_{2E}y \\ x \in L_B}} \Phi^{-1}\Phi\left(\bigvee_{\substack{y = fx \\ x \in D}} \mu_{2D}x\right)\right.$$

$$= \mu_{2D}x$$

Hence $\bar{f}^{-1}\,\bar{f}(D) \supseteq D$

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