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PATH FUNCTIONALS OF A CLASS OF LÉVY INSURANCE RISK PROCESSES

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Abstract. We study expected discounted penalty functionals for a class of Lévy processes having a component given by the difference of two independent Poisson compound processes, and a perturbation term given by an α-stable process. We obtain a formula for the Laplace transforms of the expected discounted penalty functionals, as well as explicit representations of such functionals as infinite series of convolutions of given functions. We illustrate our results in some particular examples.

1. Introduction

In this work we consider the Lévy insurance risk process $V_\alpha = \{V_\alpha(t),\, t \geq 0\}$ defined by

$$V_\alpha(t) = u + ct + Z_1(t) - Z_2(t) - \eta W_\alpha(t), \tag{1.1}$$

where u and c are nonnegative constants, $Z_1 = \{Z_1(t),\, t \geq 0\}$ and $Z_2 = \{Z_2(t),\, t \geq 0\}$ are independent Poisson compound processes, and $\{W_\alpha(t),\, t \geq 0\}$ is either an independent standard Brownian motion (if $\alpha = 2$), or else an independent standard α-stable process with index of stability $1 < \alpha < 2$ and skewness, scale and location parameters given respectively by $\beta = 1, \sigma = 1$ and $\mu = 0$. We suppose also some additional conditions on Z_1 and Z_2 which we describe in detail in Section 2.

The expected discounted penalty function for a general insurance risk process $V = \{V(t),\, t \geq 0\}$ is defined by the following path functional of V:

$$\phi_V(u) = \mathbb{E}\left[e^{-\delta\tau_0}\omega(|V(\tau_0)|, V(\tau_0-))1_{\{\tau_0<\infty\}} \,\middle|\, V(0) = u \right],$$

where $\tau_0 = \inf\{t > 0 : V(t) < 0\}$ is the time of ruin of V, $\delta \geq 0$ is a discounted force of interest and $\omega(x, y) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a given non-negative function called penalty function. This functional was introduced by Gerber and Shiu [8] in the context of the Cramèr-Lundberg risk process, generalizing in this way the concept of ruin probability, and involves the joint distribution of the time of ruin, the surplus immediately before ruin and the severity of ruin. The usefulness of the process V_α in insurance risk modeling is well-known, see e.g. Albrecher *et al.* [1] and the references therein for the case $\alpha = 2$. Furrer [6] proposed the model (1.1) with $Z_1 = 0$ and studied the ruin probabilities of this model. Albrecher *et*

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al. [1] studied the expected discounted penalty functions of (1.1) for $\alpha = 2$, while the case of $Z_1 = 0$, $1 < \alpha \leq 2$ was investigated in [11].

In this paper we obtain the Laplace transform ϕ of V_α, as well as a defective renewal equation for ϕ which allows to express ϕ as series of convolutions of given real functions. Using this we calculate several useful functionals of the process, namely: the ruin probability, the Laplace transform of the time of ruin, the first moment of the severity of ruin given by $|V(\tau_0)|$ and of the surplus prior to ruin given by $V(\tau_0-)$, the joint tail distribution and the bivariate Laplace transform of the severity of ruin and the surplus prior to ruin.

The main difficulty in working with the process V_α lies in the lack of a closed expression for the α-stable density, and the fact that we can not use the standart tool of a first step analysis to obtain a renewal equation for ϕ, because of the infinite number of jumps of the α-stable process in each time interval. Using the weak approximation for the α-stable process given in Furrer *et al.* [7], we construct a sequence of two-sided classical risk processes which weakly approximates the process V_α in the Skorokhod space, and prove the convergence of the corresponding expected discounted penalty functions. Afterward we obtain the Laplace transform of ϕ and a defective renewal equation for ϕ. The results we present here extend previous work of Furrer [6], Albrecher *et al.* [1], Tsai and Wilmott [15] and Kolkovska and Martín-González [11]. Weak approximations in risk theory have been used in Sarkar and Sen [13] in the case $\alpha = 2$ and $\lambda_2 = 0$, and in Furrer *et al.* [7] in the case $\lambda_2 = 0$ to estimate ruin probabilities within finite time horizon.

The paper is organized as follows. In section 2 we give a detailed description of the model and introduce some background results and notations. In section 3 we construct a sequence of two-sided Lévy processes that weakly converges to V_α, and prove convergence of the corresponding discounted penalty functionals and of the generalized Lundberg functions of the approximating processes. In section 4 we calculate the Laplace transform of ϕ, under Hypothesis 2.1 and Hypothesis 4.1 below. In section 5 we obtain another formula for the Laplace transform of ϕ, this time in terms of the Dickson-Hipp translation operator. We also present a defective renewal equation for ϕ from which we develop a representation of ϕ as an infinite series of convolutions of known functions. In section 6 we provide some examples that illustrate the use of our results. Several technical calculations are deferred to an appendix.

2. Description of the Model and Background Results

We study the model (1.1) under the following conditions. The processes Z_1 and Z_2 are independent and given by $Z_1(t) = \sum_{i=1}^{N_1(t)} Y_{i1}$, $Z_2(t) = \sum_{i=1}^{N_2(t)} Y_{i2}$, where for $j = 1, 2$, $\{Y_{ij},\ i = 1, 2, \dots\}$ is a sequence of independent and identically distributed random variables with a common distribution function $F_j(x)$ such that $F_j(0) = 0$. In addition, $\{N_j(t),\ t \geq 0\}$, $j = 1, 2$, are independent homogeneous Poisson processes with parameters $\lambda_j \geq 0$; $\eta \geq 0$ is constant. The constants u and c represent, respectively, initial capital and prime per unit time. The upward jumps $\{Y_{i1},\ i = 1, 2, \dots\}$ model the random gains of the insurance company, while the downward jumps $\{Y_{i2},\ i = 1, 2, \dots\}$ represent random claims. We are going to assume that F_j has density f_j, $j = 1, 2$, where f_2 is arbitrary, and the upward-jumps density f_1 is assumed to have Laplace transform given by the rational function

$$\widehat{f_1}(r) = Q(r) \prod_{i=1}^{N} (q_i + r)^{-m_i} \tag{2.1}$$

for some $N, m \in \mathbb{N}$, where $m_1, \ldots, m_N$ are natural numbers such that $m_1 + m_2 + \cdots + m_N = m$ while $0 < q_1 < q_2 < \cdots < q_N$ are real numbers and $Q(r)$ is a polynomial of degree at most $m - 1$. The family of probability distributions whose densities satisfy (2.1) is a wide class which includes the Erlang, Cox and phase-type distributions, as well as mixtures of them. The assumption $1 < \alpha < 2$ implies finiteness of the first moment of $V_\alpha(t)$ for each $t \geq 0$, while the assumption $\beta = 1$ ensures that $\{W_\alpha(t),\, t \geq 0\}$ possesses only positive jumps. Also, we assume that the following conditions hold:

Hypothesis 2.1. a) The upward density f_1 has a Laplace transform of the form (2.1),

b) The net profit condition $E[V_\alpha(1) - u] = c + \lambda_1\mu_1 - \lambda_2\mu_2 > 0$ holds,

c) There exists a positive constant B such that $\omega(x, y) \leq B$ for all $x, y \geq 0$,

d) $c \geq 0$ and $\eta > 0$.

e) We have $\mathbb{P}[(|V_\alpha(\tau_0)|, V_\alpha(\tau_0-)) \in D_\omega] = 0$ where D_ω is the set of discontinuity points of ω.

Condition c) will be relaxed in Corollary 5.8 below. Condition e) is used to apply the Continuous Mapping Theorem (see, for instance, Billingsley [2]), which is required to obtain our main results. Several relevant penalty functions which satisfy the above assumptions arise as particular instances of ω in the following way:

(1) If $\omega(x, y) \equiv a$ for some constant $a > 0$ we obtain that $\phi(u) = a\varphi_\delta(u)$, where $\varphi_\delta(u) = \mathbb{E}(e^{-\delta\tau_0} 1_{\{\tau_0 < \infty\}})$ is the Laplace transform of the time to ruin when $\delta > 0$, and if $\delta = 0$ we obtain $\phi(u) = a\psi(u)$, where $\psi(u)$ is the ruin probability.

(2) Putting $\omega(x, y) = 1_{\{x > a, y > b\}}$ for some constants $a, b > 0$ and $\delta = 0$, we obtain that ϕ is the joint tail of the severity of ruin $|V(\tau_0)|$ and the surplus prior to ruin $V(\tau_0-)$.

(3) When $\delta > 0$ and $\omega(x, y) = e^{-sx - ty}$ for fixed constants $s, t \geq 0$, then ϕ is the trivariate Laplace transform of the time of ruin τ_0, the severity of ruin $|V(\tau_0)|$ and the surplus before ruin $V(\tau_0-)$.

(4) If $\delta = 0$ and $\omega(x, y) = 1_{\{x+y > a\}}$ for some constant $a > 0$, then ϕ is the tail of the distribution of the claim that causes ruin.

(5) If $\omega(x, y) = \max\{K - e^{a-x}, 0\}$ for some constants $K, a > 0$, then ϕ is a special case of a payoff function in option pricing; see Gerber and Shiu [9].

For any nonnegative function f we denote its Laplace transform by $\widehat{f}(r)$, $r \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers. If F is a distribution function with finite first moment μ, and $F(0) = 0$, its integrated tail distribution F_I is defined by $F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\, dy$, where $\overline{F}(x) = 1 - F(x)$, $x \geq 0$, and we denote its density by $f_I(x) = \frac{1}{\mu}\overline{F}(x)$. The convolution of two nonnegative measurable functions h, g, is denoted by $h * g(x)$. We write g^{*n} for the n-th convolution of the function g with itself, where $g^{*0}(x) = 1_{\{0\}}(x)$. Let us denote by $S_\alpha(\sigma, \beta, \mu)$, the α-stable distribution with stability index $0 < \alpha \leq 2$, and scale, skewness and shift parameters $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in (-\infty, \infty)$, respectively; we write $g_{\alpha,\beta,\sigma,\mu}(x)$ for the density function of $S_\alpha(\sigma, \beta, \mu)$. The particular case of the α-stable density $g_{\alpha,\beta,1,1}$ is denoted by $g_{\alpha,\beta}$. It is well known (see e.g. Theorem 2.6.1 in Zolotarev [16]) that the Laplace transform of the α-stable density $g_{\alpha,1,\sigma,\mu}$ is given by

$$\widehat{g}_{\alpha,1,\sigma,\mu}(r) = \begin{cases} e^{\sigma[-\mu r - \mathrm{sgn}(1-\alpha) r^\alpha]} & \alpha \neq 1, \\ e^{\sigma[-\mu r + r \log r]} & \alpha = 1, \end{cases} \tag{2.2}$$

for $r \geq 0$, and its characteristic function, according to Theorem C3, page 12 in Zolotarev [16], is:

$$\mathbb{E}[e^{i\theta X}] = \begin{cases} e^{\sigma(i\mu\theta - |\theta|^\alpha \exp\{-i(\pi/2)\beta K(\alpha)\mathrm{sgn}(\theta)\})} & \text{for } \alpha \neq 1, \\ e^{\sigma[i\mu\theta - |\theta|(\pi/2 + i\beta \log|\theta| \mathrm{sgn}(\theta))]} & \text{for } \alpha = 1, \end{cases} \tag{2.3}$$

where $K(\alpha) = \alpha - 1 + \mathrm{sgn}(1-\alpha)$ and $\mathrm{sgn}(\theta) = 1_{\{\theta>0\}} + \theta 1_{\{\theta=0\}} - 1_{\{\theta<0\}}$. We denote by $W_\alpha = \{W_\alpha(t),\, t \geq 0\}$ the α-stable Lévy motion; W_α is a Lévy process such that $W_\alpha(t) - W_\alpha(s) \stackrel{d}{=} S_\alpha\left[(t-s)^{1/\alpha}, \beta, \mu\right]$ for all $0 \leq s < t < \infty$. Recall that when $1 < \alpha < 2$, only the moments of W_α with order less then α are finite, the first moment of $W_\alpha(1)$ is μ, and when $\beta = 1$, only positive jumps of W_α are possible. We refer the reader to Zolotarev [16] and Sato [14] for other properties of stable processes. Finally, we denote by D the Skorokhod space of all real-valued, càdlàg functions defined on $[0, \infty)$, endowed with the Skorokhod topology (see e.g. Billingsley [2] for definitions and properties of processes with càdlàg paths). We write $\Rightarrow$ for the weak convergence in D.

3. Weak Approximations of V_α and Convergence of Lundberg Equations

In this section we construct a sequence $\{V_n, n \geq 0\}$ of two-sided jumps classical risk processes such that $V_n \Rightarrow V_\alpha$, and prove that the Gerber-Shiu functional of V_n converges to the corresponding functional of V_α. First we have the following technical result.

Theorem 3.1. *Let* $Z_1(t) = \sum_{i=1}^{N_1(t)} Y_{i1}$ *and* $Z_2(t) = \sum_{i=1}^{N_2(t)} Y_{i2}$. *For each* $k \in \mathbb{N}$ *and any fixed constant* $c > 0$, *let* $A(k) = 1 - (k+1)^{-1}$, $\lambda_1(k) = \lambda_1/(1 - A(k))$, $b(k) = \lambda_1/[c(1 - A(k))]$ *and*

$$p_k^*(x) = \begin{cases} A(k)b(k)e^{-b(k)x} + (1 - A(k))f_1(x) & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Consider the sequence of processes

$$V_k^{[c]}(t) = u + \sum_{i=1}^{N_{1,k}(t)} Y_{ik}^* - \sum_{i=1}^{N_2(t)} Y_{i2} := u + Z_{1,k}(t) - Z_2(t), \quad k = 1, 2, \ldots, \tag{3.1}$$

where $\{Z_{1,k}(t),\, t \geq 0\}$ *is a compound Poisson process with intensity* $\lambda_1(k)$, *which is independent of* Z_2, *and* $\{Y_{ik}^*,\ i = 1, 2, \ldots\}$ *is a sequence of independent and* *identically distributed random variables with common density function* p_k^*. *Then, as* $k \to \infty$, $V_k^{[c]} \Rightarrow u + ct + Z_1(t) - Z_2(t)$.

Proof. Let $t \geq 0$ be fixed, and let $\xi_{1,k}$, ξ_1 and ξ_2 be the characteristic functions of the random variables $V_k^{[c]}(t), X_{11}$, and X_{12}, respectively. Using that

$$\xi_{1,k}(s) = \exp\left\{uis + \lambda_1(k)t\left(\frac{A(k)b(k)}{b(k)-is} + (1-A(k))\xi_1(s) - 1\right) + \lambda_2 t\left(\xi_2(-s) - 1\right)\right\},$$

and $\lim_{k\to\infty} \lambda_1(k)\left(\frac{A(k)b(k)}{b(k)-is} - 1\right) = cis - \lambda_1$ we obtain $\lim\limits_{k\to\infty} \xi_{1,k}(s)=\exp\{uis + ctis + \lambda_1 t\left(\xi_1(s)-1\right) + \lambda_2 t\left(\xi_2(-s)-1\right)\}$, which is the characteristic function of $u + ct + Z_1(t) - Z_2(t)$. From here the result follows from Theorem 15.17 in Kallenberg [10]. □

Now we construct a sequence of processes $\{V_{n,k}, n, k \geq 0\}$ for which the prime c is 0, and a sequence of processes $\{V_n, n \geq 0\}$ with prime $c \geq 0$, such that $V_{n,k} \Rightarrow V_n$ for each fixed n and, moreover, $V_n \Rightarrow V_\alpha$.

Theorem 3.2. *Let* $c_n = c + n^{1-1/\alpha}\eta^\alpha$, $n \in \mathbb{N}$, *and let the sequence of risk processes* $V_{n,k} = \{V_{n,k}(t),\ t \geq 0\}$ *be defined by* $V_{n,k}(t) = V_k^{[c_n]}(t) - \frac{1}{n^{1/\alpha}}\sum_{i=1}^{M(nt)} W_i$, *where* $\{V_k^{[c_n]}, k = 1, 2, \dots\}$ *is defined as in (3.1) with* $b(k) = \lambda_1/[c_n(1-A(k))] := b_n(k)$ *and* $\sum_{i=1}^{M(nt)} W_i$ *is a compound Poisson process independent of* $V_k^{[c_n]}$. *We assume that the Poisson process* M *has intensity* η^α *and that* $W_1, W_2, \dots$ *are independent and identically distributed random variables with common distribution* $S_\alpha(1,1,1)$. *We also define the sequence of processes* $V_n = \{V_n(t), t \geq 0\}$ *by*

$$V_n(t) = u + c_n t + Z_1(t) - Z_2(t) - \frac{1}{n^{1/\alpha}}\sum_{i=1}^{M(nt)} W_i. \tag{3.2}$$

Then $V_{n,k} \Rightarrow V_n$ *for each* $n \in \mathbb{N}$, *and* $V_n \Rightarrow V_\alpha$.

Proof. The first convergence follows from Theorem 3.1 and the independence of $\{W_i\}$, $\{Z_{1,k}\}$ and Z_2. For the proof of the second convergence, we note that since W_i, $i = 1, 2, \dots$, have common distribution $S_\alpha(1,1,1)$, from (2.3) it follows that for each $n \geq 1$, $\frac{1}{n^{1/\alpha}}\sum_{k=1}^{n}(W_k - 1) \stackrel{d}{=} W_\alpha(1,1,0)$. Hence equality (3) in Furrer et al. [7] holds with $\phi(n) = n^{1/\alpha}$, and since in our case $c^{(n)} = c + \eta^\alpha n^{1-1/\alpha}$ and $\lambda = \eta^\alpha$, the hypothesis in Theorem 1 in Furrer et al. [7] are fulfilled, and therefore it follows that $u + c_n t - n^{-1/\alpha}\sum_{i=1}^{M(nt)} W_i \Rightarrow u + ct - \eta W(t)$. Using now the independence of W, Z_1 and Z_2, we obtain the result. □

For any $1 < \alpha \leq 2$, let us denote by $\phi_{n,k}, \phi_n$ and ϕ the Gerber-Shiu functionals of the processes $V_{n,k}, V_n$ and V_α, respectively, with corresponding Laplace transforms $\widehat{\phi}_{n,k}, \widehat{\phi}_n$ and $\widehat{\phi}$. The following result can be proved similarly as in Furrer *et al.* [7].

Theorem 3.3. *Under Hypothesis 2.1,* $\lim_{k\to\infty}\phi_{n,k}(u) = \phi_n(u)$ *for all* $u \geq 0$ *and each* $n \in \mathbb{N}$. *Moreover,* $\lim_{n\to\infty}\phi_n(u) = \phi(u)$.

Using partial fractions decomposition it can be proved that, when f_1 satisfies (2.1), it admits the representation $f_1(x) = \sum_{i=1}^N \sum_{j=1}^{m_i} \beta_{ij} \frac{x^{j-1} q_i^j e^{-q_i x}}{(j-1)!}$, $x > 0$, where $\beta_{ij} = \frac{1}{q_i^j (m_i - j)!} \frac{d^{m_i - j}}{dr^{m_i - j}} \left\{ \prod_{k=1, k\neq i}^N \frac{Q(r)}{q_k + r} \right\}\Big|_{r=-q_i}$, hence

$$\widehat{f}_1(r) = \sum_{i=1}^N \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i + r)^j}. \tag{3.3}$$

We also need the following identity, which is known from interpolation theory:

Lemma 3.4. *For each $m \geq 1$ and for any different non-zero complex numbers $x_1, \dots, x_{m+1}$,*

$$\sum_{j=1}^{m+1} \left(x_j \prod_{l=1, l\neq j}^{m+1} (x_l - x_j) \right)^{-1} = \left(\prod_j x_j \right)^{-1}.$$

For $r \neq q_i$ we define the generalized Lundberg functions associated to the processes $V_{n,k}, V_n$ and V_α, respectively by

$$\begin{aligned} L_{\alpha,n,k}(r) &= \lambda_2 \widehat{f}_2(r) + \lambda_1(k) A(k) \frac{b_n(k)}{b_n(k) - r} + \lambda_1(k) (1 - A(k)) \sum_{i=1}^N \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} \\ &\quad + n\eta^\alpha \exp\left\{ -\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n} \right\} - (\lambda_1(k) + \lambda_2 + n\eta^\alpha + \delta), \quad r \neq b_n(k), \\ L_{\alpha,n}(r) &= \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^N \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + \left(c + \eta^\alpha n^{1-1/\alpha} \right) r \\ &\quad + n\eta^\alpha \exp\left\{ -\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n} \right\} - (n\eta^\alpha + \lambda_1 + \lambda_2 + \delta), \\ L_\alpha(r) &= \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr + (\eta r)^\alpha - (\lambda_1 + \lambda_2 + \delta). \end{aligned}$$

We denote $Q_1(r) = \prod_{k=1}^N (q_i - r)^{m_i}$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ and $\mathbb{C}_{++} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. For $\rho \in \mathbb{C}_+$ and $d > 0$ we put

$$B_d(\rho) = \{r \in \mathbb{C}_+ : |r - \rho| \leq d\}. \tag{3.4}$$

We have the following result.

Lemma 3.5. *The function $P(r) = ar + br^\alpha - c$, where $a \geq 0$, $b, c > 0$ and $\alpha \in (1, 2)$, has exactly one real and positive root.*

Proof. Let us suppose that there exists a root s of $P(r)$ such that $\mathrm{Re}(s) \geq 0$, $\mathrm{Im}(s) \neq 0$ and $\mathrm{Arg}(s) = \theta$. Then by De Moivre's formula we obtain $a|s|\sin(\theta) + b|s|^\alpha \sin(\alpha\theta) = 0$. We claim that $\theta = 0$. By the assumption that $\mathrm{Re}(s) \geq 0$, we have $\theta \in [-\pi/2, \pi/2]$, hence if $0 < \theta \leq \pi/2$ we obtain $\alpha\theta \in (0, \pi)$, which implies $\sin(\alpha\theta) > 0$, hence $a|s|\sin(\alpha\theta) > 0$, and similarly for the case $\theta \in (-\pi/2, 0)$. Thus, all possible roots of P are real. Since for $r \geq 0$ we have $dP(r)/dr > 0$ and $d^2P(r)/dr^2 = b\alpha(\alpha - 1)r^{\alpha-2} > 0$ for all $r > 0$, $P(r)$ is strictly increasing in the nonnegative real line, and noting that $P(0) = -c < 0$, we obtain the result. ☐

Proposition 3.6. *a)* $\lim_{k\to\infty} L_{\alpha,n,k}(r) = L_{\alpha,n}(r)$ *for all sufficiently large* $n \in \mathbb{N}$ *and* $r \neq b_n(k)$*, uniformly in* r *in sets of the form (3.4).*

b) Moreover, $\lim_{n\to\infty} L_{\alpha,n}(r) = L_\alpha(r)$*, uniformly in sets of the form (3.4).*

c) For $\delta \geq 0$*, the functions* L_α*,* $L_{\alpha,n}$ *and* $L_{\alpha,n,k}$ *have exactly one root of multiplicity one in the interval* $[0, q_1)$*, which is equal to zero if and only if* $\delta = 0$*. We denote these roots by* $\rho_{1,\delta}, \rho_{1,\delta}(n)$ *and* $\rho_{1,\delta}(n,k)$*, respectively.*

d) For $\delta \geq 0$ *and* $c+\eta > 0$*, the function* L_α *has exactly* $m+1$ *roots* $\rho_{1,\delta}, \dots, \rho_{m+1,\delta}$ *in* $\mathbb{C}_+$*. For* $\delta > 0$ *these roots are in* $\mathbb{C}_{++}$*, and if* $\delta = 0$*,* $\rho_{1,\delta} = 0$ *is the only root on the imaginary axis. Moreover, for all sufficiently large* n *and* k*, and all* $\delta \geq 0$*, the functions* $L_{\alpha,n}$ *and* $L_{\alpha,n,k}$ *also have* $m+1$ *roots in* $\mathbb{C}_+$*, which we denote respectively by* $\rho_{1,\delta}(n), \dots, \rho_{m+1,\delta}(n)$ *and* $\rho_{1,\delta}(n,k), \dots, \rho_{m+1,\delta}(n,k)$*. When* $\delta > 0$ *all these roots are in* $\mathbb{C}_{++}$*, and when* $\delta = 0$*,* $\rho_{1,0}(n,k) = \rho_{1,0}(n) = 0$ *are the only roots of* $L_{n,k}$ *and* L_n*, respectively, on the imaginary axis.*

e) Let $c + \eta > 0$*. For any* $j \in \{1, 2, \dots, m+1\}$ *there exists* $l \in \{1, 2, \dots, m+1\}$ *such that* $\lim_{k\to\infty} \rho_{j,\delta}(n,k) = \rho_{l,\delta}(n)$ *and* $\lim_{n\to\infty} \rho_{j,\delta}(n) = \rho_{l,\delta}$*.*

f) We have $\lim_{\delta\to 0} \rho_{1,\delta} = \rho_{1,0} = 0$ *.*

Proof. To prove a) it suffices to consider the closed complex semicircle $B_d := B_d(0)$. For any $r \neq b_n(k)$ and $n \in \mathbb{N}$, $\lim_{k\to\infty} L_{\alpha,n,k}(r) = L_{\alpha,n}(r)$ due to (3.3). We will show that this convergence is uniform in B_d. For $r \in B_d$ and $k > c_n d - \lambda_1$ we have $\lambda_1(k+1) - c_n r > 0$, and

$$\left| c_n r - \lambda_1 - \lambda_1(k) A(k) \frac{b_n(k)}{b_n(k) - r} + \lambda_1(k) \right| = \left| \frac{c_n r(\lambda_1 - c_n r)}{\lambda_1(k+1) - c_n r} \right| \leq \frac{c_n d\lambda_1 + c_n^2 d^2}{\lambda_1(k+1) - c_n d},$$

and the result follows.

To obtain b) first we prove that $\lim_{n\to\infty} L_{\alpha,n}(r) = L_\alpha(r)$. Expanding the exponential function we get $L_{\alpha,n}(r) = \lambda_2 \widehat{f}_2(r) + \lambda_1 \sum_{i=1}^N \sum_{j=1}^{m_i} \frac{\beta_{ij} q_i^j}{(q_i - r)^j} + cr + \eta^\alpha r^\alpha + a_n(r) - (\lambda_1 + \lambda_2 + \delta)$, where $a_n(r) = n\eta^\alpha \sum_{k=2}^\infty \left(-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n}\right)^k / k!$. For sufficiently large n and $r \in B_d$, we have

$$\begin{aligned} |a_n(r)| = n\eta^\alpha \left| \sum_{k=2}^\infty \left(-\frac{r}{n^{1/\alpha}} + \frac{r^\alpha}{n} \right)^k / k! \right| &< n\eta^\alpha \sum_{k=2}^\infty \frac{|r^\alpha + r|^k}{n^{k/\alpha}} \leq n\eta^\alpha \sum_{k=2}^\infty C(d)^k / n^{k/\alpha} \\ &= \eta^\alpha \left(C(d)^2 / n^{(2-\alpha)/\alpha} \right) \left(1 - C(d)/n^{1/\alpha} \right)^{-1}, \end{aligned}$$

where $C(d) = 2\max\{d^\alpha, d\}$ is a constant depending on d. Since $\alpha \in (1,2)$, the right-hand side in the above inequality converges to 0 as $n \to \infty$ uniformly in B_d. This completes the proof of b).

Now we prove part c). We will prove that $L_{\alpha,n}$ has one real nonnegative root in $[0, q_1]$; the cases for the functions L and $L_{\alpha,n,k}$ can be handled in a similar way. From Hypothesis 2.1.b) we get $\frac{dL_{\alpha,n}}{dr}(0) = c + n^{1-1/\alpha}\eta^\alpha + \lambda_1\mu_1 - \lambda_2\mu_2 > 0$. Moreover, $\frac{d^2 L_{\alpha,n}}{dr^2}(r) > 0$ for $r < q_1$, hence $L_{\alpha,n}(r)$ is increasing in $[0, q_1)$ with $L_{\alpha,n}(0) = -\delta$, and the result follows.

To prove d) we define $L^*(r) = Q_1(r) L_\alpha(r)$ and $L^{**}(r) = Q_1(r)[cr + \eta^\alpha r^\alpha - (\lambda_1 + \lambda_2 + \delta)]$. Now we take $\delta > 0$ and consider, for fixed $s > 0$, the contour C_s as the imaginary axis together with a semicircle of radius s, moving clockwise from $-is$ to

is. We note that $|L^*(r)-L^{**}(r)| = \left|Q_1(r)\left(\lambda_2\widehat{f}_2(r)+\lambda_1\sum_{i=1}^n\sum_{j=1}^{m_i}\beta_{ij}q_i^j(q_i-r)^{-j}\right)\right|$. Since $\lim_{|r|\to\infty}|L^{**}(r)| = \infty$ for any $c\geq 0$, for r in the semicircle and s sufficiently large, we have

$$\left|\lambda_2\widehat{f}_2(r)+\lambda_1\sum_{i=1}^n\sum_{j=1}^{m_i}\beta_{ij}q_i^j(q_i-r)^{-j}\right| \leq \lambda_1+\lambda_2 < |cr+\eta^\alpha r^\alpha-(\lambda_1+\lambda_2+\delta)|, \quad (3.5)$$

for $c\geq 0$. For r in the imaginary axis it follows that

$$\begin{aligned}&|cr+\eta^\alpha r^\alpha-(\lambda_1+\lambda_2+\delta)|\\ &= \sqrt{(\eta^\alpha|\mathrm{Re}\,(r^\alpha)|+\lambda_1+\lambda_2+\delta)^2+(c\mathrm{Im}(r)+\eta^\alpha\mathrm{Im}\,(r^\alpha))^2} \ > \ \lambda_1+\lambda_2. \quad (3.6)\end{aligned}$$

From (3.5) and the last inequality we obtain for sufficiently large s that $|L^*(r)-L^{**}(r)| < |L^{**}(r)|$. On the other hand, for $r\in\mathbb{R}\backslash\{0\}$ we obtain similarly $|L_\alpha(ir)|>0$, which implies that there are no roots on the imaginary axis and we conclude that, when $\delta=0$, the only root on the imaginary axis is $\rho_{1,\delta}=0$. Moreover, from c), such a root has multiplicity one. Now applying Rouche's theorem we conclude that $L^*(r)$ has the same number of roots as $L^{**}(r)$ in C_s. Letting s tend to infinity we obtain the result for $\mathbb{C}_{++}$. Taking $P(r)=cr+\eta^\alpha r^\alpha-\lambda_1-\lambda_2-\delta$, from Lemma 3.5 we conclude that $L^{**}(r)$ has $m+1$ roots in $\mathbb{C}_{++}$ for $c\geq 0$.

Now we prove the result about the number of roots of $L_{\alpha,n}(r)$. We take $L^{**}(r)$ as before and define $L_n^*(r) = Q_1(r)[\lambda_2\widehat{f}_2(r)+\lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\beta_{ij}q_i^j(q_i-r)^{-j}+cr+\eta^\alpha n^{1-1/\alpha}r+n\eta^\alpha\exp\{-\frac{r}{n^{1/\alpha}}+\frac{r^\alpha}{n}\}-n\eta^\alpha-(\lambda_1+\lambda_2+\delta)]$. Then, for r in a semicircle with sufficiently large radius s, $0<\varepsilon<\delta$ and n sufficiently large:

$$\begin{aligned}|L_n^*(r)-L^{**}(r)| &\leq |Q_1(r)|\left|\lambda_2\widehat{f}_2(r)+\lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\beta_{ij}q_i^j}{(q_i-r)^j}\right|\\ &\quad+|Q_1(r)|\left|\eta^\alpha n^{1-1/\alpha}r+n\eta^\alpha e^{-\frac{r}{n^{1/\alpha}}+\frac{r^\alpha}{n}}-n\eta^\alpha-\eta^\alpha r^\alpha\right|\\ &\leq |Q_1(r)|\,(\lambda_1+\lambda_2+\varepsilon)\\ &< |Q_1(r)|\,|cr+\eta^\alpha r^\alpha-(\lambda_1+\lambda_2+\delta)| = L^{**}(r),\end{aligned}$$

where the second inequality follows for sufficiently large n using the uniform convergence of $\eta^\alpha n^{1-1/\alpha}r+\eta^\alpha e^{-\frac{r}{n^{1/\alpha}}+\frac{r^\alpha}{n}}-n\eta^\alpha$ to $\eta^\alpha r^\alpha$ in B_d, which was proved in b). Now for r in the imaginary axis we use (3.6) to obtain

$$\begin{aligned}&|L^{**}(r)| > |Q_1(r)|\,(\lambda_1+\lambda_2+\delta) > |Q_1(r)|\,(\lambda_1+\lambda_2+\varepsilon)\\ &\quad> |Q_1(r)|\left|\lambda_2\widehat{f}_2(r)+\lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\beta_{ij}q_i^j}{(q_i-r)^j}+cr+\eta^\alpha n^{1-1/\alpha}r\right.\\ &\qquad\left.+\,\eta^\alpha e^{-\frac{r}{n^{1/\alpha}}+\frac{r^\alpha}{n}}-n\eta^\alpha-(\lambda_1+\lambda_2+\delta)\right| = |L_n^*(r)-L^{**}(r)|,\end{aligned}$$

and the result follows by Rouche's theorem. The proof for $L_{\alpha,n,k}$ is similar. Finally we prove parts e) and f). If $\lim_{k\to\infty}\rho_{j,\delta}(n,k)=r_j$ then from part a), $\lim_{k\to\infty}L_{\alpha,n,k}(\rho_{j,\delta}(n,k))=L_{\alpha,n}(r_j)=0$, hence r_j is a root of $L_{\alpha,n}(r)$. The second limit is obtained in the same way. The limit $\lim_{\delta\to 0}\rho_{1,\delta}$ exists due to the weak convergence of the stochastic processes having Laplace exponent $L_\alpha(r)$ with $\delta>0$, to the stochastic process having Laplace exponent $L_\alpha(r)$ with $\delta=0$. Let us suppose that $\lim_{\delta\to 0}\rho_{1,\delta} = s_0 \in [0,q_1)$. Since $L_{\alpha,\delta}(r) \to L_{\alpha,0}(r)$ uniformly when $\delta \to 0$ on $r \in [0,q_1)$, we obtain $s_0 = 0$ because, due to c), $\rho_{1,0} = 0$ is the only root of L_α in $[0,q_1)$. □

4. The Laplace Transform of ϕ

The following three lemmas will be used in the sequel; their proofs are given in appendix A.

Lemma 4.1. *The integral* $I_n = n^{1+1/\alpha}\int_{-\infty}^0 (1-e^{-rx}-rx)\, g_{\alpha,1}(n^{1/\alpha}x)\, dx$ *is finite for each* $n \in \mathbb{N}$ *and* $r \in \mathbb{C}_+$, *and satisfies* $\lim_{n\to\infty}|I_n| = 0$.

Lemma 4.2. *Under Hypothesis 2.1,* $K_0(n,k,r) := \lambda_1(k)\int_0^\infty\int_0^\infty e^{-ru}\phi_{n,k}(u+x)p_k^*(x)\, dx\, du$ *is finite for all* $r > 0$, *and admits the equivalent expression*

$$K_0(n,k,r) = \frac{P_{1,k}(r)}{Q_{1,k}(r)}\widehat{\phi}_{n,k}(r) - \frac{P_{2,k}(r)}{Q_{1,k}(r)},$$

where

$$\begin{aligned}
P_{1,k}(r) &= Q_{1,k}(r)\left[\frac{\lambda_1(k)A(k)b_n(k)}{b_n(k)-r} + \lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\beta_{ij}q_i^j}{(q_i-r)^j}\right],\\
P_{2,k}(r) &= Q_{1,k}(r)\left[\frac{\lambda_1(k)(1-A(k))b_n(k)}{b_n(k)-r}\widehat{\phi}_{n,k}(b_n(k))\right.\\
&\qquad\left. + \lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\beta_{ij}q_i^j\sum_{l=0}^{j-1}\frac{(q_i-r)^l\gamma_{l,i}(n,k)}{l!(q_i-r)^j}\right],\\
\gamma_{l,i}(n,k) &= \int_0^\infty \phi_{n,k}(z)e^{-q_iz}z^l dz, \qquad Q_{1,k}(r) = (b_n(k)-r)\prod_{j=1}^N (q_j-r)^{m_j}.
\end{aligned}$$

Let us define

$$\begin{aligned}
M_{\alpha,n}(r) &= n^{1+1/\alpha}\eta^\alpha\int_0^\infty\int_u^\infty e^{-ru}\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu,\\
M_\alpha(r) &= \frac{\eta^\alpha\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^\infty\int_u^\infty e^{-ru}\omega(x-u,u)x^{-1-\alpha}dxdu, \quad 1<\alpha<2,\\
N(r) &= \lambda_2\int_0^\infty\int_u^\infty e^{-ru}\omega(x-u,u)f_2(x)dxdu, \qquad (4.1)
\end{aligned}$$

and note that $\frac{1}{\lambda_2}N(r) = \widehat{\xi}_\omega(r)$, where $\xi_\omega(u) = \int\limits_u^\infty \omega(x-u,u)f_2(x)dx$.

Lemma 4.3. *For* $r_1, r_2 \in \mathbb{C}_+$ *with* $r_1 \neq r_2$ *there holds* $\lim\limits_{n\to\infty}(M_{\alpha,n}(r_1) - M_{\alpha,n}(r_2)) = M_\alpha(r_1) - M_\alpha(r_2)$.

In order to obtain simpler expressions for the Laplace transforms $\widehat{\phi}_{n,k}$ and $\widehat{\phi}$, we impose the following condition.

Hypothesis 4.1. All roots of $L_\alpha(r)$ in $\mathbb{C}_+$ have multiplicity 1.

Due to Proposition 3.6 e), Hypothesis 4.1 implies that for all sufficiently large n and k the roots of $L_{\alpha,n,k}$ and $L_{\alpha,n}$ have multiplicity 1. We need the following functions:

$$\begin{aligned}
\widehat{g}^{-}_{\alpha,1}(r/n^{1/\alpha}) &= \int_{-\infty}^{0} e^{-rx/n^{1/\alpha}} g_{\alpha,1}(x)\,dx,\ \widehat{g}^{+}_{\alpha,1}(r/n^{1/\alpha}) = \int_{0}^{\infty} e^{-rx/n^{1/\alpha}} g_{\alpha,1}(x)\,dx,\\
T(\phi_{n,k}) &= n^{1+1/\alpha}\eta^{\alpha}\int_{-\infty}^{0}\int_{0}^{-x}\phi_{n,k}(z)g_{\alpha,1}(n^{1/\alpha}x)\,dz\,dx,\\
A_n(r) &= n^{1+1/\alpha}\eta^{\alpha}\int_{-\infty}^{0}\int_{0}^{-x}\left(e^{-r(x+z)}-1\right)\phi_{n,k}(z)g_{\alpha,1}(n^{1/\alpha}x)\,dz\,dx\\
K(n,r) &= n^{1+1/\alpha}\eta^{\alpha}\int_{-\infty}^{0}\int_{-x}^{\infty} e^{-r(x+z)}\phi_{n,k}(z)g_{\alpha,1}(n^{1/\alpha}x)\,dz\,dx. \qquad (4.2)
\end{aligned}$$

Due to Hypothesis 2.1 c), $\phi_{n,k}(r)\le B$ hence $A_n(r)\le B\frac{n^{1+1/\alpha}\eta^{\alpha}}{r}\int_{-\infty}^{0}(1-e^{-rx}-rx)g_{\alpha,1}(n^{1/\alpha}x)\,dx$, and from Lemma 3.5 it follows that $\lim_{n\to\infty}A_n(r)=0$. In the next theorem we obtain an expression for $\widehat{\phi}_{n,k}$.

Theorem 4.4. *Assume that Hypothesis 2.1 holds and that $(c,\eta)\neq(0,0)$. Then the Laplace transform of the Gerber-Shiu penalty function $\phi_{n,k}$ of $V_{n,k}$ admits the representation*

$$L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r) = \tfrac{P_{1,k}(r)}{Q_{1,k}(r)} - N(r) - M_{\alpha,n}(r) - T(\phi_{n,k}) - A_n(r). \qquad (4.3)$$

Moreover, under Hypothesis 2.1 and 4.1, we have for all $\delta\ge 0$,

$$\begin{aligned}
L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r) &= \sum_{l=1}^{m+1}\frac{Q_1(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)}\frac{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-r)}{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-\rho_{l,\delta}(n,k))}\\
&\times\left[(b_n(k)-\rho_{l,\delta}(n,k))(N(\rho_{l,\delta}(n,k))+A_n(\rho_{l,\delta}(n,k))+M_{\alpha,n}(\rho_{l,\delta}(n,k)))\right.\\
&\left.-(b_n(k)-r)(N(r)+A_n(r)+M_{\alpha,n}(r))\right]. \qquad (4.4)
\end{aligned}$$

Proof. We consider a small time interval $(0,h)$ and condition on the first jump time and first claim size of $V_{n,k}$. This gives the equation

$$\begin{aligned}
\phi_{n,k}(u) &= e^{-(\lambda_n+\delta)h}\phi_{n,k}(u) + \lambda_1(k)\int_0^h\int_0^\infty e^{-(\lambda_1(k)+\delta)t}\phi_{n,k}(u+x)p_k^*(x)dxdt\\
&+\lambda_2\int_0^h\int_0^u e^{-(\lambda_2+\delta)t}\phi_{n,k}(u-x)f_2(x)dxdt + \lambda_2\int_0^h\int_u^\infty e^{-(\lambda_2+\delta)t}\omega(x-u,u)f_2(x)dxdt\\
&+n^{1+1/\alpha}\eta^{\alpha}\int_0^h\int_0^u e^{-(n\eta^{\alpha}+\delta)t}\phi_{n,k}(u-x)g_{\alpha,1}(n^{1/\alpha}x)dxdt\\
&+n^{1+1/\alpha}\eta^{\alpha}\int_0^h\int_{-\infty}^0 e^{-(n\eta^{\alpha}+\delta)t}\phi_{n,k}(u-x)g_{\alpha,1}(n^{1/\alpha}x)dxdt\\
&+n^{1+1/\alpha}\eta^{\alpha}\int_0^h\int_u^\infty e^{-(n\eta^{\alpha}+\delta)t}\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdt,
\end{aligned}$$

where $\lambda_n=\lambda_1(k)+\lambda_2+n\eta^{\alpha}$. Using the Taylor series of the exponential function in $e^{-(\lambda_n+\delta)h}\phi_{n,k}(u)$, dividing both sizes of the above equation by h, letting $h\to 0$

and taking Laplace transforms, we obtain:

$$
\begin{aligned}
&(\lambda_n+\delta)\widehat{\phi}_{n,k}(r) = \lambda_1(k)\int_0^\infty\int_0^\infty e^{-ru}\phi_{n,k}(u+x)p_k^*(x)dxdu\\
&+\lambda_2\left[\int_0^\infty\int_0^u e^{-ru}\phi_{n,k}(u-x)f_2(x)dxdu+\int_0^\infty\int_u^\infty e^{-ru}\omega(x-u,u)f_2(x)dxdu\right]+n^{1+\frac{1}{\alpha}}\eta^\alpha\\
&\times\left[\int_0^\infty\int_{-\infty}^u e^{-ru}\phi_{n,k}(u-x)g_{\alpha,1}(n^{1/\alpha}x)dxdu+\int_0^\infty\int_u^\infty e^{-ru}\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu\right]\\
&= K_0(n,k,r)+\lambda_2\widehat{\phi}_{n,k}(r)\widehat{f}_2(r)+N(r)+n\eta^\alpha\widehat{\phi}_{n,k}(r)\widehat{g}^+_{\alpha,1}(r/n^{1/\alpha})\\
&+K(n,r)+M_{\alpha,n}(r). \qquad (4.5)
\end{aligned}
$$

Next, we obtain a more explicit expression for the function $K(n,r)$ defined in (4.2). Changing the order of integration and setting $z=u-x$ in (4.2) yields

$$
\begin{aligned}
K(n,r) &= n^{1+1/\alpha}\eta^\alpha\int_{-\infty}^0\int_{-x}^\infty e^{-r(x+z)}\phi_{n,k}(z)g_{\alpha,1}(n^{1/\alpha}x)dzdx\\
&\qquad \pm n^{1+1/\alpha}\eta^\alpha\int_{-\infty}^0\int_0^{-x} e^{-r(x+z)}\phi_{n,k}(z)g_{\alpha,1}(n^{1/\alpha}x)dzdx\\
&= n\eta^\alpha\widehat{\phi}_{n,k}(r)\widehat{g}^-_{\alpha,1}(r/n^{1/\alpha})-n^{1+1/\alpha}\eta^\alpha\int_{-\infty}^0\int_0^{-x} e^{-r(x+z)}\phi_{n,k}(z)g_{\alpha,1}(n^{1/\alpha}x)dzdx\\
&= n\eta^\alpha\widehat{\phi}_{n,k}(r)\widehat{g}^-_{\alpha,1}(r/n^{1/\alpha})-A_n(r)-T(\phi_{n,k}).
\end{aligned}
$$

From the last equality we get

$$
n\eta^\alpha\widehat{\phi}_{n,k}(r)\widehat{g}^+_{\alpha,1}(r/n^{1/\alpha})+K(n,r)=n\eta^\alpha\widehat{\phi}_{n,k}(r)\widehat{g}_{\alpha,1}(r/n^{1/\alpha})+T(\phi_{n,k})+A_n(r),
$$

which, together with (4.5) and Lemma 4.2, yields

$$
\begin{aligned}
(\lambda_n+\delta)\widehat{\phi}_{n,k}(r) &= \frac{P_{1,k}(r)}{Q_{1,k}(r)}\widehat{\phi}_{n,k}(r)-\frac{P_{2,k}(r)}{Q_{1,k}(r)}+\lambda_2\widehat{\phi}_{n,k}(r)\widehat{f}_2(r)\\
&+N(r)+n\eta^\alpha\widehat{\phi}_{n,k}(r)\widehat{g}_{\alpha,1}(r/n^{1/\alpha})-M_{\alpha,n}(r)-T(\widehat{\phi}_{n,k})-A_n(r). \qquad (4.6)
\end{aligned}
$$

Let us note that, since $L_{\alpha,n,k}(r)=\lambda_2\widehat{f}_2(r)+\frac{P_{1,k}(r)}{Q_{1,k}(r)}+n\eta^\alpha\widehat{g}(r/n^{1/\alpha})-(\lambda_n+\delta)$, (4.3) follows from the above equality. Because of Hypothesis 4.1, all roots $\rho_{j,\delta}(n,k), j=1,\ldots,m+1$, have multiplicity 1. Substituting $r=\rho_{j,\delta}(n,k)$ in (4.6) and using Lagrange interpolation renders

$$
\begin{aligned}
P_{2,k}(r) = \sum_{l=1}^{m+1}\frac{Q_{1,k}(\rho_{l,\delta}(n,k))\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-r)}{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-\rho_{l,\delta}(n,k))}&\left[N(\rho_{l,\delta}(n,k))+M_{\alpha,n}(\rho_{l,\delta}(n,k))\right.\\
&\left.+T(\widehat{\phi}_{n,k})+A_n(\rho_{l,\delta}(n,k))\right].
\end{aligned}
$$

Hence from (4.3),

$$L_{\alpha,n,k}(r)\widehat{\phi}_{n,k}(r)$$
$$= \sum_{l=1}^{m+1} \frac{Q_{1,k}(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)} \frac{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-r)}{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-\rho_{l,\delta}(n,k))} (N(\rho_{l,\delta}(n,k)) + M_{\alpha,n}(\rho_{l,\delta}(n,k)))$$
$$+ \sum_{l=1}^{m+1} \frac{Q_{1,k}(\rho_{l,\delta}(n,k))}{Q_{1,k}(r)} \frac{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-r)}{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-\rho_{l,\delta}(n,k))} \Big(T(\widehat{\phi}_{n,k}) + A_n(\rho_{l,\delta}(n,k))\Big)$$
$$- \Big(N(r) + M_{\alpha,n}(r) + T(\widehat{\phi}_{n,k}) + A_n(r)\Big).$$

Using Lagrange interpolation and recalling that $Q_{1,k}(r) = (b_n(k) - r)Q_1(r)$, we get

$$\sum_{l=1}^{m+1}(b_n(k)-\rho_{l,\delta}(n,k))Q_1(\rho_{l,\delta}(n,k)) \frac{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-r)}{\prod_{i=1,i\neq l}^{m+1}(\rho_{i,\delta}(n,k)-\rho_{l,\delta}(n,k))} = (b_n(k)-r)Q_1(r).$$

Plugging this into the above equality we obtain (4.4). □

From Theorem 4.4 we obtain our main result in this section:

Theorem 4.5. *Suppose Hypothesis 2.1 and 4.1 hold, and $(c,\eta) \neq (0,0)$. Then for all $\delta \geq 0$ the Laplace transform of the Gerber-Shiu penalty function of the perturbed risk process V_α is given by*

$$\widehat{\phi}(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}[N(\rho_{j,\delta}) - N(r) + M_\alpha(\rho_{j,\delta}) - M_\alpha(r)]}{L_\alpha(r)\left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-r)}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}\right]}, \tag{4.7}$$

or equivalently, by

$$\widehat{\phi}(r) = \frac{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})}\left[\frac{N(\rho_{j,\delta})-N(r)}{\rho_{j,\delta}-r} + \frac{M_\alpha(\rho_{j,\delta})-M_\alpha(r)}{\rho_{j,\delta}-r}\right]}{L_\alpha(r)\left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})(\rho_{j,\delta}-r)}\right]}. \tag{4.8}$$

Proof. Since $\lim_{k\to\infty} b_n(k) = \infty$ implies

$$\lim_{k\to\infty} \frac{Q_1(\rho_{j,\delta}(n))(b_n(k)-\rho_{j,\delta}(n))}{Q_{1,k}(r)} = \lim_{k\to\infty} \frac{Q_1(\rho_{j,\delta}(n))(b_n(k)-\rho_{j,\delta}(n))}{Q_1(r)(b_n(k)-r)} = \frac{Q_1(\rho_{j,\delta}(n))}{Q_1(r)}.$$

Identity (4.7) follows now from Theorem 3.3, by taking limits in (4.4) firstly when $k \to \infty$ and afterward when $n \to \infty$ and using Proposition 3.6 e). The equality

(4.8) follows immediately from (4.7) after multiplying and dividing by $\rho_{j,\delta} - r$ the j-th term in the sums in the numerator and denominator of (4.7). □

Remark 4.6. Assume that f_1 is the hyperexponential distribution with density

$$f_1(x) = \sum_{l=1}^{m} A_l q_l e^{-q_l x}, \quad x > 0,$$

with $A_l > 0$ and $\sum_{l=1}^{m} A_l = 1$. In this case the roots of the Lundberg function L_α are all real and different; the proof of this fact is similar to that in Bowers *et al.* [3], page 422. If in addition $\eta = 0$, Theorem 4.5 above gives the result in Albrecher *et al.* [1].

5. A Renewal Equation for the Laplace Transform of ϕ

In this section we obtain expressions for the Gerber-Shiu penalty function ϕ by inverting its Laplace transform, given in Theorem 4.5. The expressions we obtain are in terms of the operator T_r introduced in Dickson-Hipp [4], which is defined by the equation

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) dy, \quad x \geq 0,$$

when the integral exists. Here f is any nonnegative function on $(0, \infty)$ and $r \in \mathbb{C}_+$. Notice that $T_r f$ exists for all $r \in \mathbb{C}_+$ when f is integrable, and satisfies $T_r f(0) = \widehat{f}(r)$ and $\widehat{T}_{r_2} f(r_1) = (\widehat{f}(r_1) - \widehat{f}(r_2))/(r_2 - r_1)$ for all $r, r_1, r_2 \in \mathbb{C}_+$ with $r_1 \neq r_2$. Hence, for $r_1, r_2 \in \mathbb{C}_{++}$, $r_1 \neq r_2$ and $m_\alpha(u) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_u^\infty \omega(x-u, u) x^{-1-\alpha}\, dx$,

$$\frac{M_\alpha(r_1) - M_\alpha(r_2)}{r_2 - r_1} = \eta^\alpha \widehat{T}_{r_2} m_\alpha(r_1). \tag{5.1}$$

In order to simplify our notation, we define

$$E(\rho_{j,\delta}) = \frac{Q_1(\rho_{j,\delta})}{\prod_{l \neq j} (\rho_{l,\delta} - \rho_{j,\delta})}, \; j = 1, 2, \ldots, m+1.$$

The following corollary is a direct consequence of (4.8) and (5.1).

Corollary 5.1. *Assume that Hypothesis 2.1 and 4.1 hold. Then*

$$\phi(r) = h_{\alpha,\delta,\omega} * W_{\alpha,\delta}(u), \; u > 0, \tag{5.2}$$

where

$$h_{\alpha,\delta,\omega}(u) = \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left[\lambda_2 \xi_\omega + \eta^\alpha m_\alpha\right](u) \tag{5.3}$$

and $W_{\alpha,\delta}(u), u > 0$ is the function with Laplace transform

$$\widehat{W}_{\alpha,\delta}(r) = \left(-L_\alpha(r) \sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1, i \neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})(\rho_{j,\delta} - r)} \right)^{-1}. \tag{5.4}$$

Our next step is to show that the function $\widehat{W}_{\alpha,\delta}(r)$ is related to the Laplace transform of the time to ruin when $\delta > 0$ and to the ruin probability when $\delta = 0$, and that it is the Laplace transform of some function $W_{\alpha,\delta}(u)$ whose explicit form is given in Proposition 5.6 below. We recall that for $c > 0$ and $\alpha \in (1,2)$, the tail of the extremal stable distribution $\zeta_{\alpha,c}$ is given by $\overline{\zeta}_{\alpha,c}(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(1+\{\alpha-1\}n)} x^{n(\alpha-1)}$, $x > 0$, and denote the density of $\zeta_{\alpha,c}$ by $z_{\alpha,c}$. Due to Lemma 1 in Furrer [6], $\widehat{z}_{\alpha,c}(r)$ exists for all $r \geq 0$ and is given by $\widehat{z}_{\alpha,c}(r) = c/(c + r^{\alpha-1})$. Since $\rho_{j,\delta}$, $j = 1, 2, \ldots, m+1$, appear in conjugate pairs, it follows that $\prod_{j=1}^{m+1} \rho_{j,\delta} > 0$ for $\delta > 0$. Using Lemma 3.4 and the change of variables $\rho^*_{j,\delta}(r) = \rho_{j,\delta} - r$, one can show that if $\rho_{1,\delta}, \ldots, \rho_{m+1,\delta}$ are different complex numbers and $P_l(x) = a_l x^l + a_{l-1}x^{l-1} + \cdots + a_1 x + q_0$ is a polynomial of degree l, then for all $l \geq 1$,

$$\sum_{j=1}^{m+1} \frac{P_l(\rho_{j,\delta})}{\prod_{l=1, l\neq j}^{m+1} (\rho_{l,\delta} - \rho_{j,\delta})} = \begin{cases} 0 & \text{if } l = 0, 1, \ldots, m-1 \\ (-1)^m a_m & \text{if } l = m. \end{cases} \tag{5.5}$$

The following two lemmas can be proved using Lemma 3.4 and (5.5), and the fact that the roots $\{\rho_{j,\delta}\}$ of L_α are in conjugate pairs.

Lemma 5.2. *For $\delta > 0$, it holds $\sum_{j=1}^{m+1} E(\rho_{j,\delta}) = 1$, and $\sum_{j=1}^{m+1} E(\rho_{j,\delta})\rho_{j,\delta}^{-1} = \dfrac{\prod_{l=1}^{N} q_l^{m_l}}{\prod_{k=1}^{m+1} \rho_{k,\delta}}$.*

Lemma 5.3. *For any function $K : (0,\infty) \to [0,\infty)$ and all $\delta \geq 0$, the functions $x \to \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} K(x)$ and $x \to \sum_{j=1}^{m+1} E(\rho_{j,\delta})\rho_{j,\delta} T_{\rho_{j,\delta}} K(x)$, are real-valued.*

We define the function $\ell_\alpha(u) := \frac{(\alpha-1)u^{-\alpha}}{\Gamma(2-\alpha)}$, $u > 0$, and note that although $\ell_\alpha(u)$ is not integrable, the function $T_r \ell_\alpha(x)$ exists and is finite for all $x > 0$ and $r > 0$. For all complex numbers $r_1, r_2 \in \mathbb{C}_+$ such that $r_1 \neq r_2$ and $\alpha \in (1,2)$, it can be proved by integration by parts (see Zolotarev [16], page 10) that

$$\int_0^\infty [e^{-r_1 x} - e^{-r_2 x}] x^{-\alpha} dx = \frac{\Gamma(2-\alpha)}{\alpha - 1} [r_2^{\alpha-1} - r_1^{\alpha-1}]. \tag{5.6}$$

It follows from (5.6) that $\widehat{T}_{r_1}\ell_\alpha(r_2) = \int_0^\infty e^{-r_2 x} \int_x^\infty e^{-r_1(y-x)} \ell_\alpha(y) dy dx = \frac{r_1^{\alpha-1} - r_2^{\alpha-1}}{r_1 - r_2}$ for $r_1 \neq r_2$, and $\widehat{T}_{r_1}\ell_\alpha(r_1) = \int_0^\infty e^{-r_1 x} \int_x^\infty e^{-r_1(y-x)} \ell_\alpha(y) dy dx = (\alpha-1) r_1^{\alpha-2}$ if $r_1 \neq 0$.

Let us define for all $x > 0$ the functions $f_{\alpha,\delta}(x) = \sum_{j=1}^{m+1} E(\rho_{j,\delta})\rho_{j,\delta} T_{\rho_{j,\delta}} \ell_\alpha(x)$ and $g_\delta(x) = \lambda_2 \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} f_2(x)$. Due to Lemma 5.3, $h_{\alpha,\delta,\omega}$, $f_{\alpha,\delta}$ and g_δ are real-valued. In the sequel we assume the following condition.

Hypothesis 5.1. The functions $h_{\alpha,\delta,\omega}, f_{\alpha,\delta}$ and g_δ defined above are nonnegative.

It is straightforward to prove that Hypothesis 5.1 holds in the case when f_1 is a hyperexponential distribution and f_2 is a general density function, because in such case $E(\rho_{j,\delta})$ and $\rho_{j,\delta}$ are nonnegative numbers.

In the following proposition we obtain an alternative representation of $\widehat{W}_{\alpha,\delta}$ which allows us to calculate its inverse Laplace transform.

Proposition 5.4. *Assume that hypothesis 2.1, 4.1 and 5.1 hold. Then,*
a). For $\eta > 0$ and $c > 0$,

$$\widehat{W}_{\alpha,\delta}(r) = \frac{\frac{1}{\eta^\alpha \theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{1 - \frac{1}{\theta_\delta}\left[\kappa_\delta \widehat{\nu}_{\alpha,\delta}(r) + \eta^{-\alpha}\widehat{g}_\delta(r)\widehat{\nu}_{\alpha,\delta}(r)\right]}, \tag{5.7}$$

where $\kappa_\delta = \frac{1}{\eta^\alpha}\widehat{g}_\delta(0) + \widehat{f}_{\alpha,\delta}(0)$, $\theta_\delta = c/\eta^\alpha + \kappa_\delta$ and

$$\widehat{\nu}_{\alpha,\delta}(r) = \frac{\widehat{z}_{\alpha,\theta_\delta}(r)}{1 + \frac{1}{\theta_\delta}\widehat{f}_{\alpha,\delta}(r)\widehat{z}_{\alpha,\theta_\delta}(r)}. \tag{5.8}$$

b). The function $\widehat{W}_{\alpha,\delta}(r)$ is related to the time to ruin and the probability of ruin $\psi(u)$ by the following equalities:

$$\widehat{\varphi}_\delta(r) = \frac{1}{r} - \frac{\delta}{r}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}}\widehat{W}_{\alpha,\delta}(r), \quad \delta > 0,$$

and

$$\widehat{\psi}(r) = \frac{1}{r} - \frac{c + \lambda_1\mu_1 - \lambda_2\mu_2}{r}\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\widehat{W}_{\alpha,0}(r),$$

where $\varphi_\delta(u) = \mathbb{E}\left[e^{-\delta\tau_0}1_{\{\tau_0<\infty\}}|V_\alpha(0) = u\right]$ is the Laplace transform of the ruin time for $\delta > 0$.

Proof. See appendix A. □

From Proposition 5.4 b) we obtain the following corollary.

Corollary 5.5. *For $u > 0$, both $\psi(u)$ and $\varphi_\delta(u)$ are tails of probability distributions with respective densities $\psi'(u) = (c + \lambda_1\mu_1 - \lambda_2\mu_2)\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}W_{\alpha,0}(u)$ and $\varphi'_\delta(u) = \delta\dfrac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}}W_{\alpha,\delta}(u)$. Hence, from (5.2) the Gerber-Shiu function is given by the expressions*

$$\phi(u) = \begin{cases} \left[(c + \lambda_1\mu_1 - \lambda_2\mu_2)\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=2}^{m+1}\rho_{j,0}}\right]^{-1} h_{\alpha,0,\omega} * \psi'(u) & \text{for } \delta = 0, \\ \left[\delta\frac{\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}}\right]^{-1} h_{\alpha,\delta,\omega} * \varphi_\delta(u) & \text{for } \delta > 0. \end{cases}$$

Now we are ready to give a representation of $W_{\alpha,\delta}$ as a series of convolutions of the functions $f_{\alpha,\delta}$, $g_\delta, \nu_{\alpha,\delta}$ defined above.

Proposition 5.6. *Under hypothesis 2.1, 4.1 and 5.1 the following properties hold.*

a) For $r \geq 0$, the function $\widehat{\nu}_{\alpha,\delta}(r)$ defined in (5.8) is the Laplace transform of the function $\nu_{\alpha,\delta}(u) = z_{\alpha,\theta_\delta} * \sum_{n=0}^{\infty}\left[-\frac{1}{\theta_\delta}\right]^n \left[f_{\alpha,\delta} * z_{\alpha,\theta_\delta}\right]^{*n}(u)$.

b) For $u \geq 0$, the function $\widehat{W}_{\alpha,\delta}(u)$ defined in (5.7) is the Laplace transform of the functions $W_{\alpha,\delta}(u) = \frac{1}{\eta^\alpha\theta_\delta}\nu_{\alpha,\delta} * \sum_{n=0}^{\infty}\theta_\delta{}^{-n}\left[\kappa_\delta\nu_{\alpha,\delta} + \eta^{-\alpha}g_\delta * \nu_{\alpha,\delta}\right]^{*n}(u)$,

c) $\nu_{\alpha,\delta}(u)$ and $\frac{1}{\theta_\delta\eta^\alpha}g_\delta * \nu_{\alpha,\delta}(u) + \frac{\kappa_\delta}{\theta_\delta}\nu_{\alpha,\delta}(u)$ are defective density functions.

Proof. Since $0 < \frac{\widehat{f}_{\alpha,\delta}(0)}{\theta_\delta} < 1$, Hypothesis 5.1 implies $0 < \frac{\widehat{f}_{\alpha,\delta}(r)}{\theta_\delta} < 1$ for all $r \geq 0$. Hence the series $\widehat{z}_{\alpha,\theta_\delta}(r) \sum_{n=0}^{\infty} \left[-\frac{1}{\theta_\delta}\right]^n \left[\widehat{f}_{\alpha,\delta}(r)\widehat{z}_{\alpha,\theta_\delta}(r)\right]^n$ is absolutely convergent for $r \geq 0$, and its limit equals the right-hand side of (5.8). We set $\mathcal{I}(u) := \int_0^\infty \sum_{n=0}^\infty \left[\frac{1}{\theta_\delta}\right]^n z_{\alpha,\theta_\delta} * [f_{\alpha,\delta} * z_{\alpha,\theta_\delta}]^{*n}(u)du$. Using monotone convergence we get

$$\begin{aligned}
\mathcal{I}(u) &= \lim_{m\to\infty} \sum_{n=0}^{m} \int_0^\infty \left[\frac{1}{\theta_\delta}\right]^n z_{\alpha,\theta_\delta} * [f_{\alpha,\delta} * z_{\alpha,\theta_\delta}]^{*n}(u)du \\
&\leq \lim_{m\to\infty} \sum_{n=0}^{m} \left| \left[-\frac{1}{\theta_\delta}\right]^n \widehat{z}_{\alpha,\theta_\delta}(0) \left[\widehat{f}_{\alpha,\delta}(0)\widehat{z}_{\alpha,\theta_\delta}(0)\right]^n \right| \\
&= \sum_{n=0}^{\infty} \left| \left[-\frac{1}{\theta_\delta}\right]^n \widehat{z}_{\alpha,\theta_\delta}(0) \left[\widehat{f}_{\alpha,\delta}(0)\widehat{z}_{\alpha,\theta_\delta}(0)\right]^n \right| < \infty,
\end{aligned}$$

which implies that the series $\sum_{n=0}^\infty \left[-\frac{1}{\theta_\delta}\right]^n z_{\alpha,\theta_\delta} * [f_{\alpha,\delta} * z_{\alpha,\theta_\delta}]^{*n}(u)$ converges absolutely. This proves a).

To prove c), due to Hypothesis 5.1 and the definition of $\widehat{\nu}_{\alpha,\delta}$ we have $\widehat{\nu}_{\alpha,\delta}(0) < 1$, hence $\nu_{\alpha,\delta}$ is a defective density function. From Proposition 5.4 and (5.7) we obtain

$$\begin{aligned}
\widehat{\phi}(r) &= \frac{\frac{1}{\eta^\alpha\theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{-L_\alpha(r)\left[\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})(\rho_{i,\delta}-r)}\right]\frac{\widehat{\nu}_{\alpha,\delta}(r)}{\eta^\alpha\theta_\delta}} \\
&= \frac{\frac{1}{\eta^\alpha\theta_\delta}\widehat{\nu}_{\alpha,\delta}(r)}{1-\frac{1}{\theta_\delta}\left[\kappa_\delta\widehat{\nu}_{\alpha,\delta}(r)+\eta^{-\alpha}\widehat{g}_\delta(r)\widehat{\nu}_{\alpha,\delta}(r)\right]}.
\end{aligned}$$

Putting $r = 0$ in the above equality and using the second equality in (5.5), it follows that

$$\begin{aligned}
1 - \frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(0) + \kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(0) &= -\frac{L_\alpha(0)\widehat{\nu}_{\alpha,\delta}(0)}{\eta^\alpha\theta_\delta} \sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1}(\rho_{i,\delta}-\rho_{j,\delta})\rho_{j,\delta}} \\
&= \frac{\widehat{\nu}_{\alpha,\delta}(0)}{\eta^\alpha\theta_\delta} \frac{\delta\prod_{i=1}^N q_i^{m_i}}{\prod_{j=1}^{m+1}\rho_{j,\delta}} > 0.
\end{aligned}$$

From the inequality above and the fact that $\widehat{\nu}_{\alpha,\delta}(0)$ and $\widehat{g}_\delta(0) + \kappa_\delta$ are always positive, it follows that $\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(0) + \kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(0) < 1$. Now using Hypothesis 5.1 we obtain $\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(r) + \kappa_\delta\right]\widehat{\nu}_{\alpha,\delta}(r) < 1$ for all $r \geq 0$, which implies that $\frac{1}{\theta_\delta\eta^\alpha} g_\delta * \nu_{\alpha,\delta}(u) + \frac{\kappa_\delta}{\theta_\delta}\nu_{\alpha,\delta}(u)$ is a defective density function. The proof of this result for $\nu_{\alpha,\delta}(u)$ is similar. The proof of b) follows from c) similarly as in a). □

From (5.2) and Proposition 5.6 we obtain the main result in this section, in which we give a representation of $\phi(u)$ in terms of an infinite series of convolutions of $h_{\alpha,\delta,\omega}$ and the functions g_δ and $\nu_{\alpha,\delta}$ defined above, and the corresponding defective renewal equation for ϕ.

Theorem 5.7. *Assume that hypothesis 2.1, 4.1 and 5.1 hold. Then, for $\eta > 0$ the Gerber-Shiu penalty function satisfies the defective renewal equation*

$$\phi(u) = \frac{1}{\theta_\delta}\int_0^u \phi(u-y)\left[\kappa_\delta\nu_{\alpha,\delta}(y) + \frac{1}{\eta^\alpha}g_\delta * \nu_{\alpha,\delta}(y)\right]dy + \frac{1}{\eta^\alpha\theta_\delta}h_{\alpha,\delta,\omega} * \nu_{\alpha,\delta}(u),$$

whose solution is given by $\phi(u) = \frac{1}{\eta^\alpha\theta_\delta} h_{\alpha,\delta,\omega} * \sum_{n=0}^{\infty} \nu_{\alpha,\delta}^{*(n+1)} * \left[\frac{\kappa_\delta}{\theta_\delta} + \frac{1}{\eta^\alpha\theta_\delta} g_\delta\right]^{*n}(u)$.

We note from Corollary 5.1 that the only dependence of ϕ on the penalty function ω appears in $h_{\alpha,\delta,\omega}(u)$, hence in order to obtain a formula for $\phi(u)$ for different penalty functions we only need to calculate the corresponding function $h_{\alpha,\delta,\omega}$. Let us take $\omega(x,y) = e^{-sx-ty}$ for $s,t \geq 0$. Using (5.3), we obtain that in this case the function $h_{\alpha,\delta,\omega}$ defined above has the form

$$h_{\alpha,\delta,\omega}(u) = \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left(\eta^\alpha f_{1,s,t} + \lambda_2 f_{2,s,t}\right)(u),$$

where $f_{1,s,t}(x) = e^{-tx}\ell_\alpha(x) - se^{-tx}T_s\ell_\alpha(x)$ and $f_{2,s,t}(x) = e^{-tx}T_s f_2(x)$. Since $-\frac{\partial}{\partial s}e^{-sx-ty}|_{s=t=0} = x$, $-\frac{\partial}{\partial t}e^{-sx-ty}|_{s=t=0} = y$ and $\frac{\partial^2}{\partial s \partial t}e^{-sx-ty}|_{s=t=0} = xy$, for $\delta > 0$ the results of the previous theorem can be extended to the cases of penalty functions

$$\omega(x,y) = x, \quad \omega(x,y) = y \quad \text{and} \quad \omega(x,y) = xy, \tag{5.9}$$

which are not bounded. This can be shown by applying the dominated convergence theorem and calculating the corresponding derivatives of $h_{\alpha,\delta,\omega}$. In this way we obtain the following result.

Corollary 5.8. *Let* $\delta > 0$ *and*

$$h_{\alpha,\delta,\omega}(u) = \begin{cases} \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left(\eta^\alpha \Lambda_\alpha + \lambda_2 \mu_2 \overline{F}_{2,I}\right)(u) & \text{if } \omega(x,y) = x, \\ \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left((\alpha-1)\eta^\alpha \Lambda_\alpha + \lambda_2 G\right)(u) & \text{if } \omega(x,y) = y, \\ \sum_{j=1}^{m+1} E(\rho_{j,\delta}) T_{\rho_{j,\delta}} \left(\eta^\alpha \Lambda^*_\alpha + \lambda_2 G^*\right)(u) & \text{if } \omega(x,y) = xy, \end{cases} \tag{5.10}$$

where $G(u) = u\overline{F_2}(u)$, $\Lambda_\alpha(u) = \int_u^\infty \ell_\alpha(x)dx$, $\Lambda^*_\alpha(u) = uT_0\ell_\alpha(u)$, $G^*(u) = u\int_u^\infty (z-u)f_2(z)dz$ *and* $F_{2,I}$ *is the integrated tail distribution of* F_2*. Then Theorem 5.7 holds also for the penalty functions (5.9), with the same functions* g_δ *and* $\nu_{\alpha,\delta}$*, and corresponding functions* $h_{\alpha,\delta,\omega}$ *given by (5.10).*

6. Examples and Conclusions

Here we illustrate how to apply the above results to two particular cases of risk processes. We assume that $\lambda_1 = \lambda_2 = \eta = 1$, $c > 0$, $1 < \alpha \leq 2$, and the penalty function ω is such that Hypothesis 2.1 holds.

Example 6.1. For given positive constants a, b, let $f_1(x) = ae^{-ax}, x > 0$ and $f_2(x) = be^{-bx}$, $x > 0$. In this case the Lundberg's equation $L_\alpha(r) - \delta = 0$ is given by $\frac{b}{b+r} + \frac{a}{a-r} + cr + r^\alpha - 2 - \delta = 0$, and it has two roots in $\mathbb{C}_{++}$, denoted by ρ_1 and ρ_2. These roots are real and satisfy the inequalities $\rho_1 < a < \rho_2 < b$. In order to obtain from (5.2) the Gerber-Shiu function ϕ for general penalty function ω, we need to calculate the functions $h_{\alpha,\delta,\omega}$ and $W_{\alpha,\delta}$. From (5.3) it follows that

$$\begin{aligned} h_{\alpha,\delta,\omega}(u) =& \frac{a-\rho_1}{\rho_2-\rho_1}\int_u^\infty e^{-\rho_1(x-u)}\int_x^\infty \omega(y-x,x)\left(\lambda_2 b e^{-by}+\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}y^{-1-\alpha}\right)dydx \\ &+\frac{\rho_2-a}{\rho_2-\rho_1}\int_u^\infty e^{-\rho_2(x-u)}\int_x^\infty \omega(y-x,x)\left(\lambda_2 b e^{-by}+\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}y^{-1-\alpha}\right)dydx, \end{aligned} \tag{6.1}$$

and from (A.7) we obtain

$$\widehat{W}_{\alpha,\delta}(r)=\frac{1}{c+r^{\alpha-1}-\frac{b}{b+r}\frac{a+b}{(b+\rho_1)(b+\rho_2)}+\frac{a-\rho_1}{\rho_2-\rho_2}\rho_1\frac{\rho_1^{\alpha-1}-r^{\alpha-1}}{\rho_1-r}+\frac{\rho_2-a}{\rho_2-\rho_1}\rho_2\frac{\rho_2^{\alpha-1}-r^{\alpha-1}}{\rho_2-r}}. \tag{6.2}$$

Since $\alpha<2$, the above formula does not admit a simple decomposition in partial fractions as in the case when $\alpha=2$. However, using the formula in Proposition 5.6 b) we obtain an expression for the inverse of $\widehat{W}_{\alpha,\delta}$. Therefore

$$W_{\alpha,\delta}(u)=\frac{1}{\eta^\alpha\theta_\delta}\nu_{\alpha,\delta}*\sum_{n=0}^\infty {\theta_\delta}^{-n}\left[\kappa_\delta\nu_{\alpha,\delta}+\int_0^\cdot e^{-bx}\nu_{\alpha,\delta}(\cdot-x)dx\right]^{*n}(u),$$

where $\kappa_\delta=\frac{a+b}{(b+\rho_1)(b+\rho_2)}+\frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1}+\frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$, $\theta_\delta=c+\frac{a+b}{(b+\rho_1)(b+\rho_2)}+\frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1}+\frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$, and the function $\nu_{\alpha,\delta}$ is given by

$$\begin{aligned} \nu_{\alpha,\delta}(u) =& z_{\alpha,\theta_\delta}*\sum_{n=0}^\infty\left[-\frac{1}{\theta_\delta}\right]^n\left[\frac{a-\rho_1}{\rho_2-\rho_1}\rho_1\int_0^\cdot z_{\alpha,\theta_\delta}(\cdot-y)\int_y^\infty e^{-\rho_1(z-y)}\frac{(\alpha-1)z^{-\alpha}}{\Gamma(2-\alpha)}dzdy\right. \\ &\left.+\frac{\rho_2-a}{\rho_2-\rho_1}\rho_2\int_0^\cdot z_{\alpha,\theta_\delta}(\cdot-y)\int_y^\infty e^{-\rho_2(z-y)}\frac{(\alpha-1)z^{-\alpha}}{\Gamma(2-\alpha)}dzdy\right]^{*n}(u); \end{aligned} \tag{6.3}$$

see Proposition 5.6 a). In view of (6.2) the expressions for the Laplace transforms of the ruin probability and of the ruin time given in Proposition 5.6 b) are simpler to calculate in this example.

Example 6.2. Let $f_1(x)$ be as in Example 6.1, and assume that $f_2(x)=b^2e^{-bx}x$, $x>0$, i.e. f_2 is an Erlang density with shape parameter $k=2$ and scale parameter $b>0$. In this case the Lundberg's equation is $(\frac{b}{b+r})^2+\frac{a}{a-r}+cr+r^\alpha-2-\delta=0$, which has two roots in $\mathbb{C}_{++}$, denoted by ρ_1,ρ_2. These roots are real and satisfy the inequalities $\rho_1<a<\rho_2$. In this case $\widehat{W}_{\alpha,\delta}(r)=\frac{1}{\widetilde{L}_\alpha(r)}$, where

$$\begin{aligned} \widetilde{L}_\alpha(r)=c+r^{\alpha-1}-\Bigg\{&\frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{b^2}{b+\rho_1}\frac{1}{(b+r)^2}+\frac{b^2}{(b+\rho_1)^2}\frac{1}{b+r}\right] \\ &+\frac{\rho_2-a}{\rho_2-\rho_1}\left[\frac{b^2}{b+\rho_2}\frac{1}{(b+r)^2}+\frac{b^2}{(b+\rho_2)^2}\frac{1}{b+r}\right]\Bigg\} \\ &+\frac{a-\rho_1}{\rho_2-\rho_1}\rho_1\frac{\rho_1^{\alpha-1}-r^{\alpha-1}}{\rho_1-r}+\frac{\rho_2-a}{\rho_2-\rho_1}\rho_2\frac{\rho_2^{\alpha-1}-r^{\alpha-1}}{\rho_2-r}. \end{aligned}$$

Again, this expression does not admit a partial fraction decomposition when $\alpha < 2$. Hence we use Proposition 5.6 b) to obtain:

$$W_{\alpha,\delta}(u) = \frac{1}{\eta^{\alpha}\theta_{\delta}}\nu_{\alpha,\delta} * \sum_{n=0}^{\infty} \theta_{\delta}{}^{-n} \left\{ \kappa_{\delta}\nu_{\alpha,\delta} + \int_0^{\cdot} \left(\frac{a-\rho_1}{\rho_2-\rho_1} \left[\frac{b^2}{b+\rho_1} e^{-bx} x + \frac{b^2}{(b+\rho_1)^2} e^{-bx} \right] \right.\right.$$
$$\left.\left. + \frac{\rho_2 - b}{\rho_2-\rho_1} \left[\frac{b^2}{b+\rho_2} e^{-bx} x + \frac{b^2}{(b+\rho_2)^2} e^{-bx} \right] \right) \nu_{\alpha,\delta}(\cdot - x) dx \right\}^{*n} (u),$$

where $\kappa_{\delta} = \frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_1} + \frac{b}{(b+\rho_1)^2}\right] + \frac{\rho_2-a}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_2} + \frac{b}{(b+\rho_2)^2}\right] + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1} + \frac{\rho_2-a}{\rho_2-\rho_1} \times \rho_2^{\alpha-1}$ and $\theta_{\delta} = c + \frac{a-\rho_1}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_1} + \frac{b}{(b+\rho_1)^2}\right] + \frac{\rho_2-a}{\rho_2-\rho_1}\left[\frac{1}{b+\rho_2} + \frac{b}{(b+\rho_2)^2}\right] + \frac{a-\rho_1}{\rho_2-\rho_1}\rho_1^{\alpha-1} + \frac{\rho_2-a}{\rho_2-\rho_1}\rho_2^{\alpha-1}$. The functions $h_{\alpha,\delta,\omega}$ and $\nu_{\alpha,\delta}$ have the same expressions as in (6.1) and (6.3), with the corresponding roots ρ_1 and ρ_2.

Although the expressions for $W_{\alpha,\delta}(u)$ presented in these two examples are difficult to work with in general, the formulas for $\widehat{W}_{\alpha,\delta}(r)$ are rather simple and their inverse Laplace transforms can be calculated by using numerical methods.

Since $h_{\alpha,\delta,\omega}$ and the constants in the expressions above can be calculated explicitly, the function $W_{\alpha,\delta}$ becomes the most interesting object of study. For instance, the expressions given in Proposition 5.6 a) and b) allow the use of theoretical tools to obtain asymptotic expressions for $\nu_{\alpha,\delta}$ and $W_{\alpha,\delta}$. These results can be used, in turn, to obtain asymptotic expressions for the ruin probability, the Laplace transform of the time to ruin, the joint tail of the severity of ruin and the surplus prior to ruin and some other important cases of Gerber-Shiu functions. The asymptotic expressions for the ruin probability and the joint tail of the severity of ruin and the surplus prior to ruin are the main topic in Kolkovska and Martín-González [12].

Finally, the function $W_{\alpha,\delta}$ is related to the density of the negative Wiener-Hopf factor of the Lévy process V_{α}, which we study in further detail and generality in a forthcoming work.

Appendix A. Remaining Proofs

Proof of Lemma 4.1. The existence of the integral I_n follows from the existence of both, $\widehat{g}_{\alpha,1}$ and the first moment of $g_{\alpha,1}$ for $\alpha \in (1,2)$. Setting $y = -n^{1/\alpha}x$ we obtain $I_n = \int_0^{\infty} n\left(1 - e^{ry/n^{1/\alpha}} + \frac{ry}{n^{1/\alpha}}\right) g_{\alpha,1}(-y)dy$, and putting $I_n^*(y,r) = n\left(1 - e^{ry/n^{1/\alpha}} + \frac{ry}{n^{1/\alpha}}\right) g_{\alpha,1}(-y)$ gives $|I_n^*(y,r)| = \left|\sum_{k=2}^{\infty} r^k \frac{y^k}{n^{k/\alpha-1}} g_{\alpha,1}(-y)\right| \leq \sum_{k=2}^{\infty} |r|^k y^k g_{\alpha,1}(-y) = \left(e^{|r|y} - 1 - |r|y\right) g_{\alpha,1}(-y)$. Since $\int_0^{\infty} \left(e^{|r|y} - 1 - |r|y\right) g_{\alpha,1}(-y) \times dy = \int_{-\infty}^{0} \left(e^{-|r|y} - 1 + |r|y\right) g_{\alpha,1}(y)dy < \infty$, using that $|I_n| \leq \int_0^{\infty} |I_n^*(x,r)|\, dx$ the result follows from the dominated convergence theorem.

Proof of Lemma 4.2. Hypothesis 2.1 implies that $\phi_{n,k}(u)$ is bounded for all $u \geq 0$, hence $K_0(n,k,r)$ is finite. By performing the change of variables $z = u + x$

we obtain

$$K_0(n,k,r) = \lambda_1(k)A(k)\int_0^\infty e^{-rz}\phi_{n,k}(z)\int_0^z b_n(k)e^{-(b_n(k)-r)x}dxdz + \lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\beta_{ij}\int_0^\infty e^{-rz}\phi_{n,k}(z)\int_0^z \frac{x^{j-1}q_i^j e^{(r-q_i)x}}{(j-1)!}dxdz.$$

Using the formula $\int_0^x \frac{y^{k-1}q_j^k e^{-q_j y}}{(k-1)!}dy = 1 - \sum_{l=0}^{k-1}\frac{e^{-q_j x}}{l!}q_j^l x^l$ we get

$$\begin{aligned}K_0(n,k,r) &= \lambda_1(k)A(k)\frac{b_n(k)}{b_n(k)-r}\int_0^\infty e^{-rz}\phi_{n,k}(z)dz - \lambda_1(k)(1-A(k))\\ &\quad\times\frac{b_n(k)}{b_n(k)-r}\int_0^\infty \phi_{n,k}(z)e^{-b_n(k)z}dz\\ &\quad+\lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\beta_{ij}q_i^j}{(q_i-r)^j}\int_0^\infty e^{-rz}\phi_{n,k}(z)dz\\ &\quad-\lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\beta_{ij}q_i^j}{(q_i-r)^j}\int_0^\infty e^{-rz}\phi_{n,k}(z)\left\{\sum_{l=0}^{j-1}\frac{e^{-q_i z+rz}}{l!}(q_i-r)^l z^l\right\}dz\\ &= \left[\frac{\lambda_1(k)A(k)b_n(k)}{b_n(k)-r} + \lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\beta_{ij}q_i^j}{(q_i-r)^j}\right]\widehat{\phi}_{n,k}(r)\\ &\quad-\frac{\lambda_1(k)(1-A(k))b_n(k)}{b_n(k)-r}\widehat{\phi}_{n,k}(b_n(k)) - \lambda_1\sum_{i=1}^N\sum_{j=1}^{m_i}\beta_{ij}q_i^j\sum_{l=0}^{j-1}\frac{(q_i-r)^l\gamma_{l,i}(n,k)}{l!(q_i-r)^j},\end{aligned}$$

and the result follows.

Proof of Lemma 4.3. Let us write $e(u;r_1,r_2) = e^{-r_1 u} - e^{-r_2 u}$ and recall that $d(\alpha) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}$. Notice that

$$\eta^{-\alpha}(M_{\alpha,n}(r_1) - M_{\alpha,n}(r_2)) = n^{1+1/\alpha}\int_0^\infty\int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu. \tag{A.1}$$

From formula (14.37), page 89 in [14], we get $\lim_{x\to\infty}\frac{g_{\alpha,1}(x)}{d(\alpha)x^{-1-\alpha}} = 1$. Hence, for every $\varepsilon > 0$ there exists a positive number $A_\varepsilon > 1$ such that for all $u > A_\varepsilon$,

$$\frac{g_{\alpha,1}(x)}{d(\alpha)x^{-1-\alpha}} < (1+\varepsilon). \tag{A.2}$$

We take $A = A_\varepsilon$ and $n > A^\alpha$, and split (A.1) as follows:

$$\begin{aligned}\eta^{-\alpha}(M_{\alpha,n}(r_1) - M_{\alpha,n}(r_2)) &= n^{1+1/\alpha}\int_1^\infty\int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu\\ &\quad+n^{1+1/\alpha}\int_{A/n^{1/\alpha}}^1\int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu\\ &\quad+n^{1+1/\alpha}\int_0^{A/n^{1/\alpha}}\int_u^\infty e(u;r_1,r_2)\omega(x-u,u)g_{\alpha,1}(n^{1/\alpha}x)dxdu.\end{aligned} \tag{A.3}$$

Noticing that $u \geq 1$ and $x \geq u$ imply $n^{1/\alpha}x > A$, from (A.2) we obtain for the first term above,

$$\left| n^{1+1/\alpha} \int_1^\infty \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) g_{\alpha,1}(n^{1/\alpha}x) dx du \right|$$
$$\leq 2d(\alpha)B(1+\varepsilon) \int_1^\infty \int_u^\infty x^{-1-\alpha} dx du = \frac{2d(\alpha)B(1+\varepsilon)}{\alpha(\alpha-1)}.$$

Hence, using dominated convergence it follows that

$$\lim_{n\to\infty} n^{1+1/\alpha} \int_1^\infty \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) g_{\alpha,1}(n^{1/\alpha}x)\, dx\, du$$
$$= \quad d(\alpha) \int_1^\infty \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) x^{-1-\alpha}\, dx\, du. \tag{A.4}$$

Now we consider the second term in (A.3). In this case $n^{1/\alpha}x \geq A$, hence

$$\left| n^{1+1/\alpha} \int_{A/n^{1/\alpha}}^1 \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) g_{\alpha,1}(n^{1/\alpha}x) dx du \right|$$
$$\leq d(\alpha)B \int_0^1 \int_u^\infty |e(u; r_1, r_2)| \frac{g_{\alpha,1}(n^{1/\alpha}x)}{d(\alpha)\,[n^{1/\alpha}x]^{-1-\alpha}} d(\alpha) x^{-1-\alpha} dx du$$
$$\leq d(\alpha)B(1+\varepsilon) \int_0^1 \sum_{k=1}^\infty \frac{|r_2^k - r_1^k| u^k}{k!} \int_u^\infty \frac{1}{x^{1+\alpha}} dx du \leq \frac{2d(\alpha)B}{\alpha} \left(\frac{|r_1 - r_2|}{2-\alpha} + e^{|r_1|} + e^{|r_2|} \right).$$

Using again dominated convergence yields

$$\lim_{n\to\infty} n^{\frac{\alpha+1}{\alpha}} \int_{\frac{A}{n^{1/\alpha}}}^1 \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) g_{\alpha,1}(n^{1/\alpha}x) dx du$$
$$= d(\alpha) \int_0^1 \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) x^{-1-\alpha} dx du. \tag{A.5}$$

For the third term in (A.3) we use the change of variables $y = n^{1/\alpha}u$ and $z = n^{1/\alpha}x$. This gives

$$\left| n^{1+1/\alpha} \int_0^{A/n^{1/\alpha}} \int_u^\infty e(u; r_1, r_2)\omega(x-u, u) g_{\alpha,1}(n^{1/\alpha}x) dx du \right|$$
$$\leq n^{1-1/\alpha} \int_0^A \int_y^\infty \left| e^{-r_1 y/n^{1/\alpha}} - e^{-r_2 y/n^{1/\alpha}} \right| \omega\left(\frac{z-y}{n^{1/\alpha}}, \frac{y}{n^{1/\alpha}} \right) g_{\alpha,1}(z) dz dy$$
$$\leq B \int_0^A \int_y^\infty \sum_{k=1}^\infty \frac{|r_1^k - r_2^k| y^k}{k! n^{(k+1-\alpha)/\alpha}} g_{\alpha,1}(z) dz dy \leq \frac{B}{n^{(2-\alpha)/\alpha}} \int_0^A \left(e^{|r_1|y} + e^{|r_2|y} \right) \overline{G}_{\alpha,1}(y) dy$$
$$\leq \frac{B}{n^{(2-\alpha)/\alpha}} A \left(e^{|r_1|A} + e^{|r_2|A} \right) \overline{G}_{\alpha,1}(0),$$

where to obtain the last inequality we used that $n \geq A^\alpha > 1$, which implies $n^{-(2-\alpha)/\alpha} \geq n^{-(k+1-\alpha)/\alpha}$ for all $k \geq 2$. From here the result follows using (A.4) and (A.5).

Proof of Proposition 5.4. In order to prove a) first we will show that $J_0(r) = 0$ for all $r \geq 0$, where $J_0(r) = \lambda_1 \sum_{j=1}^{m+1} Q_1(\rho_{j,\delta}) \prod_{i=1, i\neq j}^{m+1} \frac{\widehat{f}_1(-r) - \widehat{f}_1(-\rho_{j,\delta})}{\rho_{j,\delta} - r} (\rho_{i,\delta} - \rho_{j,\delta})^{-1}$. For fixed $r \geq 0$ we obtain, setting $\rho^*_{j,\delta}(r) = \rho_{j,\delta} - r$ and $q^*_i(r) = q_i - r$,

$$\begin{aligned} J_0(r) &= \lambda_1 \sum_{j=1}^{m+1} \frac{\prod_{h=1}^{N} (q^*_h(r) - \rho^*_{j,\delta}(r))^{m_h}}{\prod_{i=1, i\neq j}^{m+1} (\rho^*_{i,\delta}(r) - \rho^*_{j,\delta}(r))} \sum_{k=1}^{N} \sum_{l=1}^{m_k} \beta_{kl} q_k^l \frac{(q^*_k(r) - \rho^*_{j,\delta}(r))^l - (q^*_k(r))^l}{(q^*_k(r))^l (q^*_k(r) - \rho^*_{j,\delta}(r))^l \rho^*_{j,\delta}(r)} \\ &= \lambda_1 \sum_{j=1}^{m+1} \frac{\prod_{h=1}^{N} (q^*_h(r) - \rho^*_{j,\delta}(r))^{m_h}}{\prod_{i=1, i\neq j}^{m+1} (\rho^*_{i,\delta}(r) - \rho^*_{j,\delta}(r))} \sum_{k=1}^{N} \sum_{l=1}^{m_k} \beta_{kl} q_k^l \frac{P^*_l(\rho^*_{j,\delta}(r))}{(q^*_k(r))^l (q^*_k(r) - \rho^*_{j,\delta}(r))^l}, \quad \text{(A.6)} \end{aligned}$$

where P^*_l is a polynomial of degree $l-1$. We note that for each $j \in \{1, \dots, m+1\}$,

$$\sum_{k=1}^{N} \sum_{l=1}^{m_k} \beta_{kl} q_k^l \frac{P^*_l(\rho^*_{j,\delta}(r))}{(q^*_k(r))^l (q^*_k(r) - \rho^*_{j,\delta}(r))^l} = \frac{P^{**}(\rho^*_{j,\delta}(r))}{\prod_{h=1}^{N} (q^*_h(r) - \rho^*_{j,\delta}(r))^{m_h}},$$

where $P^{**}(\rho^*_{j,\delta}(r))$ is a polynomial on $\rho^*_{j,\delta}(r)$ of degree at most $m-1$. Putting the above equality in (A.6) gives $J_0 = \lambda_1 \sum_{j=1}^{m+1} \frac{P^{**}(\rho^*_{j,\delta}(r))}{\prod_{i=1,i\neq j}^{m+1} (\rho^*_{i,\delta}(r) - \rho^*_{j,\delta}(r))} = 0$ due to (5.5).

Using that $\lambda_1 + \lambda_2 + \delta = \lambda_2 \widehat{f}_2(\rho_{j,\delta}) + \lambda_1 \widehat{f}_1(-\rho_{j,\delta}) + c\rho_{j,\delta} + \eta^\alpha \rho^\alpha_{j,\delta}$, from (5.4) and the equality $\frac{\rho^\alpha_{j,\delta} - r^\alpha}{\rho_{j,\delta} - r} = \rho_{j,\delta} \frac{\rho^{\alpha-1}_{j,\delta} - r^{\alpha-1}}{\rho_{j,\delta} - r} + r^{\alpha-1}$ we get

$$\begin{aligned} &\widehat{W}_{\alpha,\delta}(r) \\ &= \frac{1}{\sum_{j=1}^{m+1} \frac{Q_1(\rho_{j,\delta})}{\prod_{i=1,i\neq j}^{m+1} (\rho_{i,\delta} - \rho_{j,\delta})} \left(-\lambda_2 \frac{(\widehat{f}_2(r) - \widehat{f}_2(\rho_{j,\delta}))}{\rho_{j,\delta} - r} + c + \eta^\alpha \frac{\rho^\alpha_{j,\delta} - r^\alpha}{\rho_{j,\delta} - r} \right)} \\ &= \frac{1}{\sum_{j=1}^{m+1} E(\rho_{j,\delta}) \left(-\lambda_2 \widehat{T}_{\rho_{j,\delta}} f_2(r) + c + \eta^\alpha \rho_{j,\delta} \frac{\rho^{\alpha-1}_{j,\delta} - r^{\alpha-1}}{\rho_{j,\delta} - r} + \eta^\alpha r^{\alpha-1} \right)} \\ &= \frac{1}{c + \eta^\alpha r^{\alpha-1} - \lambda_2 \sum_{j=1}^{m+1} E(\rho_{j,\delta}) \widehat{T}_{\rho_{j,\delta}} f_2(r) + \eta^\alpha \sum_{j=1}^{m+1} E(\rho_{j,\delta}) \rho_{j,\delta} \frac{\rho^{\alpha-1}_{j,\delta} - r^{\alpha-1}}{\rho_{j,\delta} - r}} \quad \text{(A.7)} \\ &= \frac{\frac{1}{\eta^\alpha \theta_\delta} \widehat{\nu}_{\alpha,\delta}(r)}{1 - \frac{1}{\theta_\delta} \left[\eta^{-\alpha} \widehat{g}_\delta(r) + \kappa_\delta \right] \widehat{\nu}_{\alpha,\delta}(r)} \end{aligned}$$

where the last equality is obtained by dividing the nominator and the denominator of (A.7) by $c + \eta^\alpha \kappa_\delta + \eta^\alpha r^{\alpha-1} + \eta^\alpha \sum_{j=1}^{m+1} E(\rho_{j,\delta}) \rho_{j,\delta} \widehat{T}_{\rho_{j,\delta}} \ell_\alpha(r)$. This shows a). To prove b) we define the function $\nu^*_{\alpha,\delta}(r) = (c + \eta^\alpha r^{\alpha-1} + \eta^\alpha \widehat{f}_{\alpha,\delta}(r))^{-1}$. From (5.2) and (A.7) we obtain

$$\widehat{\phi}(r) = \widehat{h}_{\alpha,\delta,\omega}(r) \nu^*_{\alpha,\delta}(r) (1 - \widehat{g}_\delta(r) \nu^*_{\alpha,\delta}(r))^{-1}. \quad \text{(A.8)}$$

First we consider the case of $\delta > 0$. We will show in this case that if $\omega(x, y) \equiv 1$, then

$$\frac{1}{r} \left[1 - \widehat{g}_\delta(r) \nu^*_{\alpha,\delta}(r) - \delta R \nu^*_{\alpha,\delta}(r) \right] = \widehat{h}_{\alpha,\delta,\omega}(r) \nu^*_{\alpha,\delta}(r), \quad \text{(A.9)}$$

where $R = \left(\prod_{i=1}^{N} q_i^{m_i}\right)\left(\prod_{j=1}^{m+1}\rho_{j,\delta}\right)^{-1}$. Using that $L_\alpha(\rho_{j,\delta}) = 0$ and Lemma 5.2 we obtain

$$\begin{aligned}\sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\delta}{\rho_{j,\delta}} &= \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^\alpha\rho_{j,\delta}^{\alpha-1} + \frac{\lambda_1\left(\widehat{f}_1(-\rho_{j,\delta})-1\right)}{\rho_{j,\delta}}\right]\\ &= \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^\alpha\rho_{j,\delta}^{\alpha-1}\right]\\ &\quad + \lambda_1\sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\widehat{f}_1(-\rho_{j,\delta})}{\rho_{j,\delta}} - \lambda_1\sum_{j=1}^{m+1}\frac{E(\rho_{j,\delta})}{\rho_{j,\delta}}. \end{aligned} \tag{A.10}$$

From Lemma 5.2 it follows that $\lambda_1\sum_{j=1}^{m+1}\frac{E(\rho_{j,\delta})}{\rho_{j,\delta}} = \lambda_1 R$. On the other hand, due to (2.1), $\widehat{f}_1(-\rho_{j,\delta}) = \frac{Q(-\rho_{j,\delta})}{\prod_{i=1}^{N}(q_i-\rho_{j,\delta})^{m_i}}$, hence from the definition of $E(\rho_{j,\delta})$ we obtain $\lambda_1\sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\widehat{f}_1(-\rho_{j,\delta})}{\rho_{j,\delta}} = \lambda_1\sum_{j=1}^{m+1}\frac{Q(-\rho_{j,\delta})}{\prod_{k=1,k\neq j}^{m+1}(\rho_{k,\delta}-\rho_{j,\delta})\rho_{j,\delta}}$. Since Q is a polynomial in $\rho_{j,\delta}$ of degree at most $m-1$ and constant term $\prod_{i=1}^{N} q_i^{m_i}$, it follows that $Q(-\rho_{j,\delta})\rho_{j,\delta}^{-1} = (\rho_{j,\delta})^{-1}\prod_{i=1}^{N} q_i^{m_i} + Q_0(-\rho_{j,\delta})$, where $Q_0(r)$ is a polynomial of degree at most $m-2$. Hence, applying (5.5) we get $\lambda_1\sum_{j=1}^{m+1} E(\rho_{j,\delta})\frac{\widehat{f}_1(-\rho_{j,\delta})}{\rho_{j,\delta}} = \lambda_1 R$. Using this and the equality $\lambda_1\sum_{j=1}^{m+1}\frac{E(\rho_{j,\delta})}{\rho_{j,\delta}} = \lambda_1 R$, (A.10) simplifies to

$$\delta R = \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^\alpha\rho_{j,\delta}^{\alpha-1}\right]. \tag{A.11}$$

Lemma 5.2 yields $1-\widehat{g}_\delta(r)\nu^*_{\alpha,\delta}(r) = \sum_{j=1}^{m+1} E(\rho_{j,\delta})\left(1-\lambda_2\widehat{T}_{\rho_{j,\delta}}f_2(r)\nu^*_{\alpha,\delta}(r)\right)$. From this equality and (A.11),

$$\begin{aligned}&\frac{1}{r}\left[1-\widehat{g}_\delta(r)\nu^*_{\alpha,\delta}(r) - \delta R\nu^*_{\alpha,\delta}(r)\right] \qquad \text{(A.12)}\\ &= \frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left\{1-\lambda_2\widehat{T}_{\rho_{j,\delta}}f_2(r)\nu^*_{\alpha,\delta}(r) - \left[\frac{\lambda_2\left(\widehat{f}_2(\rho_{j,\delta})-1\right)}{\rho_{j,\delta}} + c + \eta^\alpha\rho_{j,\delta}^{\alpha-1}\right]\nu^*_{\alpha,\delta}(r)\right\}.\end{aligned}$$

Using that $\nu^*_{\alpha,\delta}(r) = \dfrac{1}{c+\eta^\alpha r^{\alpha-1} + \eta^\alpha\widehat{f}_{\alpha,\delta}(r)}$ and Lemma 5.2,

$$\sum_{j=1}^{m+1} E(\rho_{j,\delta})\left[c + \eta^\alpha r^{\alpha-1} + \eta^\alpha\rho_{j,\delta}\frac{\rho_{j,\delta}^{\alpha-1} - r^{\alpha-1}}{\rho_{j,\delta}-r}\right]\nu^*_{\alpha,\delta}(r) = 1. \tag{A.13}$$

From (A.13) and (A.12) we obtain, for $r \neq \rho_{j,\delta}$,

$$\begin{aligned}
&\frac{1}{r}\left[1-g_\delta(r)\nu^*_{\alpha,\delta}(r)-\delta R\nu^*_{\alpha,\delta}(r)\right]\\
&= \frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left[c+\eta^\alpha r^{\alpha-1}+\eta^\alpha\rho_{j,\delta}\frac{\rho_{j,\delta}^{\alpha-1}-r^{\alpha-1}}{\rho_{j,\delta}-r}\right]\nu^*_{\alpha,\delta}(r)\\
&\quad -\frac{1}{r}\lambda_2\sum_{j=1}^{m+1}E(\rho_{j,\delta})\widehat{T}_{\rho_{j,\delta}}f_2(r)\nu^*_{\alpha,\delta}(r)\\
&\quad -\frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left(\frac{\lambda_2(\widehat{f}_2(\rho_{j,\delta})-1)}{\rho_{j,\delta}}+c+\eta^\alpha\rho_{j,\delta}^{\alpha-1}\right)\nu^*_{\alpha,\delta}(r)\\
&= \frac{1}{r}\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left\{\left(\eta^\alpha r\frac{\rho_{j,\delta}^{\alpha-1}-r^{\alpha-1}}{\rho_{j,\delta}-r}+\lambda_2 r\frac{\frac{1-\widehat{f}_2(r)}{r}-\frac{1-\widehat{f}_2(\rho_{j,\delta})}{\rho_{j,\delta}}}{\rho_{j,\delta}-r}\right)\nu^*_{\alpha,\delta}(r)\right\}\\
&= \sum_{j=1}^{m+1}E(\rho_{j,\delta})\left(\eta^\alpha\widehat{T}_{\rho_{j,\delta}}\ell_\alpha(r)+\lambda_2\widehat{T}_{\rho_{j,\delta}}\overline{F}_2(r)\right)\nu^*_{\alpha,\delta}(r).
\end{aligned}$$

Since $\xi_\omega(u)=\overline{F}_2(u)$ when $\omega(x,y)=1$, from (5.3), (5.1) and the above equality we obtain

$$\widehat{h}_{\alpha,\delta,\omega}(r)=\sum_{j=1}^{m+1}E(\rho_{j,\delta})\left(\eta^\alpha\widehat{T}_{\rho_{j,\delta}}\ell_\alpha(r)+\lambda_2\widehat{T}_{\rho_{j,\delta}}\overline{F}_2(r)\right)=\frac{1}{r}\frac{\left[1-\widehat{g}_\delta(r)\nu^*_{\alpha,\delta}(r)-\delta R\nu^*_{\alpha,\delta}(r)\right]}{\nu^*_{\alpha,\delta}(r)},$$

which proves (A.9). From (A.9) and (A.8) it follows that

$$\begin{aligned}
\widehat{\varphi}_\delta(r) &= \frac{1}{r}\frac{1-\widehat{g}_\delta(r)\nu^*_{\alpha,\delta}(r)-\delta R\nu^*_{\alpha,\delta}(r)}{1-\widehat{g}_\delta(r)\nu^*_{\alpha,\delta}(r)}=\frac{1}{r}-\frac{1}{r}\left[\frac{\delta R\nu^*_{\alpha,\delta}(r)}{1-\widehat{g}_\delta(r)\nu^*_{\alpha,\delta}(r)}\right]\\
&= \frac{1}{r}-\frac{1}{r}\left[\frac{\frac{\delta R}{\eta^\alpha}}{\theta_\delta+r^{\alpha-1}+\widehat{f}_{\alpha,\delta}(r)-\eta^{-\alpha}\widehat{g}_\delta(r)-\kappa_\delta}\right]\\
&= \frac{1}{r}-\frac{1}{r}\left[\frac{\frac{\delta R}{\eta^\alpha\theta_\delta}\nu_{\alpha,\delta}(r)}{1-\frac{1}{\theta_\delta}\left[\eta^{-\alpha}\widehat{g}_\delta(r)+\kappa_\delta\right]\nu_{\alpha,\delta}(r)}\right],\\
&= \frac{1}{r}-\frac{1}{r}\delta R\widehat{W}_{\alpha,\delta}(r),
\end{aligned}$$

where the last equality follows from (5.7). This shows b).

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