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STATIONARY SOLUTIONS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN THE SPACE OF TEMPERED DISTRIBUTIONS

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Abstract. In Rajeev (Indian J. Pure Appl. Math. **44** (2013), no. 2, 231–258.), 'Translation invariant diffusion in the space of tempered distributions', it was shown that there is an one-to-one correspondence between solutions of a class of finite dimensional stochastic differential equations (SDEs) and solutions of a class of stochastic parial differential equations (SPDEs) in the space of tempered distributions, which are driven by the same Brownian motion. Coefficients of the SDEs were related to coefficients of the SPDEs through convolution with the initial value of the SPDEs.

In this paper, we consider the situation where solutions of the SDEs are stationary and ask whether solutions of the corresponding SPDEs are also stationary. We provide an affirmative answer, when the initial random variable takes value in a certain set $\mathcal{C}$, which ensures that coefficients of the SDEs are related to coefficients of the SPDEs in the above manner.

1. Introduction

The topology of infinite dimensional spaces plays a major role in the study of stochastic differential equations (SDEs) in those spaces. These topological vector spaces are usually taken to be countably Hilbertian nuclear spaces (see [14, 16]) and in particular real separable Hilbert spaces (see [3, 4, 7]). In [21], a correspondence was shown between finite dimensional SDEs and stochastic partial differential equations (SPDEs) in $\mathcal{S}'(\mathbb{R}^d)$ (where $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions) via an Itô formula. The results there involves deterministic initial conditions in some Hermite Sobolev space $\mathcal{S}_p(\mathbb{R}^d)$. In this paper we extend this correspondence to random initial conditions. Assuming the existence of stationary solutions of finite dimensional SDEs, we show the existence of stationary solutions of infinite dimensional SPDEs, via an Itô formula which is used in proving the correspondence.

1.1. Main results. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions. Let $\{B_t\}$ be a d dimensional $(\mathcal{F}_t)$ standard Brownian motion. By $(\mathcal{F}_t^B)$ we denote the filtration generated by $\{B_t\}$. Let δ be an

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arbitrary state, viewed as an isolated point of $\hat{\mathcal{S}}_p(\mathbb{R}^d) := \mathcal{S}_p(\mathbb{R}^d) \cup \{\delta\}$. For an initial condition $\psi \in \mathcal{S}_p(\mathbb{R}^d)$, consider the SPDE

$$dY_t = A(Y_t)\, dB_t + L(Y_t)\, dt; \quad Y_0 = \psi, \tag{1.1}$$

where

(i) the operators $A := (A_1, \cdots, A_d), L$ on $\mathcal{S}_p(\mathbb{R}^d)$ as follows: for $\phi \in \mathcal{S}_p(\mathbb{R}^d)$

$$A_i\phi := -\sum_{j=1}^{d} \langle \sigma\, ,\, \phi\rangle_{ji}\, \partial_j\phi,\ i = 1, \cdots, d \tag{1.2}$$

and

$$L\phi := \frac{1}{2}\sum_{i,j=1}^{d} (\langle \sigma\, ,\, \phi\rangle\, \langle \sigma\, ,\, \phi\rangle^t)_{ij}\, \partial_{ij}^2\phi - \sum_{i=1}^{d} \langle b\, ,\, \phi\rangle_i\, \partial_i\phi, \tag{1.3}$$

(ii) $\sigma = (\sigma_{ij})_{d\times d}, b = (b_1, b_2, \cdots, b_d)$ with $\sigma_{ij}, b_i \in \mathcal{S}_{-p}(\mathbb{R}^d), \forall i, j = 1, 2, \cdots, d$. For any $\phi \in \mathcal{S}_p(\mathbb{R}^d)$, by $\langle \sigma\, ,\, \phi\rangle$ we denote the $d \times d$ matrix with entries $\langle \sigma\, ,\, \phi\rangle_{ij} := \langle \sigma_{ij}\, ,\, \phi\rangle$. Similarly $\langle b\, ,\, \phi\rangle$ is a vector in $\mathbb{R}^d$ with $\langle b\, ,\, \phi\rangle_i := \langle b_i\, ,\, \phi\rangle$.

Given $\psi \in \mathcal{S}_p(\mathbb{R}^d)$, let $\bar{\sigma}(\cdot\,; \psi) : \mathbb{R}^d \to \mathbb{R}^{d^2}$ and $\bar{b}(\cdot\,; \psi) : \mathbb{R}^d \to \mathbb{R}^d$ be denote the functions $\bar{\sigma}(x; \psi) := (\langle \sigma_{ij}\, ,\, \tau_x\psi\rangle)$ and $\bar{b}(x; \psi) := (\langle b_i\, ,\, \tau_x\psi\rangle)$. Let $\tau_x, x \in \mathbb{R}^d$ denote the translation operators (see Section 2). The next result is about the existence and uniqueness of a strong solution of (1.1).

Theorem 1.1 ([21, Theorem 3.4 and Lemma 3.6])**.** *Let $\psi, \sigma_{ij}, b_i, \{B_t\}$ be as above. Suppose that the functions $x \mapsto \bar{\sigma}(x; \psi)$ and $x \mapsto \bar{b}(x; \psi)$ are locally Lipschitz. Then equation* (1.1) *has a unique $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued $(\mathcal{F}_t^B)$ adapted strong solution given by*

$$Y_t = \tau_{Z_t}(\psi), \ \ \textit{for}\ \ 0 \le t < \eta,$$

where η is an $(\mathcal{F}_t^B)$ adapted stopping time and $\{Z_t\}$ solves the SDE

$$dZ_t = \bar{\sigma}(Z_t; \psi)\, dB_t + \bar{b}(Z_t; \psi)\, dt; \quad Z_0 = 0. \tag{1.4}$$

We extend this result in Theorem 3.12 and Theorem 3.17 which allows initial conditions to be random. Furthermore, in Theorem 4.2, assuming the existence of stationary solutions of the finite dimensional SDEs, we show the existence of stationary solutions of the corresponding SPDEs. In Theorem 4.5, under an additional assumption on the initial condition of the SPDE, viz. that it takes values in a specific set, we show that stationary solutions of (possibly) unrelated finite dimensional SDEs can be lifted to stationary solutions of the given SPDE. Note that in this result, we do not require the SDEs to be in correspondence with the SPDE. However, this is done at the cost of the additional assumption on the initial condition.

1.2. Layout of the paper. In this paper, we use the same techniques as those used in [21]. In Section 2, we list basic properties of Hermite Sobolev spaces which are used throughout the paper.

In Section 3, we extend the Itô formula [20, Theorem 2.3] to Theorem 3.2 involving random initial conditions and in Theorem 3.4 and Theorem 3.6 prove existence and uniqueness results for the solutions of finite dimensional SDE (3.7). These results allow us to extend the said correspondence (see Theorem 3.12, Theorem 3.14, Theorem 3.17). Note that the equation for Z involves the initial condition for Y i.e. Y_0, but with $Z_0 = 0$. To discuss existence and uniqueness for Z, we use a Lipschitz-type criteria, which depends on Y_0. We need control on the norm of Y_0 to make the usual proof via Picard iteration work. In Proposition 3.15 and Proposition 3.16 we prove $\mathcal{L}^2$ estimates on the supremum of the norms of the solutions of (3.1), in terms of the initial condition.

In Proposition 4.1, we show that $\mathbb{R}^d$ valued stationary processes can be lifted to $\mathcal{S}_p(\mathbb{R}^d)$ valued stationary processes via the translations operators τ_x. We then show in Theorem 4.2 the existence of a stationary solution of infinite dimensional SPDE (4.3) given that the corresponding finite dimensional SDE (3.7) has a stationary solution. We present a method to lift stationary solutions of (possibly) unrelated finite dimensional SDEs to stationary solutions of (4.3). We do this by describing conditions on a random variable ξ that appears in the initial condition of (4.3). We define a subset $\mathcal{C}$ of the Hermite Sobolev space with the following property: if the random variable ξ takes values in the set $\mathcal{C}$, then the corresponding finite dimensional SDEs are all the same. This property is observed in Lemma 4.4 and using which, in Theorem 4.5, we construct stationary solutions of SPDEs in our class. To guarantee non-explosion for finite dimensional SDEs with locally Lipschitz coefficients, we use a 'Liapunov' type criteria (see [25, 7.3.14 Corollary]). Two examples of stationary solutions are given in Example 4.7 and in Proposition 4.8, we obtain $\mathcal{L}^1$ estimates on the supremum of the norms of the stationary solutions, in terms of the initial condition.

2. Topologies on $\mathcal{S}$ and $\mathcal{S}'$

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of smooth rapidly decreasing $\mathbb{R}$-valued functions on $\mathbb{R}^d$ with the topology given by L. Schwartz (see [27]) and let $\mathcal{S}'(\mathbb{R}^d)$ denote the dual space, known as the space of tempered distributions. For any $p \in \mathbb{R}$, let $\mathcal{S}_p(\mathbb{R}^d)$ be the completion of $\mathcal{S}(\mathbb{R}^d)$ in the inner product $\langle \cdot\,,\,\cdot \rangle_p$ which is defined in terms of the $\mathcal{L}^2(\mathbb{R}^d)$ inner product $\langle \cdot\,,\,\cdot \rangle$ (see [14, Chapter 1.3] for the details). The spaces $\mathcal{S}_p(\mathbb{R}^d), p \in \mathbb{R}$ are separable Hilbert spaces and are known as the Hermite-Sobolev spaces. We write $\mathcal{S}, \mathcal{S}', \mathcal{S}_p$ instead of $\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}), \mathcal{S}_p(\mathbb{R})$.

Note that $\mathcal{S}_0(\mathbb{R}^d) = \mathcal{L}^2(\mathbb{R}^d)$ and for $p > 0$, $\mathcal{S}_p(\mathbb{R}^d) \subset \mathcal{L}^2(\mathbb{R}^d)$ (i.e. these distributions are given by functions) and $(\mathcal{S}_{-p}(\mathbb{R}^d), \|\cdot\|_{-p})$ is dual to $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$. Furthermore,

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{p \in \mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{p \in \mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p).$$

Given $\psi \in \mathcal{S}(\mathbb{R}^d)$ (or $\mathcal{S}_p(\mathbb{R}^d)$) and $\phi \in \mathcal{S}'(\mathbb{R}^d)$ (or $\mathcal{S}_{-p}(\mathbb{R}^d)$), the action of ϕ on ψ will be denoted by $\langle \phi\,,\,\psi \rangle$.

Let $\{h_n : n \in \mathbb{Z}^d_+\}$ be the Hermite functions (see [14, Chapter 1.3]), where $\mathbb{Z}^d_+ := \{n = (n_1, \cdots, n_d) : n_i \text{ non-negative integers}\}$. If $n = (n_1, \cdots, n_d)$, we define $|n| := n_1 + \cdots + n_d$. Note that $\{h^p_n : n \in \mathbb{Z}^d_+\}$ forms an orthonormal basis for $\mathcal{S}_p(\mathbb{R}^d)$, where $h^p_n := (2|n| + d)^{-p} h_n$.

Consider the derivative maps denoted by $\partial_i : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ for $i = 1, \cdots, d$. We extend these maps by duality to $\partial_i : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ as follows: for $\psi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\langle \partial_i \psi \, , \, \phi \rangle := - \langle \psi \, , \, \partial_i \phi \rangle \, , \, \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

Let $\{e_i : i = 1, \cdots, d\}$ be the standard basis vectors in $\mathbb{R}^d$. Then for any multi-index $n = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$ we have (see [12, Appendix A.5])

$$\partial_i h_n = \sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i + 1}{2}} h_{n+e_i},$$

with the convention that for a multi-index $n = (n_1, \cdots, n_d)$, if $n_i < 0$ for some i, then $h_n \equiv 0$. Above recurrence implies that $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator.

For $x \in \mathbb{R}^d$, let τ_x denote the translation operators on $\mathcal{S}(\mathbb{R}^d)$ defined by $(\tau_x \phi)(y) := \phi(y - x), \forall y \in \mathbb{R}^d$. Extend this operator to $\tau_x : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \tau_x \phi \, , \, \psi \rangle := \langle \phi \, , \, \tau_{-x} \psi \rangle \, , \, \forall \psi \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 2.1. *The translation operators $\tau_x, x \in \mathbb{R}^d$ have the following properties:*

(a) ([22, Theorem 2.1]) *For $x \in \mathbb{R}^d$ and any $p \in \mathbb{R}$, $\tau_x : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is a bounded linear map. In particular, there exists a real polynomial P_k of degree $k = 2([|p|] + 1)$ such that*

$$\|\tau_x \phi\|_p \leq P_k(|x|) \|\phi\|_p, \, \forall \phi \in \mathcal{S}_p(\mathbb{R}^d),$$

where $|x|$ denotes the standard Euclidean norm of x.

(b) For $x \in \mathbb{R}^d$ and any $i = 1, \cdots, d$ we have $\tau_x \partial_i = \partial_i \tau_x$.

(c) Fix $\phi \in \mathcal{S}_p(\mathbb{R}^d)$. Then $x \mapsto \tau_x \phi$ is continuous.

Proof. We verify part (b) for elements of $\mathcal{S}(\mathbb{R}^d)$ and then extend to elements of $\mathcal{S}'(\mathbb{R}^d)$ via duality. Proof of part (c) is contained in the proof of [23, Proposition 3.1]. □

On $\mathcal{S}(\mathbb{R}^d)$ consider the multiplication operators $M_i, i = 1, \cdots, d$ defined by

$$(M_i \phi)(x) := x_i \phi(x), \, \phi \in \mathcal{S}(\mathbb{R}^d), x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By duality, extend these operators to $M_i : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. By [12, Appendix A.5, equation (A.26)], $x_i h_n(x) = \sqrt{\frac{n_i+1}{2}} h_{n+e_i}(x) + \sqrt{\frac{n_i}{2}} h_{n-e_i}(x)$ and hence $M_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator, for any $p \in \mathbb{R}$. For dimension $d = 1$, we write M_x instead of M_1.

3. Stochastic Differential Equations in $\mathcal{S}'$

We use the following terminology and notations. We say $\{\eta_n\}$ is a localizing sequence, if each η_n is an $(\mathcal{F}_t)$ stopping time with $\eta_n \uparrow \infty$. We use stochastic integration in the Hilbert spaces $\mathcal{S}_p(\mathbb{R}^d)$ (as in [21]). Note that $\mathcal{S}_p(\mathbb{R}^d)$ valued stochastic integrals $\int_0^t G_s \, dX_s$ can be defined for $\mathcal{S}_p(\mathbb{R}^d)$ valued predictable, locally norm-bounded processes $\{G_t\}$ and real semimartingales $\{X_t\}$. For any $x \in \mathbb{R}^n$, by $|x|$ we denote the standard Euclidean norm of x. The dimension will be clear from the context where this notation is used.

Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$-measurable random variable. Now consider the SPDE

$$dY_t = A(Y_t)\, dB_t + L(Y_t)\, dt; \quad Y_0 = \xi, \tag{3.1}$$

where the operators $A = (A_1, \cdots, A_d), L$ are as in (1.2), (1.3). We want to extend the results of [21, Section 3] to the case of $\mathcal{S}_p(\mathbb{R}^d)$ valued random initial conditions. We have the following Itô formula (see [20, Theorem 2.3]).

Proposition 3.1. *Let $p \in \mathbb{R}$ and $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued continuous $(\mathcal{F}_t)$ adapted semimartingale. Then we have the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.*

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\phi\, d[X^i, X^j]_s,\ \forall t \geq 0. \tag{3.2}$$

We need to extend above result to allow random ϕ.

Theorem 3.2. *Let $p \in \mathbb{R}$. Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|\xi\|_p^2 < \infty$. Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued continuous semimartingale. Then we have the following equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$, a.s.*

$$\tau_{X_t}\xi = \tau_{X_0}\xi - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i, X^j]_s,\ \forall t \geq 0. \tag{3.3}$$

Proof. Fix $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then $\phi \in \mathcal{S}_{-p+1}(\mathbb{R}^d)$ and by the previous Proposition, we have in $\mathcal{S}_{-p}(\mathbb{R}^d)$ a.s. for all $t \geq 0$

$$\tau_{-X_t}\phi = \tau_{-X_0}\phi + \sum_{i=1}^d \int_0^t \partial_i \tau_{-X_s}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{-X_s}\phi\, d[X^i, X^j]_s.$$

Then a.s.

$$\begin{aligned} \langle \xi\,,\, \tau_{-X_t}\phi\rangle = \langle \xi\,,\, \tau_{-X_0}\phi\rangle &+ \left\langle \xi\,,\, \sum_{i=1}^d \int_0^t \partial_i \tau_{-X_s}\phi\, dX_s^i \right\rangle \\ &+ \left\langle \xi\,,\, \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{-X_s}\phi\, d[X^i, X^j]_s \right\rangle,\ \forall t \geq 0. \end{aligned} \tag{3.4}$$

Now using [20, Proposition 1.3(a)] and Lemma 2.1(b), we obtain

$$\left\langle \xi\,,\, \sum_{i=1}^d \int_0^t \partial_i \tau_{-X_s}\phi\, dX_s^i \right\rangle = \left\langle -\sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i\,,\, \phi \right\rangle$$

and

$$\left\langle \xi\,,\, \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{-X_s}\phi\, d[X^i, X^j]_s \right\rangle = \left\langle \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i, X^j]_s\,,\, \phi \right\rangle.$$

Using (3.4) we get a P-null set $\mathcal{N}$ such that for $\omega \in \Omega \setminus \mathcal{N}$ and for any multi-index $n = (n_1, \cdots, n_d)$ we have

$$\langle \tau_{X_t}\xi\,,\,h_n\rangle = \langle \tau_{X_0}\xi\,,\,h_n\rangle - \left\langle \sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i\,,\,h_n \right\rangle + \left\langle \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i, X^j]_s\,,\,h_n \right\rangle,\ \forall t \geq 0,$$

where h_n are the Hermite functions which form a total set in $\mathcal{S}_{p-1}(\mathbb{R}^d)$. Since $\{\tau_{X_t}\xi - \tau_{X_0}\xi + \sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i - \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i, X^j]_s\}$ is an $\mathcal{S}_{p-1}(\mathbb{R}^d)$ valued process, we have the equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$ a.s.

$$\tau_{X_t}\xi - \tau_{X_0}\xi + \sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i, X^j]_s = 0,\ t \geq 0.$$

This completes the proof. □

Alternative proof of Theorem 3.2. We make two observations, including a property of stochastic integrals, viz. (3.5).

(a) Given any $\mathcal{F}_0$-measurable set F, an $\mathcal{S}_p(\mathbb{R}^d)$ valued predictable step process $\{G_t\}$ and a continuous $\mathbb{R}^d$ valued semimartingale $\{X_t\}$, we have a.s.

$$\mathbb{1}_F \int_0^t G_s\, dX_s = \int_0^t \mathbb{1}_F G_s\, dX_s,\ t \geq 0. \tag{3.5}$$

Extend the above equality to the case involving $\mathcal{S}_p(\mathbb{R}^d)$ valued norm-bounded predictable process $\{G_t\}$.

(b) Given any $\mathcal{F}_0$-measurable set F, $\phi \in \mathcal{S}_p(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned} \langle \mathbb{1}_F \tau_x \phi\,,\,\psi\rangle &= \mathbb{1}_F\,\langle \tau_x\phi\,,\,\psi\rangle = \mathbb{1}_F\,\langle \phi\,,\,\tau_{-x}\psi\rangle \\ &= \langle \mathbb{1}_F\phi\,,\,\tau_{-x}\psi\rangle = \langle \tau_x(\mathbb{1}_F\phi)\,,\,\psi\rangle \end{aligned} \tag{3.6}$$

and hence $\mathbb{1}_F\tau_x\phi = \tau_x(\mathbb{1}_F\phi)$. Similarly $\mathbb{1}_F\tau_x\phi = \tau_{\mathbb{1}_F x}(\mathbb{1}_F\phi)$.

Using Proposition 3.1 and equations (3.5), (3.6), we establish Theorem 3.2 when X is bounded and ξ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$-measurable simple function. When ξ is square integrable, the result is proved by approximating ξ using $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$-measurable simple functions. Finally, via localization under stopping times, we prove the result for unbounded X. □

We need an existence and uniqueness of solution to the following SDE:

$$dZ_t = \bar{\sigma}(Z_t;\xi)\, dB_t + \bar{b}(Z_t;\xi)\, dt; \quad Z_0 = \zeta, \tag{3.7}$$

where ξ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$-measurable random variable and ζ is an $\mathbb{R}^d$ valued $\mathcal{F}_0$-measurable random variable. Unless stated otherwise, we assume that both ξ, ζ are independent of the Brownian motion $\{B_t\}$. Let $(\mathcal{G}_t)$ denote the filtration generated by ξ, ζ and $\{B_t\}$. Let $\mathcal{G}_\infty$ denote the smallest sub σ-field of $\mathcal{F}$ containing

$\mathcal{G}_t$ for all $t \geq 0$. Let $\mathcal{G}_\infty^P$ be the P-completion of $\mathcal{G}_\infty$ and let $\mathcal{N}^P$ be the collection of all P-null sets in $\mathcal{G}_\infty^P$. Define

$$\mathcal{F}_t^{\xi,\zeta} := \bigcap_{s>t} \sigma(\mathcal{G}_s \cup \mathcal{N}^P),\ t \geq 0,$$

where $\sigma(\mathcal{G}_s \cup \mathcal{N}^P)$ denotes the smallest σ-field generated by the collection $\mathcal{G}_s \cup \mathcal{N}^P$. This filtration satisfies the usual conditions. $\mathcal{F}_\infty^{\xi,\zeta}$ will denote the σ field generated by the collection $\bigcup_{t\geq 0} \mathcal{F}_t^{\xi,\zeta}$. If ζ is a constant, then we write $(\mathcal{F}_t^{\xi})$ instead of $(\mathcal{F}_t^{\xi,\zeta})$.

Proposition 3.3. *Suppose the following conditions are satisfied:*

(i) ξ *is norm-bounded in* $\mathcal{S}_p(\mathbb{R}^d)$*, i.e. there exists a constant* $K > 0$ *such that* $\|\xi\|_p \leq K$.

(ii) $\mathbb{E}|\zeta|^2 < \infty$.

(iii) (Globally Lipschitz in x, locally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$*, the functions* $x \mapsto \bar{\sigma}(x;y)$ *and* $x \mapsto \bar{b}(x;y)$ *are globally Lipschitz functions in* x *and the Lipschitz coefficient is independent of* y *when* y *varies over any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$*; i.e. for any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$ *there exists a constant* $C(G) > 0$ *such that for all* $x_1, x_2 \in \mathbb{R}^d, y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \leq C(G)|x_1 - x_2|.$$

Then SDE (3.7) *has a continuous* $(\mathcal{F}_t^{\xi,\zeta})$ *adapted strong solution* $\{X_t\}$ *with the property that* $\mathbb{E}\int_0^T |X_t|^2\, dt < \infty$ *for any* $T > 0$*. Pathwise uniqueness of solutions also holds, i.e. if* $\{X_t^1\}$ *is another solution, then* $P(X_t = X_t^1,\ t \geq 0) = 1$.

Proof. We follow the proof in [19, Theorem 5.2.1] with appropriate modifications. First we show the uniqueness of the strong solution.

Let $\{Z_t^1\}$ and $\{Z_t^2\}$ be two strong solutions of (3.7). Define two processes

$$a(t,\omega) := \bar{\sigma}(Z_t^1(\omega);\xi(\omega)) - \bar{\sigma}(Z_t^2(\omega);\xi(\omega)),$$
$$\gamma(t,\omega) := \bar{b}(Z_t^1(\omega);\xi(\omega)) - \bar{b}(Z_t^2(\omega);\xi(\omega)).$$

Since ξ is norm-bounded, then by our hypothesis

$$|a(t,\omega)|^2 \leq C^2 \left|Z_t^1(\omega) - Z_t^2(\omega)\right|^2,\ |\gamma(t,\omega)|^2 \leq C^2 \left|Z_t^1(\omega) - Z_t^2(\omega)\right|^2$$

with $C = C(Range(\xi))$. Using Itô isometry and Cauchy-Schwarz Inequality, we get

$$\begin{aligned}\mathbb{E}\left|Z_t^1 - Z_t^2\right|^2 &\leq 2\,\mathbb{E}\int_0^t |a(s)|^2\, ds + 2t\,\mathbb{E}\int_0^t |\gamma(s)|^2\, ds \\ &\leq 2C^2(1+t)\int_0^t \mathbb{E}\left|Z_s^1 - Z_s^2\right|^2\, ds\end{aligned} \tag{3.8}$$

By Gronwall's inequality, $\mathbb{E}\left|Z_t^1 - Z_t^2\right|^2 = 0, \forall t \geq 0$. This proves the uniqueness.

For the existence of solution we use a Picard type iteration. Set $Z_t^{(0)} = \zeta$ and then successively define

$$Z_t^{(k+1)} := \zeta + \int_0^t \bar{\sigma}(Z_s^{(k)};\xi)\, dB_s + \int_0^t \bar{b}(Z_s^{(k)};\xi)\, ds,\ \forall k \geq 0. \tag{3.9}$$

Fix any compact time interval $[0, N]$. For $k \geq 1, t \in [0, N]$ we have

$$\mathbb{E}|Z_t^{(k+1)} - Z_t^{(k)}|^2 \leq 2C^2(1+N) \int_0^t \mathbb{E}|Z_s^{(k)} - Z_s^{(k-1)}|^2 \, ds. \tag{3.10}$$

Using the Lipschitz continuity for any $x \in \mathbb{R}^d, y \in Range(\xi)$ we have, $|\bar{\sigma}(x;y) - \bar{\sigma}(0;y)| + |\bar{b}(x;y) - \bar{b}(0;y)| \leq C|x|$. But $|\bar{\sigma}(0;y)| = |\langle \sigma \, , \, y \rangle| \leq \|\sigma_{ij}\|_{-p} \|y\|_p$ and $|\bar{b}(0;y)| = |\langle b \, , \, y\rangle| \leq \|b_i\|_{-p}\|y\|_p$. This shows $\bar{\sigma}, \bar{b}$ has linear growth in x, i.e. there exists a constant $D = D(Range(\xi)) > 0$ such that $|\bar{\sigma}(x;y)| \leq D(1+|x|), |\bar{b}(x;y)| \leq D(1+|x|)$ for $x \in \mathbb{R}^d, y \in Range(\xi)$. Since $Z_t^{(0)} = \zeta$, using (3.10) we get

$$\begin{aligned} \mathbb{E}|Z_t^{(1)} - Z_t^{(0)}|^2 &\leq 2\,\mathbb{E}\int_0^t |\bar{\sigma}(\zeta;\xi)|^2 \, ds + 2t\,\mathbb{E}\int_0^t |\bar{b}(\zeta;\xi)|^2 \, ds \\ &\leq 4D^2(1+N)(1+\mathbb{E}|\zeta|^2)t, \, \forall t \in [0,N]. \end{aligned} \tag{3.11}$$

Now we use an induction on k with (3.10) as the recurrence relations and (3.11) as our base step. Then there exists a constant $R > 0$ such that

$$\mathbb{E}|Z_t^{(k+1)} - Z_t^{(k)}|^2 \leq \frac{(Rt)^{k+1}}{(k+1)!}, \forall k \geq 0, t \in [0,N]. \tag{3.12}$$

Let λ denote the Lebesgue measure on $[0, N]$. We are going to show that the iteration converges in $\mathcal{L}^2(\lambda \times P)$ and the limit satisfy (3.7). As in [19, Theorem 5.2.1] we show

$$\|Z^{(m)} - Z^{(n)}\|_{\mathcal{L}^2(\lambda\times P)} = \sum_{k=n}^{m-1} \left(\frac{(RN)^{k+2}}{(k+2)!} \right)^{\frac{1}{2}},$$

where m, n are positive integers with $m > n$. Observe that $\|Z^{(m)} - Z^{(n)}\|_{\mathcal{L}^2(\lambda\times P)} \to 0$ as $m, n \to \infty$. Using completeness of $\mathcal{L}^2(\lambda \times P)$ we have a limit, which we denote by $\{X_t\}_{t\in[0,N]}$. Using (3.12), we also have $\lim_{n\to\infty} Z_t^{(n)} \overset{\mathcal{L}^2(P)}{=} X_t$ for each $t \in [0, N]$.

This $\{X_t\}$ is measurable and $(\mathcal{F}_t^{\xi,\zeta})$ adapted. Now using the linear growth of $x \mapsto \bar{\sigma}(x; y)$ (for every fixed $y \in \mathcal{S}_p(R^d)$) we have

$$\begin{aligned} \mathbb{E}\int_0^N \bar{\sigma}(X_s;\xi)^2 \, ds &\leq D^2\,\mathbb{E}\int_0^N (1+|X_s|)^2 \, ds \\ &\leq 2D^2\,\mathbb{E}\int_0^N (1+|X_s|^2)\, ds = 2D^2N + 2D^2\|X\|^2_{\mathcal{L}^2(\lambda\times P)} < \infty. \end{aligned}$$

One defines stochastic integrals with respect to a Brownian motion for adapted integrands satisfying the above integrability condition (see [17, Chapter 3, Remark 2.11]). Hence $\{\int_0^t \bar{\sigma}(X_s;\xi)\, dB_s\}_{t\in[0,N]}$ exists. Since $\mathbb{E}\int_0^N |X_s|^2\, ds < \infty$, we have $\int_0^N |X_s|^2 \, ds < \infty$ almost surely. Using linear growth of $x \mapsto \bar{b}(x;y)$ (for every fixed $y \in \mathcal{S}_p(R^d)$) and Cauchy-Schwarz inequality, existence of the process $\{\int_0^t \bar{b}(X_s;\xi)\, ds\}_{t\in[0,N]}$ is established. Using Itô isometry and Lipschitz continuity of $\bar{\sigma}$ we get

$$\mathbb{E}\left|\int_0^t \bar{\sigma}(Z_s^{(k)};\xi)\,dB_s - \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s\right|^2 \leq C^2\,\mathbb{E}\int_0^N |Z_s^{(k)} - X_s|^2\,ds.$$

Using Jensen's inequality and the Lipschitz property of $\bar{b}$ we get

$$\mathbb{E}\left|\int_0^t \bar{b}(Z_s^{(k)};\xi)\,ds - \int_0^t \bar{b}(X_s;\xi)\,ds\right|^2 \leq C^2 N\,\mathbb{E}\int_0^N |Z_s^{(k)} - X_s|^2\,ds.$$

Using above estimates, for each $t \in [0,N]$ we have

$$\int_0^t \bar{\sigma}(Z_s^{(k)};\xi)\,dB_s \xrightarrow[k\to\infty]{\mathcal{L}^2(P)} \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s$$

and

$$\int_0^t \bar{b}(Z_s^{(k)};\xi)\,dB_s \xrightarrow[k\to\infty]{\mathcal{L}^2(P)} \int_0^t \bar{b}(X_s;\xi)\,dB_s.$$

From (3.9) we conclude that for each $t \in [0,N]$, a.s.

$$X_t = \zeta + \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s + \int_0^t \bar{b}(X_s;\xi)\,ds.$$

By [19, Theorem 3.2.5], the integral $\int_0^t \bar{\sigma}(X_s;\xi)\,dB_s$ has a continuous version. We denote the continuous version of $\{\zeta + \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s + \int_0^t \bar{b}(X_s;\xi)\,ds\}_{t\in[0,N]}$ by $\{\widetilde{X}_t\}_{t\in[0,N]}$. Then for each $t \in [0,N]$, a.s.

$$\widetilde{X}_t = \zeta + \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s + \int_0^t \bar{b}(X_s;\xi)\,ds = X_t, \text{ a.s.}$$

In particular, for all $t \in [0,N]$ we have $\mathbb{E}|X_t - \widetilde{X}_t|^2 = 0$. Then using the Lipschitz property of $\bar{\sigma}$, $\int_0^t \bar{\sigma}(X_s;\xi)\,dB_s = \int_0^t \bar{\sigma}(\widetilde{X}_s;\xi)\,dB_s$ a.s. We also obtain $\int_0^t \bar{b}(X_s;\xi)\,dB_s = \int_0^t \bar{b}(\widetilde{X}_s;\xi)\,dB_s$ a.s. for each $t \in [0,N]$. Then for each $t \in [0,N]$, a.s.

$$\widetilde{X}_t = \zeta + \int_0^t \bar{\sigma}(\widetilde{X}_s;\xi)\,dB_s + \int_0^t \bar{b}(\widetilde{X}_s;\xi)\,ds, \text{ a.s.}$$

Since $\{\widetilde{X}_t\}$ is continuous, we have, a.s.

$$\widetilde{X}_t = \zeta + \int_0^t \bar{\sigma}(\widetilde{X}_s;\xi)\,dB_s + \int_0^t \bar{b}(\widetilde{X}_s;\xi)\,ds,\ t \in [0,N].$$

So we have obtained a continuous $(\mathcal{F}_t^{\xi,\zeta})$ adapted solution for time interval $[0,N]$ for any positive integer N. The uniqueness of this continuous solution follows from the proof of uniqueness given at the beginning of this proof.

Let $\{X_t^{(N)}\}$ and $\{X_t^{(N+1)}\}$ be the solutions up to time N and $N+1$ respectively. Then $\{X_{t\in[0,N]}^{(N+1)}\}$ is also a continuous solution up to time N and hence by the uniqueness, is indistinguishable from $\{X_t^{(N)}\}$ on $[0,N]$. Using this consistency, we patch up the solutions $\{X_t^{(N)}\}$ to obtain the solution of (3.7) on the time interval $[0,\infty)$. ∎

We now come to a main result regarding the existence and uniqueness of solutions of (3.7).

Theorem 3.4. *Suppose the following are satisfied:*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) $\zeta = c$, *where* c *is some element in* $\mathbb{R}^d$.

(iii) (Globally Lipschitz in x, locally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, *the functions* $x \mapsto \bar{\sigma}(x;y)$ *and* $x \mapsto \bar{b}(x;y)$ *are globally Lipschitz functions in* x *and the Lipschitz coefficient is independent of* y *when* y *varies over any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$*; i.e. for any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$ *there exists a constant* $C(G) > 0$ *such that for all* $x_1, x_2 \in \mathbb{R}^d, y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \leq C(G)|x_1 - x_2|.$$

Then SDE (3.7) *has a continuous* $(\mathcal{F}_t^\xi)$ *adapted strong solution* $\{X_t\}$ *such that there exists a localizing sequence of stopping times* $\{\eta_n\}$ *with* $\mathbb{E}\int_0^{T\wedge\eta_n} |X_t|^2\, dt < \infty$ *for any* $T > 0$. *Pathwise uniqueness of solutions also holds, i.e. if* $\{\tilde{X}_t\}$ *is another solution, then* $P(X_t = \tilde{X}_t,\, t \geq 0) = 1$.

Remark 3.5. Theorem 3.4 is also true if ζ is an $\mathbb{R}^d$ valued $\mathcal{F}_0$-measurable square integrable random variable, which is also independent of the Brownian motion $\{B_t\}$. However, we only need the version for $\zeta = 0$, which is used in Theorem 3.12.

Proof. For all positive integers k, define $\xi^{(k)} := \xi \mathbb{1}_{(\|\xi\|_p \leq k)}$. Note that $\xi^{(k)} \xrightarrow[\mathcal{L}^2]{k\to\infty} \xi$ and the convergence is also almost sure. Also, $\mathbb{1}_{(\|\xi\|_p \leq k)}\xi^{(k+1)} = \xi^{(k)}$. By (3.6), we have for any $x \in \mathbb{R}^d, y \in \mathcal{S}_p(\mathbb{R}^d), F \in \mathcal{F}$,

$$\begin{aligned} \mathbb{1}_F\bar{\sigma}(x;y) &= \bar{\sigma}(x;\mathbb{1}_F y) = \bar{\sigma}(\mathbb{1}_F x;\mathbb{1}_F y),\\ \mathbb{1}_F\bar{b}(x;y) &= \bar{b}(x;\mathbb{1}_F y) = \bar{b}(\mathbb{1}_F x;\mathbb{1}_F y). \end{aligned} \tag{3.13}$$

By Proposition 3.3 we have the $(\mathcal{F}_t^{\xi^{(k)}})$ adapted (and hence $(\mathcal{F}_t^\xi)$ adapted) strong solution denoted by $\{Z_t^{(k)}\}$, satisfying a.s.

$$Z_t^{(k)} = c + \int_0^t \bar{\sigma}(Z_s^{(k)};\xi^{(k)})\, dB_s + \int_0^t \bar{b}(Z_s^{(k)};\xi^{(k)})\, ds,\ t \geq 0.$$

Using (3.5) and (3.13), we have a.s. for all $t \geq 0$

$$\begin{aligned} \mathbb{1}_{(\|\xi\|_p \leq k)} Z_t^{(k)} = \mathbb{1}_{(\|\xi\|_p \leq k)} c &+ \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k)};\xi^{(k)})\, dB_s\\ &+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k)};\xi^{(k)})\, ds. \end{aligned}$$

and

$$\begin{aligned} \mathbb{1}_{(\|\xi\|_p \leq k)} Z_t^{(k+1)} = \mathbb{1}_{(\|\xi\|_p \leq k)} c &+ \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k+1)};\mathbb{1}_{(\|\xi\|_p \leq k)}\xi^{(k+1)})\, dB_s\\ &+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k+1)};\mathbb{1}_{(\|\xi\|_p \leq k)}\xi^{(k+1)})\, ds \end{aligned}$$

$$= \mathbb{1}_{(\|\xi\|_p \le k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \le k)} Z_s^{(k+1)}; \xi^{(k)})\, dB_s$$
$$+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \le k)} Z_s^{(k+1)}; \xi^{(k)})\, ds.$$

Using the uniqueness obtained in Proposition 3.3 (applied to $(\mathcal{F}_t^\xi)$ adapted processes), we have a.s.

$$\mathbb{1}_{(\|\xi\|_p \le k)} Z_t^{(k+1)} = \mathbb{1}_{(\|\xi\|_p \le k)} Z_t^{(k)},\ t \ge 0, \tag{3.14}$$

with the null set possibly depending on k. Let $\tilde{\Omega}_k$ be the set of probability 1 where the above relation holds. Then on $\Omega' := \bigcap_{k=1}^\infty \tilde{\Omega}_k$, which is a set of probability 1, (3.14) holds for all k.

Note that $(\|\xi\|_p < \infty) = \Omega$ and hence for any $\omega \in \Omega$, there exists a positive integer k such that $\|\xi(\omega)\|_p \le k$. Then write $\Omega' = \bigcup_{k=1}^\infty (\Omega' \cap (\|\xi\|_p \le k))$. Now Ω' is an element of $\mathcal{F}$ with probability 1 and hence $(\Omega')^c$ is a null set in $\mathcal{F}$. Since $(\mathcal{F}_t)$ satisfies the usual conditions, we have $(\Omega')^c \in \mathcal{F}_0$ and hence $\Omega' \in \mathcal{F}_0$.

We define a process $\{X_t\}$ as follows: for any $t \ge 0$

$$X_t(\omega) := \begin{cases} Z_t^{(k)}(\omega), \text{ if } \omega \in \Omega' \cap (\|\xi\|_p \le k),\ k = 1, 2, \cdots \\ 0, \text{ if } \omega \in (\Omega')^c. \end{cases}$$

From equation (3.14), $Z_t^{(k+1)} = Z_t^{(k)}$, $\forall t \ge 0$ on $\Omega' \cap (\|\xi\|_p \le k)$ and hence $\{X_t\}$ is well-defined. Furthermore $\{X_t\}$ is $(\mathcal{F}_t^\xi)$ adapted and has continuous paths. We now show that $\{X_t\}$ solves equation (3.7). On Ω' we have

$$\mathbb{1}_{(\|\xi\|_p \le k)} X_t = \mathbb{1}_{(\|\xi\|_p \le k)} Z_t^{(k)},\ \forall t \ge 0, k = 1, 2, \cdots \tag{3.15}$$

i.e. above relation holds almost surely. Then for each $k = 1, 2, \cdots$, a.s. $t \ge 0$

$$\begin{aligned} \mathbb{1}_{(\|\xi\|_p \le k)} X_t &= \mathbb{1}_{(\|\xi\|_p \le k)} Z_t^{(k)} \\ &= \mathbb{1}_{(\|\xi\|_p \le k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \le k)} Z_s^{(k)}; \xi^{(k)})\, dB_s \\ &\quad + \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \le k)} Z_s^{(k)}; \xi^{(k)})\, ds \\ &= \mathbb{1}_{(\|\xi\|_p \le k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \le k)} X_s; \xi^{(k)})\, dB_s \\ &\quad + \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \le k)} X_s; \xi^{(k)})\, ds, \text{ (using (3.15))} \\ &= \mathbb{1}_{(\|\xi\|_p \le k)} c + \int_0^t \mathbb{1}_{(\|\xi\|_p \le k)} \bar{\sigma}(X_s; \xi)\, dB_s \\ &\quad + \int_0^t \mathbb{1}_{(\|\xi\|_p \le k)} \bar{b}(X_s; \xi)\, ds, \text{ (using (3.13))} \\ &= \mathbb{1}_{(\|\xi\|_p \le k)} c + \mathbb{1}_{(\|\xi\|_p \le k)} \int_0^t \bar{\sigma}(X_s; \xi)\, dB_s \end{aligned}$$

$$+ \mathbb{1}_{(\|\xi\|_p \le k)} \int_0^t \bar{b}(X_s;\xi)\, ds, \text{ (using (3.5))}$$

Let $\bar{\Omega}_k$ denote the set of probability 1 where the above relation holds. Then $\bar{\Omega} := \bigcap_{k=1}^{\infty} \bar{\Omega}_k$ is also a set of probability 1 and on $\bar{\Omega}$, for all $k = 1, 2, \cdots$ and for all $t \ge 0$

$$\mathbb{1}_{(\|\xi\|_p \le k)} X_t = \mathbb{1}_{(\|\xi\|_p \le k)} \left(c + \int_0^t \bar{\sigma}(X_s;\xi)\, dB_s + \int_0^t \bar{b}(X_s;\xi)\, ds \right).$$

Then on $\bar{\Omega} \cap (\|\xi\|_p \le k)$ we have for all $t \ge 0$

$$X_t = c + \int_0^t \bar{\sigma}(X_s;\xi)\, dB_s + \int_0^t \bar{b}(X_s;\xi)\, ds.$$

But $\bar{\Omega} \cap (\|\xi\|_p \le k) \uparrow \bar{\Omega}$ and hence on $\bar{\Omega}$ above relation holds for all $t \ge 0$. So $\{X_t\}$ is a solution of (3.7). Taking $\eta_n := \inf\{t \ge 0 : |X_t| \ge n\}$ it follows that $\mathbb{E} \int_0^{t \wedge \eta_n} |X_t|^2\, dt < \infty$ for any $t > 0$.

To prove the uniqueness, let $\{\tilde{X}_t\}$ be a continuous $(\mathcal{F}_t^{\xi})$ adapted strong solution of (3.7). Then a.s. for all $t \ge 0$

$$\begin{aligned}
&\mathbb{1}_{(\|\xi\|_p \le k)} \tilde{X}_t \\
&= \mathbb{1}_{(\|\xi\|_p \le k)} \left(c + \int_0^t \bar{\sigma}(\tilde{X}_s;\xi)\, dB_s + \int_0^t \bar{b}(\tilde{X}_s;\xi)\, ds \right) \\
&= \mathbb{1}_{(\|\xi\|_p \le k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \le k)} \tilde{X}_s;\xi^{(k)})\, dB_s + \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \le k)} \tilde{X}_s;\xi^{(k)})\, ds.
\end{aligned}$$

From the uniqueness obtained in Proposition 3.3 and equation (3.15), we now conclude a.s. for all $t \ge 0$, $\mathbb{1}_{(\|\xi\|_p \le k)} \tilde{X}_t = \mathbb{1}_{(\|\xi\|_p \le k)} Z_t^{(k)} = \mathbb{1}_{(\|\xi\|_p \le k)} X_t$. Since $(\|\xi\|_p \le k) \uparrow \Omega$, this proves $P(X_t = \tilde{X}_t,\, t \ge 0) = 1$. □

In Theorem 3.4 we assume locally Lipschitz nature of the coefficients $\bar{\sigma}, \bar{b}$ instead of those being globally Lipschitz. The extension from globally Lipschitz to locally Lipschitz is a well-known technique in literature (see [15, Theorem 18.3 and the discussion in page 340 about explosion], [24, Chapter IX, Exercise 2.10], [13, Theorem 2.3 and 3.1]). The one point compactification of $\mathbb{R}^d$ is denoted by $\widehat{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$. We state the next result without proof.

Theorem 3.6. *Suppose the following are satisfied:*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) $\zeta = 0$.

(iii) (Locally Lipschitz in x, locally in y) for any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$ the functions $x \mapsto \bar{\sigma}(x;y)$ and $x \mapsto \bar{b}(x;y)$ are locally Lipschitz functions in x and the Lipschitz coefficient is independent of y when y varies over any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$; i.e. for any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$ and any positive integer n there exists a constant $C(G,n) > 0$ such that for all $x_1, x_2 \in B(0,n), y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \le C(G,n)|x_1 - x_2|,$$

where $B(0,n) = \{x \in \mathbb{R}^d : |x| \le n\}$.

Then there exists an $(\mathcal{F}^\xi_t)$ *stopping time* η *and an* $(\mathcal{F}^\xi_t)$ *adapted* $\widehat{\mathbb{R}^d}$ *valued process* $\{X_t\}$ *such that*

(a) $\{X_t\}$ *solves SDE* (3.7) *upto* η *i.e. a.s.*
$$X_t = \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s + \int_0^t \bar{b}(X_s;\xi)\,ds,\ 0 \le t < \eta$$
and $X_t = \infty$ *for* $t \ge \eta$.

(b) $\{X_t\}$ *has continuous paths on the interval* $[0,\eta)$.

(c) $\eta = \lim_m \theta_m$ *where* $\{\theta_m\}$ *are* $(\mathcal{F}^\xi_t)$ *stopping times defined by* $\theta_m := \inf\{t \ge 0 : |X_t| \ge m\}$.

This is also pathwise unique in this sense: if $(\{X'_t\},\eta')$ *is another solution satisfying* $(a),(b),(c)$, *then* $P(X_t = X'_t, 0 \le t < \eta \wedge \eta') = 1$.

In Proposition 3.8 we show that stronger assumption on ξ implies a 'local Lipschitz' condition. We use this result to obtain Theorem 3.9, which is a version of Theorem 3.6. Fix $p > d + \frac{1}{2}$ and $y \in \mathcal{S}_p(\mathbb{R}^d)$. By [23, Theorem 4.1], $\delta_x \in \mathcal{S}_{-p}(\mathbb{R}^d)$, $\forall x \in \mathbb{R}^d$. Hence $x \mapsto \langle \delta_x\,,\,y\rangle : \mathbb{R}^d \to \mathbb{R}$ is well-defined. Abusing notation, we denote this function by y. Next result is about the continuity and differentiability of the function y.

Proposition 3.7. *Let* p, y *be as above. Then the first order partial derivatives of function* y *exist and the distribution* y *is given by the differentiable function* y. *Furthermore, the first order distributional derivatives of* y *are given by the first order partial derivatives of* y, *which are continuous functions.*

Proof. Express the tempered distribution y as $y \overset{\mathcal{S}_p(\mathbb{R}^d)}{=} \sum_{k=0}^\infty \sum_{|n|=k} y_n h_n$ for some $y_n \in \mathbb{R}$. Note that

(1) The Hermite functions h_n are uniformly bounded (see [26]).
(2) From $\|y\|_p^2 = \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{2p} y_n^2$, we get $|y_n| \le \|y\|_p (2|n|+d)^{-p}$.
(3) By [6, Chapter II, Section 5], the cardinality $\#\{n \in \mathbb{Z}^d_+ : |n| = \binom{k+d-1}{d-1}$. Hence $\#\{n \in \mathbb{Z}^d_+ : |n| = k\} \le C'.(2k+d)^{d-1}$ for some $C' > 0$.

Using these estimates we show that the convergences of $\sum_{k=0}^\infty \sum_{|n|=k} y_n h_n(x)$ and $\sum_{k=0}^\infty \sum_{|n|=k} y_n \partial_i h_n(x)$ are uniform in x. The required continuity and differentiability follows from properties of uniform convergence. □

Proposition 3.8. *Let* $p > d + \frac{1}{2}$ *and* $\sigma \in \mathcal{S}_{-p}(\mathbb{R}^d)$. *Then for any bounded set* G *in* $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ *and any positive integer* n *there exists a constant* $C(G,n) > 0$ *such that for all* $x_1, x_2 \in B(0,n), y \in G$
$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| \le C(G,n)|x_1 - x_2|,$$
where $B(0,n) = \{x \in \mathbb{R}^d : |x| \le n\}$.

Proof. Let $x^1 = (x^1_1,\cdots,x^1_d), x^2 = (x^2_1,\cdots,x^2_d) \in B(0,n)$. Then for any $y \in \mathcal{S}_p(\mathbb{R}^d)$,
$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| \le \|\sigma\|_{-p}\|\tau_{x_1}y - \tau_{x_2}y\|_p. \tag{3.16}$$
The target of the subsequent computations is to obtain an estimate of $\|\tau_{x_1}y - \tau_{x_2}y\|_p$. By Proposition 3.7, first order distributional derivatives of y are given by

the first order partial derivatives of y, which are continuous functions. For any $1 \leq i \leq d$ and $t = \lambda_i x_i^1 + (1-\lambda_i)x_i^2$ with $\lambda_i \in [0,1]$ we have

$$|(x_1^1, \cdots, x_{i-1}^1, t, x_{i+1}^2, \cdots, x_d^2)| \leq 2n.$$

Let $y \in \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$. Then by Lemma 2.1, there exist constants $C_n > 0, \tilde{C}_n > 0$ independent of i such that

$$\begin{aligned} \|\tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)}\partial_i y\|_p &= \|\partial_i \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} y\|_p \\ &\leq C_n \, \|\tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} y\|_{p+\frac{1}{2}} \\ &\leq \tilde{C}_n \, \|y\|_{p+\frac{1}{2}} \end{aligned} \tag{3.17}$$

The following is an equality of continuous functions.

$$\begin{aligned} &\tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^1,x_{i+1}^2,\cdots,x_d^2)} y(\cdot) - \tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^2,x_{i+1}^2,\cdots,x_d^2)} y(\cdot) \\ &= \int_{x_i^2}^{x_i^1} \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y(\cdot)\, dt. \end{aligned}$$

In view of (3.17), we have the equality of distributions in $\mathcal{S}_p(\mathbb{R}^d)$

$$\begin{aligned} &\tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^1,x_{i+1}^2,\cdots,x_d^2)} y - \tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^2,x_{i+1}^2,\cdots,x_d^2)} y \\ &= \int_{x_i^2}^{x_i^1} \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y\, dt \end{aligned}$$

and

$$\begin{aligned} &\|\tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^1,x_{i+1}^2,\cdots,x_d^2)} y - \tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^2,x_{i+1}^2,\cdots,x_d^2)} y\|_p \\ &\leq \left| \int_{x_i^2}^{x_i^1} \|\tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y\|_p\, dt \right| \leq \tilde{C}_n \, \|y\|_{p+\frac{1}{2}} \, |x_i^1 - x_i^2|. \end{aligned}$$

Now

$$\begin{aligned} \tau_{x_1} y - \tau_{x_2} y = &\ \tau_{(x_1^1,\cdots,x_{d-1}^1,x_d^1)} y - \tau_{(x_1^1,\cdots,x_{d-1}^1,x_d^2)} y \\ &+ \tau_{(x_1^1,\cdots,x_{d-1}^1,x_d^2)} y - \tau_{(x_1^1,\cdots,x_{d-2}^1,x_{d-1}^2,x_d^2)} y \\ &+ \cdots \\ &+ \tau_{(x_1^1,x_2^2,\cdots,x_d^2)} y - \tau_{(x_1^2,\cdots,x_d^2)} y \end{aligned}$$

and hence $\|\tau_{x_1} y - \tau_{x_2} y\|_p \leq \tilde{C}_n \, \|y\|_{p+\frac{1}{2}} \sum_{i=1}^d |x_i^1 - x_i^2| \leq d\tilde{C}_n \, \|y\|_{p+\frac{1}{2}} \, |x_1 - x_2|$. Using this estimate in (3.16), we have $|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| \leq d\tilde{C}_n \, \|\sigma\|_{-p} \|y\|_{p+\frac{1}{2}} \, |x_1 - x_2|$. In particular, if G is a bounded set in $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$, then for any $y \in G$

$$\|\tau_{x_1} y - \tau_{x_2} y\|_p \leq d\tilde{C}_n \, \|\sigma\|_{-p} \sup_{y \in G} (\|y\|_{p+\frac{1}{2}}) \, |x_1 - x_2|,$$

i.e. the function $x \mapsto \bar{\sigma}(x; y)$ is locally Lipschitz in x for any $y \in G$ and the Lipschitz constant can be taken uniformly in $y \in G$. □

Using Proposition 3.8, we get the following version of Theorem 3.6.

Theorem 3.9. *Let $p > d + \frac{1}{2}$. Suppose the following are satisfied:*

(1) $\sigma, b \in \mathcal{S}_{-p}(\mathbb{R}^d)$.

(2) ξ *is* $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ *valued and* $\mathbb{E}\|\xi\|^2_{p+\frac{1}{2}} < \infty$.
(3) $\zeta = 0$.

Then there exists an $(\mathcal{F}^\xi_t)$ *stopping time* η *and an* $(\mathcal{F}^\xi_t)$ *adapted* $\widehat{\mathbb{R}^d}$ *valued process* $\{X_t\}$ *such that*

(a) $\{X_t\}$ *solves SDE* (3.7) *upto* η *i.e. a.s.*

$$X_t = \int_0^t \bar{\sigma}(X_s;\xi)\,dB_s + \int_0^t \bar{b}(X_s;\xi)\,ds,\ 0 \le t < \eta$$

and $X_t = \infty$ *for* $t \ge \eta$.

(b) $\{X_t\}$ *has continuous paths on the interval* $[0,\eta)$.

(c) $\eta = \lim_m \theta_m$ *where* $\{\theta_m\}$ *are* $(\mathcal{F}^\xi_t)$ *stopping times defined by* $\theta_m := \inf\{t \ge 0 : |X_t| \ge m\}$.

This is also pathwise unique in this sense: if (X'_t, η') *is another solution satisfying* $(a),(b),(c)$*, then* $P(X_t = X'_t, 0 \le t < \eta \wedge \eta') = 1$.

We are ready to prove the main result of this section. We make two definitions extending [21, Definition 3.1 and Definition 3.3]. Note that ξ is assumed to be independent of the Brownian motion $\{B_t\}$ and $\hat{\mathcal{S}}_p(\mathbb{R}^d) = \mathcal{S}_p(\mathbb{R}^d) \cup \{\delta\}$, where δ is an isolated point (as described in Section 1).

Definition 3.10. (A) We say $\{Y_t\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued *strong solution* of SPDE (3.1), if $\{Y_t\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $(\mathcal{F}^\xi_t)$ adapted continuous process such that a.s. the following equality holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$,

$$Y_t = \xi + \int_0^t A(Y_s)\,dB_s + \int_0^t L(Y_s)\,ds;\ t \ge 0.$$

(B) By an $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued *strong local solution* of SPDE (3.1), we mean a pair $(\{Y_t\},\eta)$ where η is an $(\mathcal{F}^\xi_t)$ stopping time and $\{Y_t\}$ an $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued $(\mathcal{F}^\xi_t)$ adapted continuous process such that

(1) for all $\omega \in \Omega$, the map $Y_\cdot(\omega) : [0,\eta(\omega)) \to \mathcal{S}_p(\mathbb{R}^d)$ is continuous and $Y_t(\omega) = \delta$, $t \ge \eta(\omega)$.

(2) a.s. the following equality holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$,

$$Y_t = \xi + \int_0^t A(Y_s)\,dB_s + \int_0^t L(Y_s)\,ds;\ 0 \le t < \eta.$$

Definition 3.11. (A) We say strong solutions of SPDE (3.1) are *pathwise unique* if given any two $\mathcal{S}_p(\mathbb{R}^d)$ valued strong solutions $\{Y^1_t\}$ and $\{Y^2_t\}$, we have $P(Y^1_t = Y^2_t,\ t \ge 0) = 1$.

(B) We say strong local solutions of SPDE (3.1) are *pathwise unique* if given any two $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued strong solutions $(\{Y^1_t\},\eta^1)$ and $(\{Y^2_t\},\eta^2)$, we have $P(Y^1_t = Y^2_t,\ 0 \le t < \eta^1 \wedge \eta^2) = 1$.

Now we prove the existence and uniqueness of solutions to (3.1).

Theorem 3.12. *Suppose the following conditions are satisfied:*

(i) $\mathbb{E}\|\xi\|^2_p < \infty$.

(ii) (Globally Lipschitz in x, locally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, the functions $x \mapsto \bar{\sigma}(x;y)$ and $x \mapsto \bar{b}(x;y)$ are globally Lipschitz functions in x and the Lipschitz coefficient is independent of y when y varies over any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$; i.e. for any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$ there exists a constant $C(G) > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d, y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \leq C(G)|x_1 - x_2|.$$

Then SPDE (3.1) has an $(\mathcal{F}_t^\xi)$ adapted continuous strong solution. The solutions are pathwise unique.

To prove the above result, we first show a characterization of the solution of equation (3.1) in Theorem 3.14. We also require the monotonicity inequality described below. Let $\rho = (\rho_{ij})$ be a constant $d \times r$ matrix with $(a_{ij}) = (\rho\rho^t)_{ij}$ and $\theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d$. For $\phi \in \mathcal{S}$, we define

$$\begin{aligned} \widetilde{L}\phi &:= \tfrac{1}{2}\textstyle\sum_{i,j=1}^d a_{ij}\partial_{ij}^2\phi - \sum_{i=1}^d \theta_i\partial_i\phi, \\ \widetilde{A}_i\phi &:= -\textstyle\sum_{j=1}^d \rho_{ji}(\partial_j\phi),\ i = 1, \cdots, r \\ \widetilde{A}\phi &:= (\widetilde{A}_1\phi, \ldots, \widetilde{A}_r\phi) \end{aligned}$$

Theorem 3.13 ([9, Theorem 2.1 and Remark 3.1])**.** *For every $p \in \mathbb{R}$ there exists a constant $C = C(p, d, (\rho_{ij}), (\theta_i)) > 0$, such that*

$$2\left\langle \phi\,,\, \widetilde{L}\phi \right\rangle_p + \|\widetilde{A}\phi\|_{HS(p)}^2 \leq C.\|\phi\|_p^2, \tag{3.18}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $\|\widetilde{A}\phi\|_{HS(p)}^2 := \sum_{i=1}^r \|\widetilde{A}_i\phi\|_p^2$. Furthermore, by density arguments the above inequality extends to all $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$. The constant C depends on $(\rho_{ij}), (\theta_i)$ through the upper bound of $|\rho_{ij}|, |\theta_i|$ and hence the inequality extends to the case where ρ, θ are bounded processes parametrized by some set.

Theorem 3.14. *Let $\xi, \bar{\sigma}, \bar{b}$ be as in Theorem 3.12. Let $\{Y_t\}$ be an $(\mathcal{F}_t^\xi)$ adapted $\mathcal{S}_p(\mathbb{R}^d)$ valued strong solution of SPDE (3.1). Define a process $\{Z_t\}$ as follows:*

$$Z_t := \int_0^t \langle \sigma\,,\, Y_s\rangle\; dB_s + \int_0^t \langle b\,,\, Y_s\rangle\; ds,\ t \geq 0.$$

Then a.s. $Y_t = \tau_{Z_t}\xi$ for $t \geq 0$ and consequently, Z solves SDE (3.7) with $Z_0 = 0$.

Proof. This is an extension of [21, Lemma 3.6] to random initial condition ξ. The arguments are similar and we include the details for completeness. First we define linear operator valued processes $\{\bar{L}(t)\}$ and $\{\bar{A}_j(t)\}$, $j = 1, \cdots, d$. For $\phi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\bar{L}(t,\omega)\phi := \frac{1}{2}\sum_{i,j=1}^d (\langle \sigma\,,\, Y_t(\omega)\rangle\, \langle \sigma\,,\, Y_t(\omega)\rangle^t)_{ij}\, \partial_{ij}^2\phi - \sum_{i=1}^d (\langle b\,,\, Y_t(\omega)\rangle)_i\, \partial_i\phi,$$

$$\bar{A}_j(t,\omega)\phi := -\sum_{i=1}^d (\langle \sigma\,,\, Y_t(\omega)\rangle)_{ij}\, \partial_i\phi.$$

Note that $\bar{L}(t,\omega), \bar{A}_j(t,\omega)$ are linear operators from $\mathcal{S}_p(\mathbb{R}^d)$ to $\mathcal{S}_{p-1}(\mathbb{R}^d)$. We write $Z_t = (Z_t^1, \cdots, Z_t^d)$ and $\bar{A}(t) = (\bar{A}_1(t), \cdots, \bar{A}_d(t))$. By Theorem 3.2, we have the following equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$: a.s. $t \geq 0$

$$\tau_{Z_t}\xi = \xi + \int_0^t \bar{A}(s)(\tau_{Z_s}\xi)\, dB_s + \int_0^t \bar{L}(s)(\tau_{Z_s}\xi)\, ds.$$

Since $\{Y_t\}$ is a solution of (3.1), we also have a.s. $t \geq 0$

$$Y_t = \xi + \int_0^t \bar{A}(s)(Y_s)\, dB_s + \int_0^t \bar{L}(s)(Y_s)\, ds.$$

Define a localizing sequence $\{\eta_n\}$ as

$$\eta_n := \inf\{t \geq 0 : |\langle \sigma_{ij}\,,\, Y_t\rangle| \geq n, \text{ or} \,|\langle b_i\,,\, Y_t\rangle| \geq n,\, i,j = 1,\cdots,d\},\; n \geq 1.$$

Now define $X_t^{(n)} := Y_{t\wedge\eta_n} - \tau_{Z_{t\wedge\eta_n}}\xi$. Recall that $\langle\cdot\,,\cdot\rangle$ denotes the duality action of $\mathcal{S}_{-p}$ on $\mathcal{S}_p$ and h_k denotes Hermite functions. For all multi-index k, we have $\{\langle X_s^{(n)}, h_k\rangle\}$ is a continuous semimartingale and $\|X_t^{(n)}\|_{p-1}^2 = \sum_k (2|k| + d)^{2(p-1)}\langle X_s^{(n)}, h_k\rangle^2$. Then applying Itô formula, we get a.s.

$$\begin{aligned}\|X_t^{(n)}\|_{p-1}^2 = &\int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\,,\, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)}\\ &+ \int_0^{t\wedge\eta_n} \left[2\left\langle X_s^{(n)}\,,\, \bar{L}(s)X_s^{(n)}\right\rangle_{p-1} + \|\bar{A}(s)X_s^{(n)}\|_{HS(p-1)}^2\right] ds,\end{aligned}$$

where $\{\int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\,,\, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)}\}$ is a continuous martingale and the coefficients $\{\langle\sigma_{ij}\,,\, Y_t\rangle\}$ and $\{\langle b_i\,,\, Y_t\rangle\}$ are uniformly bounded for $t \leq \eta_n$. Since the coefficients are bounded, by Theorem 3.13, there exists a constant $C_n > 0$ such that a.s.

$$\begin{aligned}\|X_{t\wedge\eta_n}^{(n)}\|_{p-1}^2 &\leq \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\,,\, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)} + C_n\int_0^{t\wedge\eta_n} \|X_s^{(n)}\|_{p-1}^2\, ds\\ &\leq \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\,,\, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)} + C_n\int_0^{t} \|X_{s\wedge\eta_n}^{(n)}\|_{p-1}^2\, ds.\end{aligned}$$

Taking expectation, we obtain $\mathbb{E}\|X_{t\wedge\eta_n}^{(n)}\|_{p-1}^2 \leq C_n\int_0^t \mathbb{E}\|X_{s\wedge\eta_n}^{(n)}\|_{p-1}^2\, ds$ for all $t \geq 0$. By the Gronwall's inequality we get $\mathbb{E}\|X_{t\wedge\eta_n}^{(n)}\|_{p-1}^2 = 0$, which implies the equality a.s. $Y_{t\wedge\eta_n} = \tau_{Z_{t\wedge\eta_n}}\xi, t \geq 0$. Since $\eta_n \uparrow \infty$, we have a.s. $Y_t = \tau_{Z_t}\xi, t \geq 0$. This implies a.s. $t \geq 0$

$$\begin{aligned}Z_t &= \int_0^t \langle\sigma\,,\, Y_s\rangle\; dB_s + \int_0^t \langle b\,,\, Y_s\rangle\; ds\\ &= \int_0^t \langle\sigma\,,\, \tau_{Z_s}\xi\rangle\; dB_s + \int_0^t \langle b\,,\, \tau_{Z_s}\xi\rangle\; ds\\ &= \int_0^t \bar{\sigma}(Z_s;\xi).\, dB_s + \int_0^t \bar{b}(Z_s;\xi)\, ds\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.12. The proof is similar to that of [21, Theorem 3.4]. By Theorem 3.4, we have a solution $\{Z_t\}$ of (3.7) with initial condition $Z_0 = 0$. Then using the Itô formula in Theorem 3.2, we observe that the process $\{\tau_{Z_t}\xi\}$ is a solution.

To prove the uniqueness, let $\{Y_t^1\}, \{Y_t^2\}$ be two solutions. Then define $\{Z_t^1\}$ and $\{Z_t^2\}$ corresponding to $\{Y_t^1\}, \{Y_t^2\}$ as in Theorem 3.14. The uniqueness part in Theorem 3.4 implies a.s. $Z_t^1 = Z_t^2, \forall t \geq 0$ and hence a.s. $Y_t^1 = Y_t^2, \forall t \geq 0$. □

Since $Y_t = \tau_{Z_t}\xi$ solves SPDE (3.1) (notations as in Theorem 3.12), we have $\mathbb{E}\|Y_0\|_p^2 = \mathbb{E}\|\xi\|_p^2 < \infty$. Now we prove estimates on Y using two different techniques.

Proposition 3.15. *There exists a localizing sequence $\{\eta_n\}$ such that*

$$\mathbb{E}\sup_{t\geq 0}\|Y_t^{\eta_n}\|_p^2 \leq C_n.\mathbb{E}\|Y_0\|_p^2,$$

where the constant C_n depends only on n.

Proof. Consider the process $\{Z_t\}$ as in Theorem 3.14. Define a localizing sequence $\{\eta_n\}$ as follows: $\eta_n := \inf\{t \geq 0 : |Z_t| \geq n\}, n \geq 1$. Now using Lemma 2.1(a) there exists a polynomial Q of degree $2([|p|] + 1)$ such that

$$\|Y_t^{\eta_n}\|_p \leq \|\xi\|_p.Q(|Z_t^{\eta_n}|) \leq \|\xi\|_p \sup_{\{x:|x|\leq n\}} Q(|x|).$$

Hence $\sup_{t\geq 0}\|Y_t^{\eta_n}\|_p^2 \leq C_n\,\|\xi\|_p^2$ with $C_n = (\sup_{\{x:|x|\leq n\}} Q(|x|))^2$. This implies the required estimate. □

Following [8, Lemma 1], we get the next estimate.

Proposition 3.16. *There exists a localizing sequence $\{\eta_n\}$ such that for any positive real number T,*

$$\mathbb{E}\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2 \leq C.\mathbb{E}\|Y_0\|_{p-1}^2,$$

where the constant C depends only on n and T.

Proof. Define three localizing sequences. For any positive integer n, consider

$$\bar{\eta}_n := \inf\{t \geq 0 : \|Y_t - Y_0\|_p \geq n\},$$

and

$$\eta_n' := \inf\{t \geq 0 : |\langle\sigma\,,\, Y_t\rangle| \geq n, \text{ or } |\langle b\,,\, Y_t\rangle| \geq n\},$$

and $\eta_n := \bar{\eta}_n \wedge \eta_n'$. Now using Itô formula for $\|\cdot\|_{p-1}^2$ we obtain a.s. $t \geq 0$

$$\begin{aligned}\|Y_t^{\eta_n}\|_{p-1}^2 = \|Y_0\|_{p-1}^2 &+ \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \langle Y_s^{\eta_n}\,,\, A_iY_s^{\eta_n}\rangle_{p-1}\, dB_s^{(i)} \\ &+ \int_0^{t\wedge\eta_n} \left[2\,\langle Y_s^{\eta_n}\,,\, LY_s^{\eta_n}\rangle_{p-1} + \sum_{i=1}^d \|A_iY_s^{\eta_n}\|_{p-1}^2\right] ds\end{aligned} \tag{3.19}$$

where $\{\int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \langle Y_s^{\eta_n}\,,\, A_i Y_s^{\eta_n}\rangle_{p-1}\, dB_s^{(i)}\}$ is a continuous martingale and $B_t^{(i)}$ denotes the i-th component of B_t. Then using the monotonicity inequality in Theorem 3.13 and taking expectation in (3.19), we have

$$\mathbb{E}\|Y_t^{\eta_n}\|_{p-1}^2 \leq \mathbb{E}\|Y_0\|_{p-1}^2 + \gamma \int_0^t \mathbb{E}\|Y_s^{\eta_n}\|_{p-1}^2\, ds$$

where the constant γ depends only on η_n. Then Gronwall's inequality implies

$$\mathbb{E}\|Y_t^{\eta_n}\|_{p-1}^2 \leq e^{\gamma t}.\mathbb{E}\|Y_0\|_{p-1}^2,\ t \geq 0. \tag{3.20}$$

Let $\{M_t\}$ and $\{V_t\}$ respectively denote the martingale term and the finite variation term on the right hand side of (3.19). Then using the monotonicity inequality and (3.20), we get

$$\mathbb{E}\sup_{t\leq T} V_t \leq \gamma\, \mathbb{E}\sup_{t\leq T}\int_0^t \|Y_s^{\eta_n}\|_{p-1}^2\, ds = \gamma \int_0^T \mathbb{E}\|Y_s^{\eta_n}\|_{p-1}^2\, ds \leq \widetilde{C}\, \mathbb{E}\|Y_0\|_{p-1}^2 \tag{3.21}$$

for some constant $\widetilde{C}$ depending only on η_n and T.

By [1, Theorem 2.5], for each $1 \leq i \leq d$, there exists a bounded operator $\mathbb{T}_i : \mathcal{S}_{p-1}(\mathbb{R}^d) \to \mathcal{S}_{p-1}(\mathbb{R}^d)$ such that

$$\begin{aligned}
|2\, \langle Y_t^{\eta_n}\,,\, A_i Y_t^{\eta_n}\rangle_{p-1}\,| &= \left|-2\sum_{j=1}^d \langle \sigma_{ji}\,,\, Y_t^{\eta_n}\rangle\, \langle Y_t^{\eta_n}\,,\, \partial_j Y_t^{\eta_n}\rangle_{p-1}\right| \\
&= \left|\sum_{j=1}^d \langle \sigma_{ji}\,,\, Y_t^{\eta_n}\rangle\, \langle Y_t^{\eta_n}\,,\, \mathbb{T}_j Y_t^{\eta_n}\rangle_{p-1}\right| \\
&\leq n\sum_{j=1}^d |\, \langle Y_t^{\eta_n}\,,\, \mathbb{T}_j Y_t^{\eta_n}\rangle_{p-1}\,| \leq \beta \|Y_t^{\eta_n}\|_{p-1}^2.
\end{aligned} \tag{3.22}$$

where $\beta = nd\, \max\{\|\mathbb{T}_j\|_{\mathcal{S}_{p-1}(\mathbb{R}^d)\to\mathcal{S}_{p-1}(\mathbb{R}^d)}\ |\ 1 \leq j \leq d\}$. Using (3.22), we obtain (as in [8, Lemma 1])

$$\mathbb{E}\sup_{t\leq T}|M_t| \leq \frac{1}{2}\mathbb{E}\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2 + C.\mathbb{E}\|Y_0\|_{p-1}^2 \tag{3.23}$$

for some $C > 0$ depending only on η_n and T. Using (3.19), (3.21) and (3.23) we get the desired estimate. □

The counterpart of Theorem 3.12 involving locally Lipschitz coefficients is as follows. This result is an extension of [21, Theorem 3.4].

Theorem 3.17. *Suppose the following conditions are satisfied:*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) (Locally Lipschitz in x, locally in y) for any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$ the functions $x \mapsto \bar{\sigma}(x;y)$ and $x \mapsto \bar{b}(x;y)$ are locally Lipschitz functions in x and the Lipschitz coefficient is independent of y when y varies over any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$; i.e. for any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$ and any positive

integer n there exists a constant $C(G,n) > 0$ such that for all $x_1, x_2 \in B(0,n), y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \leq C(G,n)|x_1 - x_2|,$$

where $B(0,n) = \{x \in \mathbb{R}^d : |x| \leq n\}$.

Then an $(\mathcal{F}_t^\xi)$ adapted continuous strong local solution of SPDE (3.1) *exists. The solutions are also pathwise unique.*

4. Stationary Solutions

In this section, we investigate existence of stationary solutions of the SPDE (3.1). First we show that finite dimensional stationary processes can be lifted to infinite dimensional stationary processes via the translation operators τ_x.

Proposition 4.1. *Let $\{Z_t\}$ be an $\mathbb{R}^d$ valued stationary process. Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued random variable (for some $p \in \mathbb{R}$), which is independent of $\{Z_t\}$. Then the process $\{Y_t\}$ defined by $Y_t := \tau_{Z_t}\xi$ is also stationary.*

Proof. Let $\stackrel{\mathcal{L}}{=}$ denote equality in law. Since $\{Z_t\}$ is stationary, for time points $s, t_1, t_2, \cdots, t_n \geq 0$ we have

$$(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n}) \stackrel{\mathcal{L}}{=} (Z_{s+t_1}, Z_{s+t_2}, \cdots, Z_{s+t_n}).$$

Let $\psi \in \mathcal{C}$. Following the proof of [23, Proposition 3.1], we have $x \mapsto \tau_x\psi$ is continuous and hence is measurable. Using this fact and the stationarity of $\{Z_t\}$, for Borel sets $G_1, \cdots, G_n$ in $\mathcal{S}_p(\mathbb{R}^d)$, we have

$$\begin{aligned} &P((\tau_{Z_{t_1}}\psi, \tau_{Z_{t_2}}\psi, \cdots, \tau_{Z_{t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n) \\ &= P((\tau_{Z_{s+t_1}}\psi, \tau_{Z_{s+t_2}}\psi, \cdots, \tau_{Z_{s+t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n). \end{aligned} \tag{4.1}$$

Let μ_ξ denote the law of ξ on $\mathcal{S}_p(\mathbb{R}^d)$. Then using conditional probability and the independence of $\{Z_t\}$ and ξ, we have

$$\begin{aligned} &P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n) \\ &= \int_{\mathcal{S}_p} P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n | \xi = \psi)\, \mu_\xi(d\psi) \\ &= \int_{\mathcal{C}} P((\tau_{Z_{t_1}}\psi, \tau_{Z_{t_2}}\psi, \cdots, \tau_{Z_{t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\, \mu_\xi(d\psi). \end{aligned}$$

Similarly,

$$\begin{aligned} &P((\tau_{Z_{s+t_1}}\xi, \tau_{Z_{s+t_2}}\xi, \cdots, \tau_{Z_{s+t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n) \\ &= \int_{\mathcal{C}} P((\tau_{Z_{s+t_1}}\psi, \tau_{Z_{s+t_2}}\psi, \cdots, \tau_{Z_{s+t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\, \mu_\xi(d\psi). \end{aligned}$$

Using (4.1) we have

$$\begin{aligned} &P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n) \\ &= P((\tau_{Z_{s+t_1}}\xi, \tau_{Z_{s+t_2}}\xi, \cdots, \tau_{Z_{s+t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n) \end{aligned} \tag{4.2}$$

i.e. $\{Y_t\}$ is stationary. □

As a consequence of the previous result, stationary solutions of finite dimensional SDEs can be lifted to stationary solutions of corresponding infinite dimensional SPDEs.

Theorem 4.2. *Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$-measurable random variable and independent of $\{B_t\}$. Let $\{Z_t\}$ be a stationary solution of SDE* (3.7)*. Then the process $\{Y_t\}$ defined by $Y_t := \tau_{Z_t}\xi$ is a stationary solution of the SPDE*

$$dY_t = A(Y_t)\,dB_t + L(Y_t)\,dt; \quad Y_0 = \tau_{Z_0}\xi. \tag{4.3}$$

Proof. Using the Itô formula in Theorem 3.2, we show that $Y_t = \tau_{Z_t}\xi$ solves (4.3). By Proposition 4.1, $\{Y_t\}$ is stationary, since $\{Z_t\}$ is stationary. □

Theorem 4.2 allows us to construct stationary solutions of SPDE (4.3) from those of SDE (3.7). In practice, however, it might be difficult to obtain stationary solutions of the SDEs (3.7). These difficulties may arise from the coefficients $\bar{\sigma}, \bar{b}$, i.e. from the interplay of σ, b and the random variable ξ. In Theorem 4.5, we present a method of constructing stationary solutions of SPDE (4.3) from those of finite dimensional SDEs (4.4) by modifying the random variable ξ. Since there is no relation between the coefficients σ, b and f, g, a connection is made between the SDE (4.4) and the SPDE (4.3) at the cost of an additional assumption on the initial condition ξ.

Assume that

(1) $f : \mathbb{R}^d \to \mathbb{R}^{d\times d}, g : \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions such that the SDE
$$dZ_t = f(Z_t)dB_t + g(Z_t)dt, \, \forall t \geq 0 \tag{4.4}$$
has a continuous, stationary solution and we denote the corresponding invariant measure by ν. Let $f = (f_{ij}), g = (g_i), 1 \leq i, j \leq d$ be the component functions of f, g.

(2) σ_{ij}, b_i (for $i, j = 1, \cdots, d$) are tempered distributions given by continuous functions.

Remark 4.3. Typically f, g will be locally Lipschitz functions such that explosions do not happen in finite time. This non-explosion is usually guaranteed by a 'Liapunov' type criteria. See for example, [25, 7.3.14 Corollary].

Note that there exists a $p > 0$ such that $\sigma_{ij}, b_i \in \mathcal{S}_{-p}(\mathbb{R}^d)$ for all i, j. Fix such a $p > 0$. Consider the following subset of $\mathcal{S}_p(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{C} = \Bigg\{\psi \in \mathcal{S}_p(\mathbb{R}^d) : &\int_{\mathbb{R}^d} \sigma_{ij}(y+x)\psi(y)\,dy = f_{ij}(x), \forall x \in \mathbb{R}^d;\\ &\int_{\mathbb{R}^d} b_i(y+x)\psi(y)\,dy = g_i(x), \forall x \in \mathbb{R}^d, i, j = 1, \cdots, d\Bigg\}.\end{aligned} \tag{4.5}$$

Note that $\mathcal{C}$ is a closed and convex. The motivation behind above conditions requires clarification. Firstly, we want to choose a subset $\mathcal{C}$ of $\mathcal{S}_p(\mathbb{R}^d)$ such that the resultant equation (1.4) is the same for all deterministic initial conditions ψ as ψ varies over the set $\mathcal{C}$. This allows us to think of $\bar{\sigma}(x;\psi)$ and $\bar{b}(x;\psi)$ as just $\bar{\sigma}(x)$ and $\bar{b}(x)$. Secondly, we want $\bar{\sigma} = f$ and $\bar{b} = g$ which is a choice that allows us to use the invariant measure ν of (4.4). The set $\mathcal{C}$ considered above provides exactly those conditions, which is pointed out in the next result.

Lemma 4.4. *Let $\psi \in \mathcal{C}$. Then $\bar{\sigma}(x;\psi) = f(x)$ and $\bar{b}(x;\psi) = g(x)$ for all $x \in \mathbb{R}^d$.*

We present the main result of this section.

Theorem 4.5. *Let ξ be a $\mathcal{C}$-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|\xi\|_p^2 < \infty$ and independent of $\{B_t\}$. Then the process $\{Y_t\}$ defined by $Y_t := \tau_{Z_t}\xi$, is a stationary process and solves SPDE* (4.3)*, where $\{Z_t\}$ is the stationary solution of SDE* (4.4)*.*

Proof. Using Lemma 4.4 and the Itô formula in Theorem 3.2, we show that $Y_t = \tau_{Z_t}\xi$ solves (4.3). By Proposition 4.1, $\{Y_t\}$ is stationary. □

The next result will be used in Example 4.7.

Lemma 4.6. *The tempered distribution x given by the function $x \in \mathbb{R} \mapsto x$ belongs to $\mathcal{S}_{-p}$ for any $p > \frac{3}{4}$. The tempered distribution given by the function $b(x) := x^3, x \in \mathbb{R}$ belongs to $\mathcal{S}_{-p}$ for any $p > \frac{7}{4}$.*

Proof. First we show that the tempered distribution 1 given by the constant function 1 belongs to $\mathcal{S}_{-p}$ for any $p > \frac{1}{4}$. The Hermite-Sobolev spaces $\mathcal{S}_p(\mathbb{R};\mathbb{C})$ can be defined corresponding to the Schwartz space $\mathcal{S}(\mathbb{R};\mathbb{C})$, where the functions are complex valued. The Fourier transform $\widehat{\cdot} : \mathcal{S}(\mathbb{R};\mathbb{C}) \to \mathcal{S}(\mathbb{R};\mathbb{C})$ defined by

$$\widehat{\phi}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{-ixy}\phi(y)\,dy, \quad \forall \phi \in \mathcal{S}(\mathbb{R};\mathbb{C})$$

extends to $\widehat{\cdot} : \mathcal{S}(\mathbb{R};\mathbb{C}) \to \mathcal{S}(\mathbb{R};\mathbb{C})$ via duality. By [12, Appendix A.5, equation (A.27)], $\widehat{h}_n = (-i)^n h_n$, which leads to an isometry $\widehat{\cdot} : \mathcal{S}_p(\mathbb{R};\mathbb{C}) \to \mathcal{S}_p(\mathbb{R};\mathbb{C})$. If $T \in \mathcal{S}'(\mathbb{R};\mathbb{C})$ is such that $\langle T\,,\,\phi\rangle \in \mathbb{R}, \forall \phi \in \mathcal{S}(\mathbb{R})$ then we have $\|\widehat{T}\|_{\mathcal{S}_p(\mathbb{R};\mathbb{C})} = \|T\|_{\mathcal{S}_p(\mathbb{R};\mathbb{C})} = \|T\|_{\mathcal{S}_p}$. By [23, Theorem 4.1(a)], $\delta_0 \in \mathcal{S}_{-p}$ for any $p > \frac{1}{4}$ and we have $1 \in \mathcal{S}_{-p}$ for any $p > \frac{1}{4}$.

The operator M_x was defined in Section 2. Observe that for any $\phi \in \mathcal{S}$ and $p > \frac{1}{4}$,

$$|\langle x\,,\,\phi\rangle| = |\langle 1\,,\,M_x\phi\rangle| \leq \|1\|_{-p}\|M_x\phi\|_p \leq \|1\|_{-p}\|M_x\|_{\mathcal{S}_{p+\frac{1}{2}}\to\mathcal{S}_p}\cdot\|\phi\|_{p+\frac{1}{2}}.$$

This implies $x \in \mathcal{S}_{-p}$ for $p > \frac{3}{4}$. Proof for b is similar. □

Example 4.7. We present two examples where the stationary solutions of SDE (4.4) can be lifted to stationary solutions of SPDE (4.3) via Theorem 4.5.

(1) Take $d = 1, f(x) \equiv 1, g(x) = -x, \forall x, \sigma = f, b = g$. It is well-known that (4.4) (the Ornstein-Uhlenbeck diffusion) has a stationary solution with the following initial condition:

$$dZ_t = dB_t - Z_t\,dt; \quad Z_0 \sim N\big(0, \tfrac{1}{2}\big), \tag{4.6}$$

where $N(0, \frac{1}{2})$ denotes the law of a Gaussian random variable with mean 0 and variance $\frac{1}{2}$ (see [17, Example 6.8]). By Lemma 4.6, $\sigma \in \mathcal{S}_{-p}$ for $p > \frac{1}{4}$ and $b \in \mathcal{S}_{-p}$ for $p > \frac{3}{4}$. Take $p > \frac{3}{4}$. It is easy to check that $\mathcal{C} = \{\psi \in \mathcal{S}_p : \int_{\mathbb{R}} \psi = 1, \int_{\mathbb{R}} t\psi(t)\,dt = 0\}$. $\mathcal{C}$ is non empty since (centered) Gaussian densities satisfy such conditions.

(2) We take $\Omega = C([0,\infty),\mathbb{R})$ and use the setup of [25, Chapter VII, Sections 3 and 5]. Consider $f(x) \equiv 1, g(x) = -x^3, \forall x \in \mathbb{R}, \sigma = f, b = g$. Then (4.4) has an invariant measure, say ν, given by

$$\nu(B) := c \int_B \exp\left(-\frac{x^4}{2}\right) dx$$

for any Borel set B in $\mathbb{R}$, where $c = 2^{\frac{3}{4}} \left(\Gamma(\frac{1}{4})\right)^{-1}$ is the normalization constant. By Lemma 4.6, $\sigma, b \in \mathcal{S}_{-p}$ for $p > \frac{7}{4}$. Then

$$\mathcal{C} = \left\{\psi \in \mathcal{S}_p : \int_{\mathbb{R}} \psi = 1, \int_{\mathbb{R}} t\psi(t)\,dt = 0, \int_{\mathbb{R}} t^2\psi(t)\,dt = 0, \int_{\mathbb{R}} t^3\psi(t)\,dt = 0\right\}.$$

$\mathcal{C}$ is non-empty since $\psi_1, \psi_2 \in \mathcal{C}$ where

$$\psi_1(t) = \exp(-t^2)\left[\frac{3}{2\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}t^2\right], \psi_2(t) = \exp\left(-\frac{t^2}{2}\right)\left[\frac{3}{2\sqrt{2\pi}} - \frac{1}{2\sqrt{2\pi}}t^2\right].$$

We now prove an estimate of a stationary solution $\{Y_t\}$.

Proposition 4.8. *Let $\xi, \{Z_t\}, \{Y_t\}$ be as in Theorem 4.5. In addition assume that ξ is norm-bounded and Z_0 has moments of orders upto $4([|p|]+1)$ and f, g are Lipschitz continuous. Then*

(a) $\mathbb{E}\|Y_0\|_p^2 = \mathbb{E}\|\tau_{Z_0}\xi\|_p^2 < \infty$.
(b) $\mathbb{E}\sup_{t\leq T}\|Y_t\|_p \leq C\,(\mathbb{E}\|Y_0\|_p^2)^{\frac{1}{2}}$, *where C is a positive constant depending only on f, g and T.*

Proof. By Lemma 2.1, we have $\mathbb{E}\|\tau_{Z_0}\xi\|_p^2 \leq R\,\mathbb{E}P(|Z_0|)$ where $R > 0$ and P is a polynomial of degree $4([|p|]+1)$. Then by our assumption, $\mathbb{E}\|\tau_{Z_0}\xi\|_p^2 < \infty$.

Observe that $Y_t = \tau_{Z_t}\xi = \tau_{Z_t - Z_0}\tau_{Z_0}\xi = \tau_{Z_t - Z_0}Y_0$. Using Lemma 2.1(a) we have

$$\|Y_t\|_p \leq \|Y_0\|_p\, P_k(|Z_t - Z_0|),$$

where P_k is some real polynomial of degree $k = 2([|p|]+1)$ with non-negative coefficients. We use the following estimate to establish the result.

$$\mathbb{E}\sup_{t\leq T}\|Y_t\|_p \leq (\mathbb{E}\|Y_0\|_p^2)^{\frac{1}{2}}\,(\mathbb{E}\sup_{t\leq T}P_k(|Z_t - Z_0|)^2)^{\frac{1}{2}}. \quad (4.7)$$

Now a.s. $Z_t - Z_0 = \int_0^t f(Z_s)\,dB_s + \int_0^t g(Z_s)\,ds,\ t \geq 0$. Using stationarity of Z and the BDG inequalities ([15, Proposition 15.7]), we show $\mathbb{E}\sup_{t\leq T}P_k(|Z_t - Z_0|)^2$ is bounded. In this estimate we use the assumption on the moments of Z_0 and linear growth of f, g. This completes the proof. □

Remark 4.9. We make a few observations.

(1) If the convex set $\mathcal{C}$ (as in equation (4.5)) has more than one element, then consider probability measures on $\mathcal{C}$ which are convex combinations of Dirac measures on $\mathcal{C}$. By Theorem 4.5, we have the existence of infinitely many stationary solutions corresponding to each of these probability measures. This may be happening due to $\mathcal{C}$ being not translation invariant.

(2) The set $\mathcal{C}$ may be non-compact. Consider the special case $d = 1, f(x) \equiv 1, g(x) = -x, \sigma = f, b = g$ and take p sufficiently large so that the tempered distribution given by the function $x \mapsto x^2$ is in $\mathcal{S}_{-p}$. Then the image of $\mathcal{C}$ under this tempered distribution (a continuous linear functional on $\mathcal{S}_p$) contains $(0, \infty)$, the variances of centered Gaussian densities. So $\mathcal{C}$ is unbounded and non-compact.

Remark 4.10. Existence of invariant measures of finite dimensional diffusions and Markov processes has been studied by many authors (to cite only a few see [2, 5, 10, 11, 18], [25, Chapter VII, Section 5]).

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