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A Fascinated Connection Between Partitions And Divisors

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Abstract : The partition function $p(n)$ plays a vital role in Additive Number Theory like as sum of divisor function $\sigma(n)$ in Multiplicative Number Theory. The two arithmetic functions are defined. Euler pentagonal number theorem and a recursive formula for $p(n)$ are discussed, while later Euler proved that $\sigma(n)$ also satisfy the same recursive relation as $p(n)$. In the last a formula connecting $p(n)$ and $\sigma(n)$ is being derived.

1. INTRODUCTION

Number Theory is one of the interesting, oldest and best loved areas of Mathematics. Number Theory can be divided into many branches but out of these main two branches: The Multiplicative Number Theory and The Additive Number Theory

Multiplicative Number Theory is too old, the most work was done on Multiplicative Number Theory some 2600 years ago. Additive Number Theory, on the other hand is much younger, going back to Euler, less than 300 years ago. Although it is an area of intensive research today.

In Multiplicative Number Theory we are concerned with how the positive integer can be written as product of primes. An important arithmetic function sum of divisor of 'n' is divisor function $\sigma(n)$

For example let $n = 6$

Then divisors of 6 are 1, 2, 3, 6

And $\sigma(6) = 1 + 2 + 3 + 6 = 12$

On the other hand Additive Number Theory deals with having, a positive integer can be written as sum of positive integers. One of the interesting topic is Partition of given integer 'n'

For example : Let $n = 6$ then partition of n is as follows

$$6 = 1 + 1 + 1 + 1 + 1 + 1$$
$$6 = 2 + 1 + 1 + 1 + 1$$
$$6 = 2 + 2 + 1 + 1$$
$$6 = 2 + 2 + 2$$
$$6 = 3 + 1 + 1 + 1$$
$$6 = 3 + 2 + 1$$
$$6 = 3 + 3$$
$$6 = 4 + 1 + 1$$
$$6 = 4 + 2$$
$$6 = 5 + 1$$
$$6 = 6$$

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There are eleven ways such that 6 can be written as sum of positive integers so $p(6) = 11$

For these two arithmetic functions there is one common recursive formula given by

$$p(n) = p(n-1) - p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots \quad (1)$$

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) + \ldots \quad (1)$$

2. THE PARTITION FUNCTION

The partition function $p(n)$ is very important in Number Theory. The partition function to be studied is the number of ways such that a positive integer 'n' can be written as sum of positive integers $\leq n$

i.e. number of solution of

$n = a_1 + a_2 + a_3 + \ldots + a_n$ Where $1 \leq a_i \leq n$ $\forall\ i = 1, 2, 3 \ldots n$

is called partition function and is denoted by $p(n)$

The number of summand is unrestricted, repetition is allowed and order of summands is not taken into account

For example: $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(10) = 42$, $p(15) = 176$, $p(22) = 1001$

While it is simple to find the value of $p(n)$ for small values of 'n' but for large value of 'n' $p(n)$ grows quickly very large, so for large value of 'n' there is an easier way to calculate the value of $p(n)$ given by recursive formula in (1) where $p(0) = 1$ and $p(n) = 0$ if $n < 0$ $n < 0$

For example : $p(8) = p(7) + p(6) - p(3) - p(1) = 22$

3. PARTITION FUNCTION THROUGH GENERATING FUNCTION

In 1748, Euler was the first who discovered generating function

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n$$

The infinite product on the left side generates the value of $p(n)$ as the coefficient of power series on the right side

4. EULER PENTAGONAL NUMBER THEOREM AND RECURSIVE FORMULA

In fact a careful analysis of generating function and some calculus techniques, Euler gave

$$\prod_{n=1}^{\infty} (1-x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n [x^{x(n)} + x^{w(-n)}] = \sum_{n=-\infty}^{\infty} (-1)^n x^{w(n)} \quad \ldots(3)$$

Where $w(n)\ \frac{3n^2 - n}{2}$ and $w(-n) = \frac{3n^2 + n}{2}$

And give recursive formula $p(n) = \sum_{k=1}^{\infty} \{p(n - 2(k)) + p(n - 2(k))\}$

where $w(k) = \frac{3k^2 - k}{2}$

Or

$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots$

Here $w(k)$ and $w(-k)$ are pentagonal numbers

If we introduce number e_n where $e_n = \begin{cases} 1 & n = 0 \\ (-1)^k & n = \dfrac{k(3k \pm 1)}{2} \\ 0 & \text{otherwise} \end{cases}$

From (3) we get $$g(x) = \prod_{n=1}^{\infty}(1-x^n)\sum_{n=0}^{\infty} e_n x^n$$

And recursive formula can be written as

$$p(n) = -\sum_{k=1}^{\infty} e_k\, p(n-k)$$

And as $$\left[\prod_{n=1}^{\infty}(1-x^n)\right]\left[\prod_{n=1}^{\infty}(1-x^n)^{-1}\right] = 1$$

So $$\left[\sum_{n=0}^{\infty} p(n)x^n\right]\left[\sum_{n=0}^{\infty} e_n x^n\right] = 1$$

Using standard Cauchy product $$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} e_k\, p(n-k)\right)x^n = 1 \qquad \text{...(4)}$$

Comparing coefficient we get $$\sum_{k=0}^{\infty} e_k\, p(n-k) = 1$$

For $n = 0$ and as $p(0) = 1$

5. THE DIVISOR FUNCTION

For positive integer n, the divisor function $\sigma(n)$ = sum of positive divisor of n. Here we are not surprising that partition satisfy a recursion relation, but we do not expect this of $\sigma(n)$. What do the divisor of n have to do with the divisors of $n-1$, $n-2$, ...? Euler showed that $\sigma(n)$ satisfy the same recursion relation that $\sigma(n)$ does, with only $\sigma(0)$ different from $p(0)$

For example : $\sigma(n) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 7$, $\sigma(10) = 18$, $\sigma(15) = 124$, $\sigma(22) = 36$

We use the convention $\sigma(0) = 0$, and if p is prime then $\sigma(p) = p + 1$ and $\sigma(p^2) = 1 + p + p^2 = \dfrac{p^3 - 1}{p - 1}$ continuing in this way we get some interesting results given below

Result–1 : For any prime p and positive integer k

$$\sigma(p^k) = 1 + p + p^2 + ... + p^k = \frac{p^k - 1}{p - 1}$$

Result–2 : σ is multiplicative *i.e.* if m & n are relatively prime then $\sigma(mn) = \sigma(m) = \sigma(n)$

Result–3: if $n = p_1^{k_1}\ p_2^{k_2}\, p_3^{k_3} \ldots p_m^{k_m}$

Then $\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1}\frac{p_2^{k_2+1}-1}{p_2-1}\cdots\frac{p_m^{k_m+1}-1}{p_m-1} = \prod_{i=1}^{m}\frac{p_i^{k_i+1}-1}{p_i-1}$

Where $p_1, p_2,...p_m$ are distinct primes and $k_1, k_2, ...\ k_m$ are positive integers

If n is large number then recursive formula given in (2) will be used to find the value

Result – 4 :

$$\sum_{n=1}^{\infty} n\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty}\sigma(n)x^n \qquad ...(5)$$

6. CONNECTION BETWEEN $p(n)$ AND $\sigma(n)$

As

$$g(x) = \prod_{n=1}^{\infty}(1-x^n)$$

Taking log on both sides and differentiating then multiplying by x, we get

$$x\frac{g'(x)}{g(x)} = -\sum_{n=1}^{\infty} n\frac{x^n}{1-x^n} \qquad ...(6)$$

And also as $g(x) = \sum_{n=0}^{\infty} e_n x^n$ which implies $xg'(x) = \sum_{n=0}^{\infty} ne_n x^n$

And also

$$\frac{1}{g(x)} = \sum_{n=0}^{\infty} p(n)x^n$$

So from equation (5) and (6)

$$\left(\sum_{n=0}^{\infty} p(n)x^n\right)\left(\sum_{n=0}^{\infty} ne_n x^n\right) = -\sum_{n=1}^{\infty}\sigma(n)x^n$$

i.e.

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n}(ke_k\, p(n-k))x^n = -\sum_{n=1}^{\infty}\sigma(n)x^n$$

Comparing coefficient both sides we get

$$\sigma(n) = -\sum_{k=0}^{\infty} ke_k\, p(n-k)$$

Which is connection between $p(n)$ and $\sigma(n)$

7. RECURSIVE RELATION FOR $\sigma(n)$

As

$$x\frac{g'(x)}{g(x)} = -\sum_{n=1}^{\infty} n\frac{x^n}{1-x^n} = -\sum_{n=1}^{\infty}\sigma(n)x^n - \sum_{n=0}^{\infty}\sigma(n)x^n$$

i.e.

$$-xg'(x) = g(x)\sum_{n=0}^{\infty}\sigma(n)x^n$$

And as

$$g(x) = \sum_{n=0}^{\infty} e_n x^n$$

This implies $$-\sum_{n=0}^{\infty} ne_n x^n = \left(\sum_{n=0}^{\infty} e_n x^n\right)\left(\sum_{n=0}^{\infty} \sigma(n) x^n\right)$$

Equating coefficient of x^n we get $$-ne_n = \sum_{k=0}^{n} e_{n-k}\sigma(k)$$

i.e. $$\sigma(n) - ne_n = \sum_{k=0}^{n-1} e_{n-k}\sigma(k) \qquad \text{... (7)}$$

This gives recursive relation of $\sigma(n)$

8. FORMULA RELATES σ(*n*) AND *p*(*n*)

Let $$\mathrm{F}(x) = \frac{1}{g(x)}$$

Then $$\mathrm{F}'(x) = -\frac{g'(x)}{(g(x))^2} = -\frac{g'(x)}{g(x)}\mathrm{F}(x) \qquad \text{... (8)}$$

As $$-\frac{g(x)}{g(x)} = \sum_{n=1}^{\infty} \sigma(n) x^{n-1}$$

and $$\mathrm{F}(x) = \sum_{n=0}^{\infty} p(n) x^n$$

and $$\mathrm{F}'(x) = \sum_{n=0}^{\infty} np(n) x^{n-1}$$

So from (8) we get $$\sum_{n=0}^{\infty} np(n) x^{n-1} = \left(\sum_{n=0}^{\infty} \sigma(n) x^n\right)\left(\sum_{n=0}^{\infty} p(n) x^n\right)$$

Multiplying these infinite series on the right hand side and comparing coefficients we get

$$np(n) = \sum_{k=0}^{n} \sigma(k)\, p(n-k)$$

Which gives the formula for $p(n)$ and $\sigma(n)$

9. A FINAL LOOK AT THE UNION OF TWO FUNCTION

We now take the surprising union of $p(n)$ and $\sigma(n)$ given by almost identical relation

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots$$

And

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) + \ldots$$

and equation (7) gives

$$\sigma(n) = p(n-1) + 2p(n-2) - 5p(n-5) - 7\,p(n-7) + 12\,p(n-12) + 15p(n-15) - \ldots$$

And equation (9) gives

$$np(n) = \sigma(1)p(n-1) - \sigma(2)p(n-2) + \sigma(3)p(n-3) + \ldots$$

These recursion relations are so simple even pentagonal numbers are not needed

10. CONCLUSION

The partition function $p(n)$ of additive number theory satisfy a recursive formula. It is not surprising, but one cannot expect this of sum divisor function $\sigma(n)$ of Multiplicative Number Theory. The mystery is only partially revealed how can the problem of expressing a number as a sum, be at all related to expressing same number as product ? The only difference between two is $p(0) \neq \sigma(0)$. Euler was first who gave remarkable relation connecting a function of Multiplicative Number Theory with one of the Additive Number Theory.

11. REFERENCES

1. Andrews, G.E., The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Vol. 2, Addison-Wesley, Reading, 1976 (Reissued: Cambridge University Press, Cambridge, 1985).
2. G. H. Hardy, and E. M. Wright, An Introduction to the Theory of Numbers, 5th ed., Clarendon Press, 1979
3. Niven, Zuckerman & Montgomery ,An Introduction to the Theory of Numbers , 5th ed., New York: John Wiley and Sons,1991
4. T. M. Apostol , Introduction to Analytic Number Theory , Springer-Verlag (New York),1976
5. G. E. Andrews, Euler's pentagonal number theorem, Mathematics Magazine 56 (1983) 279–284
6. David ,M.Burton,Elementry Number Theory,2nd Edition ,UBS Publishers.
7. H. Rademacher, On the expansion of the partition function in a series, Ann. of Math. 4(1943), 416–422.
8. K. Knopp, Theory and Application of Infinite Series, Dover, 1990
9. R. M. Young, Excursions in Calculus, An Interplay of the Continuous and the Discrete, MAA, 1992