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WATER WAVES DIFFRACTION BY A CIRCULAR PLATE IN THE PRESENCE OF CRACKED–BARRIER NEAR IT

A. Aghili & A. Ansari

Abstract: In this article, we consider the velocity potential of water waves in the framework of linear theory. The circular plate lies on the free surface of water of finite depth with the circular cracked–barrier near it. The problem is reduced to a system of dual integral equation.

In order to solve this dual integral equation, we convert it into a Fredholm's integral equation of second kind. Representations for far diffraction field and for kernels of Fredholm's integral equation of deep or shallow water and long wave are given.

Keywords: Water waves, Dual integral equations, Fredholm's integral equation.

1. INTRODUCTION

The circular plate with the radius r located on the free surface of water with finite constant depth d. Water waves are scattered by a partly immersed circular plate and uniform wave train is incident on the plate.

This train can be written by a complex value potential [5, 11] as

$$\phi_0 = \frac{ig\eta_0}{\sigma} \frac{\cos h\, k_1(z+d)}{\cos h\, k_1 d} e^{ik_1 r \cos\theta} \tag{1.1}$$

where η_0 is amplitude of the wave train, g is gravitational constant, k is the wave number and σ the frequency, so that $\sigma^2 = gk_1 \tan h\,(k_1 d)$.

The velocity potential can be determined through complex–valued potential as Re $[\phi e^{i\sigma t}]$. We consider the solution of the problem in the form

$$\phi = \frac{ig\eta_0}{\sigma} (\Phi_0 + \Phi) \tag{1.2}$$

where Φ_0, Φ are the dimensionless potential of the main and perturbed motion.

It can be noted that the distribution of the pressure in water $\overline{p} = \frac{p}{\rho g h}$ (ρ is density of fluid) can be expressed through the potential as

$$\overline{p} = \Phi_0 + \Phi \tag{1.3}$$

velocity potential of this problem [DP (dock-problem)] will be obtained through boundary conditions on free surface and condition of scattering waves at infinity.

2. FORMULATION OF THE DP IN PRESENCE OF A BARRIER NEAR IT

By setting the cracked–barrier near the circular plate then, in cylindrical coordinate system (r, θ, z) in which (r, θ)-plane coincides with the undisturbed free surface and z-axis is oriented vertically upwards and center of the system is coincides with the center of the plate.

By introducing parameter $v = \frac{\sigma^2}{g}$ and writing the Laplace's equation in cylindrical coordinates for $0 < r < \infty$, $-d < z < 0$, $0 < \theta < \pi$ we have

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{2.1}$$

Following conditions present the behavior of the waves on free surface and circular line $r = a$ and barrier $r = b$ for $0 \le \theta \le \pi/2 - \alpha$, $\pi/2 + \alpha \le \theta - \pi$, $0 < \alpha < \pi/2$ (region. A)

$$\frac{\partial \Phi}{\partial z} = -k_1 \tan h\,((k_1 d)\, e^{ik_1 r \cos\theta} \qquad 0 < r < a, \quad z = 0 \tag{2.2A}$$

$$\frac{\partial \Phi}{\partial z} = -k_2 \tan h\,((k_2 d)\, e^{ik_2 r \cos\theta} \qquad a < r < b, \quad z = 0 \tag{2.3A}$$

$$\frac{\partial \Phi}{\partial z} = k_2 \tan h\,(k_2 d)\,\Phi, \qquad b < r < \infty, \quad z = 0 \tag{2.4A}$$

and for $\pi/2 - \alpha \le \theta \le \pi/2 + \alpha$ (region. B)

$$\frac{\partial \Phi}{\partial z} = -k_1 \tan h\,((k_1 d)\, e^{ik_1 r \cos\theta}, \qquad 0 < r < a, \quad z = 0 \tag{2.2B}$$

$$\frac{\partial \Phi}{\partial z} = k_1 \tan h\,(k_1 d)\,\Phi, \qquad a < r < \infty, \quad z = 0 \tag{2.3B}$$

Next condition shows no flow through the bottom of the plate

$$\frac{\partial \Phi}{\partial z} = 0, \quad \mathrm{z} = -d \tag{2.5}$$

The following conditions show finite potential in circular lines $r = a$, $r = b$ and $r = 0$

$$\Phi < \infty, \quad r \to a, b, \quad z = 0 \tag{2.6}$$

$$\Phi < \infty, \quad r \to 0 \tag{2.7}$$

And sommerfeld radiation condition at infinity which indicates no flow of water at far diffraction field.

$$\sqrt{r}\left(\frac{\partial \Phi}{\partial r}+ik_2\Phi\right)\to 0 \quad r\to\infty \quad \text{For re.A} \tag{2.8A}$$

$$\sqrt{r}\left(\frac{\partial \Phi}{\partial r}+ik_1\Phi\right)\to 0 \quad r\to\infty \quad \text{For re.B} \tag{2.8B}$$

Equations (2.2)-(2.8) indicate a mixed boundary value problem that has a solution with following form.

Potential and Dual Integral Equation

We consider Φ, solution of the *DP* as a Fourier series [6, 7, 8] for region A:

$$\Phi = \sum_{m=0}^{\infty} f^m \Phi^m(r, z)\cos m\theta \tag{2.9}$$

$$f^m = \begin{cases} 1 & m = 0 \\ 2i^m & m \geq 1 \end{cases}$$

where

$$\Phi^m = \int_0^{\infty^*} A_m(\lambda)\, s(\lambda)\, \frac{\cos h[\lambda(z+d)]}{\cos h\, \lambda d} J_m(\lambda r)\, d\lambda \tag{2.10}$$

We need to obtain the function $s(\lambda)$ that satisfies in sommerfeld radiation condition and asymptotic behavior as follows:

$$\frac{A_m(\lambda)\, s(\lambda)}{\sqrt{\lambda}} \to 0, \quad \lambda\to\infty \tag{2-11}$$

defining $s(\lambda)$ as

$$s(\lambda) = \frac{k_1 \tan h(k_1 d)(1 - H(\lambda - a)) + k_2 \tan h(k_2 d) H(\lambda - a)}{k_2 \tan h(k_2 d) - \lambda \tan h(\lambda d)} \tag{2-12A}$$

(where $H(x)$ is unit step function).

It is bounded on range $(0, \infty)$, near zero and at infinity, so that

$s(0) = \dfrac{k_1 \tan h(k_1 d)}{k_2 \tan h(k_2 d)}$, $s(\infty) = 0$ and has a singular point $\lambda = k_2$.

and for region B: the function has the following form

$$s(\lambda) = \frac{k_1 \tan h(k_1 d)}{k_1 \tan h(k_1 d) - \lambda \tan h(\lambda d)} \tag{2.12B}$$

Also, $s(\lambda)$ is bounded on range (0. ∞) and has a singular point $\lambda = k_1$.

The symbol $\int_0^{\infty^*}$ stands for the contour integration in λ-complnex plane whose path of integration must run beneath the pole of the integrand. It can be shown that equations (2.9), (2.10) satisfy Laplace's equation.

Now, we need to obtain the function $A_m(\lambda)$, by substitution of (2.9) into (2.2A), (2.3A) and (2.4A) and using (A.1) that yields dual integral equation with Bessel kernel for region A.

$$\begin{aligned} &\int_0^{\infty^*} A_m(\lambda)\,[1 - s(\lambda)]\, J_m(\lambda r)\, d\lambda \\ &= J_m(k_1 r)\,[1 - H(r-a)] + J_m(k_2 r)\, H(r-a) \qquad 0 < r < b \\ &= \bar{V}_{2m}(r) \end{aligned}$$

$$\int_0^{\infty} A_m(\lambda)\, J_m(\lambda r)\, d\lambda = 0 \qquad b < r < \infty \tag{2.13 B}$$

In the same way, for the region B:

$$\int_0^{\infty^*} A_m(\lambda)[1 - s(\lambda)]\, J_m(\lambda r)\, d\lambda = J_m(k_1 r) = \bar{V}_{1m}(r) \qquad 0 < r < a$$

$$\int_0^{\infty} A_m(\lambda)\, J_m(\lambda r)\, d\lambda = 0 \qquad 0 < r < \infty \tag{2.13 B}$$

3. FREDHOLM'S INTEGRAL EQUATION

Now, we introduce $r_1 = a$, $r_2 = b =$ and setting $i = 2$ for region A and $i = 1$ for region B. The method of solution (2.13) consists of finding a Fredholm's integral equation for the auxiliary function $\Lambda_m(\xi)$ relatred to the required function $A_m(\lambda)$.

$$A_m(\lambda) = \sqrt{\frac{\pi\lambda}{2}} \int_0^{r_i} \sqrt{\xi}\, \Lambda_m(\xi)\, J_{m-1/2}(\lambda\xi)\, d\xi \tag{3.1}$$

It can be shown that the equation (3.1) is satisfied identically by homogeneous equation (3.13). In order to prove this identity, we consider the following integral

$$I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} A_m(\lambda)\, J_m(\lambda r)\, d\lambda = \int_0^{\infty} \lambda\sqrt{\lambda}\, J_m(\lambda r) \int_0^{r_i} \sqrt{\xi}\, \Lambda_m(\xi)\, J_{m-1/2}(\lambda\xi)\, d\xi \tag{3.2}$$

substitution of the representation for $\lambda J_{m-1/2}$ by (A.2) and integration (3.2) by parts yields

$$I = \int_0^\infty \sqrt{\lambda} J_m(\lambda r)\left\{\sqrt{\xi}\Lambda_m(\xi) J_{m+1/2}(\lambda\xi)\Big|_0^{r_i} - \int_0^1 \xi^{m+1/2} J_{m+1/2}(\lambda\xi)\frac{d}{d\xi}[\xi^{-m}\Lambda_m(\xi)]d\xi\right\}d\lambda \quad (3.3)$$

With (A.3) the quantity I is equal to zero identically.

Now, we transform the inhomogeneous equation (2.13), and introduce a new function $\hat{V}^m(r)$ as:

$$\hat{V}^m(r) = \bar{V}^m(r) + \int_0^{\infty^*} A_m(\bar{\lambda})\, s(\bar{\lambda})\, J_m(\bar{\lambda} r)\, d\bar{\lambda} \quad (3.4)$$

inhomogeneous equation (2.13) takes the following form,

$$\int_0^\infty A_m(\lambda)\, J_m(\lambda r)\, d\lambda = \hat{V}^m(r), \qquad 0 < r < r_i \quad (3.5)$$

using inverse Hankel transform of (3.5), the function Am(l) can be expressed as $\hat{V}^m(r)$.

$$A_m(\lambda) = \lambda \int_0^{r_i} \hat{V}^m(r)\, J_m(\lambda r)\, r dr \quad (3.6)$$

Now, we insert the integral representation (A.4) for Bessel kernel into (3.5) and interchange the order of integration, we get

$$A_m(\lambda) = \lambda\sqrt{\frac{2\lambda}{\pi}} \int_0^{r_i} \hat{V}^m(r)\, r\sqrt{r}\, dr \int_0^r J_{m-1/2}(\lambda\xi)\left(\frac{\xi}{r}\right)^{m+1/2} \frac{1}{\sqrt{r^2-\xi^2}}\, d\xi$$

$$= \lambda\sqrt{\frac{2\lambda}{\pi}} \int_0^{r_i} J_{m-1/2}(\lambda\xi)\, \xi^{m+1/2}\, d\xi \int_\xi^{r_i} \hat{V}^m(r)\, r^{1-m} \frac{1}{\sqrt{r^2-\xi^2}}\, d\xi \quad (3.7)$$

Comparison of formulas (3.1) and (3.7) yields expression for function $\hat{V}^m(r)$.

$$\Lambda_m(\xi) = \frac{2}{\pi}\xi^m \int_\xi^{r_i} \hat{V}^m(r)\, \mu^{1-m} \frac{d\mu}{\sqrt{\mu^2-\xi^2}} \quad (3.8)$$

Now, we rewrite expression (3.4) into the following form

$$\hat{V}^m(\mu) = \hat{V}^m(\mu) + \sqrt{\frac{\pi}{2}} \int_0^{\infty^*} \lambda\sqrt{\lambda}\, s(\lambda)\, J_m(\lambda\mu)\, d\lambda \int_0^{r_i} \sqrt{\chi}\, \Lambda_m(\chi)\, J_{m-1/2}(\lambda\chi)\, d\chi \quad (3.9)$$

substitution of (3.9) into (3.8), yields Fredholm's integral equation of the second kind [2, 3] for the function Λ_m.

$$\Lambda_m(\xi) \int_0^{r_i} K_m(\xi, \chi)\, \Lambda_m(\chi)\, d\chi = F_m(\xi) \tag{3.10}$$

Representation of kernel of this equation has the following form

$$K_m(\xi, \chi) = \sqrt{\frac{2}{\pi}}\, \sqrt{\chi} \int_0^{\infty^*} \lambda \sqrt{\lambda}\, s(\lambda)\, J_{m-1/2}(\lambda\chi)\, L_m(\lambda, \xi)\, d\lambda \tag{3.11}$$

where

$$L_m(\lambda, \xi) = \xi^m \int_\xi^{r_i} \frac{\mu^{1-m}}{\sqrt{\mu^2 - \xi^2}}\, J_m(\lambda\mu)\, d\mu$$

the function F_m of the R.H.S is defined by

$$F_m(\xi) = \frac{2}{\pi}\, \xi^m \int_\xi^{r_i} \frac{\mu^{1-m}}{\sqrt{\mu^2 - \xi^2}}\, \bar{V}_i^m(\lambda\mu)\, d\mu \tag{3.12}$$

where, for region B, R.H.S has the following shape

$$F_m(\xi) = \frac{2}{\pi}\, L_m(k_1, \xi)$$

Asymptotic Behavior of far Diffraction Field

Let's determine asymptotic behavior of the diffraction field at infinity, now we apply Cauchy's theorem to the integral (2.10) and calculate it at singular point $\lambda = k_i$

$$\Phi_\infty^m = A_m(k_i)\, U(k_i)\, H_m^{(2)}(k_i r)\, \frac{\cos h\,(k_i[z+d])}{\cos h\,(k_i d)} \tag{3.13}$$

where

$$U(k_2) = -2\pi i\, \frac{k_1 \tan h\,(k_1 d)\,[1 - H(k_2 - a)] + k_2 \tan h\,(k_2 d)\, H\,(k_2 - a)}{\tan h\,(k_2 d) + (1 - \tan h^2 K_2 d)\, k_2},$$

$$U(k_1) = -\pi i k_1\, \frac{1}{1 + \dfrac{2k_1 d}{\sin h\,(2k_1 d)}}$$

the kernel of the integral equation (3.11), in this case the number wave is k_i.

$$K_{\infty}^{m}(\xi,\chi)=\sqrt{\frac{2}{\pi}}\,\sqrt{k_i\chi}\;k_i\,U(k_i)\,J_{m-1/2}(k_i\chi)\,\sqrt{\chi}\;L_m(k_i,\xi) \tag{3.14}$$

Substitution of the kernel into (3.10) yields the integral equation with separable kernel

$$\Lambda_m^{\infty}(\xi)-\frac{2}{\pi}\,k_i\,\sqrt{k_i}\;U(k_i)\,L_m\,(k_i,\xi)\int_0^{r_i}\Lambda_m^{\infty}(\chi)\,J_{m-1/2}(k_i\chi)\,\sqrt{\chi}\;d\chi=F_m(\xi) \tag{3.15}$$

for solving this equation, by multiplication of (3.15) by $k_i\sqrt{\frac{\pi k_i}{2}}\sqrt{\xi}\,J_{m-1/2}(k_i\xi)$ and using (3.1) and integration over the range $\xi\in[0,r_i]$ gives the closed form for $A_m(k_i)$.

$$A_m(k_2)=\frac{\hat{F}_m(k_2)}{\left[1-\sqrt{\frac{2}{\pi}}\,U(k_2)\,\hat{L}_m(k_2)\right]} \tag{3.16A}$$

where

$$\hat{L}_m(k_2)\;=\;\sqrt{\frac{2}{\pi}}\,k_2\sqrt{k_2}\,\int_0^{b}L_m(k_2,\xi)\,J_{m-1/2}(k_2\xi)\,\sqrt{\xi}\;d\xi$$

$$\hat{F}_m(k_2)\;=\;k_2\sqrt{\frac{\pi k_2}{2}}\,\int_0^{b}F_m(\xi)\sqrt{\xi}\;J_{m-1/2}(k_2\xi)\,d\xi$$

and

$$A_m(k_1)=\frac{\hat{L}_m(k_1)}{\left[1-\sqrt{\frac{2}{\pi}}\,U(k_1)\hat{L}_m(k_1)\right]} \tag{3.16B}$$

that is,

$$\hat{L}_m(k_1)\;=\;\sqrt{\frac{2}{\pi}}\,k_1\sqrt{k_1}\,\int_0^{a}L_m(k_1,\xi)\,J_{m-1/2}(k_1\xi)\,\sqrt{\xi}\;d\xi$$

4. TRANSFORMATION OF KERNEL

Let's transform the kernel (3.10), in this case we have symmetric kernel. We use (A.4), the integral representation for Bessel kernel in $L_m(\lambda,\xi)$ and interchange $(r,\xi)\to(\mu,\varepsilon)$ in it and introducing a new variable of integration $y=\mu^2$ we obtain

$$L_m(\lambda, \xi) = \sqrt{\frac{2\lambda}{\pi}} \int_0^{r_i} G_m(\xi, \varepsilon)\, J_{m-1/2}(\lambda\varepsilon) \sqrt{\varepsilon}\, d\varepsilon \tag{4.1}$$

where

$$G_m(\xi, \varepsilon) = \frac{1}{2} (\xi\varepsilon)^m \int_{\xi^2}^{r_i} \frac{dy}{y^m \sqrt{(y-\xi^2)(y-\varepsilon^2)}} \quad \xi \neq \in \tag{4.2}$$

we have the following closed form representation for the function G_m $(m = 0, 1, 2, ...)$

$$G_0(\xi, \varepsilon) = \ln \frac{\left(\sqrt{(r_i^2 - \xi^2} + \sqrt{r_i^2 - \varepsilon^2}\right)}{\sqrt{\left|\xi^2 - \varepsilon^2\right|}}, \qquad G_1(\xi, \varepsilon) = \ln \frac{\left|\xi \sqrt{(r_i^2 - \varepsilon^2} - \sqrt{r_i^2 - \xi^2}\right|}{\sqrt{\left|\xi^2 - \varepsilon^2\right|}}$$

$$G_m(\xi, \varepsilon) = -\frac{\sqrt{(r_i^2 - \xi^2)(r_i^2 - \varepsilon^2)}}{2} (\xi\varepsilon)^{m-2} + \frac{(2m-3)}{2} \frac{(\xi^2 + \varepsilon^2)}{\xi\varepsilon} G_{m-1} - (m-2) G_{m-2} \tag{4.3}$$

It is clear that the function G_m is symmetric with respect to the parameters where (ξ, ε) and possesses a weak integrable singularity on the diagonal $\xi = \varepsilon$ of the square $(\xi, \varepsilon) \in [0, r] \times [0, r]$.

The kernel of integral equation (3.11) can be expressed through the function G_m

$$K_m(\xi, \chi) = \int_0^{r_i} G_m(\xi, \varepsilon)\, \Pi_m(\chi, \varepsilon, k_2)\, d\varepsilon \tag{4.4}$$

where

$$\Pi_m(\chi, \varepsilon, k_2) = \frac{2}{\pi} \sqrt{\chi\varepsilon} \int_0^{\infty^*} \lambda s(\lambda)\, J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon)\, \lambda d\lambda \tag{4.5}$$

The function Π_m is a symmetric with respect to the parameter χ, ε and reduce it as:

$$\Pi_m = \Pi_m^v + \Pi_m^p \tag{4.6}$$

where Π_m^v is principle-value integral and Π_m^p is the pole of the integrand, and we have

$$\Pi_m^v = \frac{2}{\pi} \sqrt{\chi\varepsilon}\, P.V. \int_0^{\infty} s(\lambda)\, J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon)\, \lambda^2 d\lambda \tag{4.7}$$

$$\Pi_m^p = \frac{2k_i^2}{\pi} \sqrt{\chi\varepsilon}\, U(k_i)\, J_{m-1/2}(k_i\chi)\, J_{m-1/2}(k_i\varepsilon) \tag{4.8}$$

Also, we may note the quantity $\hat{L}_m(k_i)$ of the asymptotic representation (3.16) through G_m.

$$\hat{L}_m(k_i) = \frac{2}{\pi} k_i^2 \int_0^{r_i} \int G_m(\xi, \varepsilon)\, J_{m-1/2}(k_i\xi)\, J_{m-1/2}(k_i\varepsilon) \sqrt{\xi} \sqrt{\varepsilon}\, d\xi\, d\varepsilon \tag{4.9}$$

In case $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and using (A.5), the function $\hat{L}_m(k_i)$ can be expressed as two dimensional Fourier transform for the functio $G_m(\xi, \varepsilon)$.

$$\hat{L}_m(k_i) = \frac{4}{\pi^2} k_i \int_0^{r_i} \int G_m(\xi, \varepsilon) \begin{pmatrix} \cos k_i\xi \\ \sin k_i\xi \end{pmatrix} \begin{pmatrix} \cos k_i\varepsilon \\ \sin k_i\varepsilon \end{pmatrix} d\xi d\varepsilon \tag{4.10}$$

5. DEEP WATER APPROXIMATION

Transformation of the Kernel

For this case k_1d, $k_2d \gg 1$ and the functions $s(\lambda)$ and $\lambda s(\lambda)$ take the following form In region A:

$$s(\lambda) = \frac{k_1[1 - H(\lambda - a)] + k_2 H(\lambda - a)}{k_2 - \lambda} \tag{5.1A}$$

$$\lambda s(\lambda) = -\{k_1[1 - H(\lambda - a)] + k_2 H(\lambda - a)\} \left\{1 + \frac{k_2}{\lambda - k_2}\right\} \tag{5.2A}$$

in region B:

$$s(\lambda) = \frac{k_1}{k_1 - \lambda} \tag{5.1B}$$

$$\lambda s(\lambda) = -k_1 \left\{1 + \frac{k_1}{k - \lambda}\right\} \tag{5.2B}$$

and relation (4.7) yields

$$\Pi_m^v = -\frac{2}{\pi}\sqrt{\chi\varepsilon}\,(k_1 + k_2)\,P.V.\int_0^\infty \left(1 + \frac{k_2}{\lambda - k_2}\right) J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon)\,\lambda d\lambda \tag{5.3A}$$

$$\Pi_m^v = -\frac{2}{\pi}\sqrt{\chi\varepsilon}\, k_1 P.V.\int_0^\infty J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon) \left(1 + \frac{k_1}{\lambda - k_1}\right) d\lambda \tag{5.3B}$$

using δ-function expansion for Hankel integral transform (A.7) for Π_m^v

$$\Pi_m^v(\chi, \varepsilon) = -\frac{2}{\pi}(k_1 + k_2)\sqrt{\frac{\varepsilon}{\chi}}\,\delta(\varepsilon - \chi) - (k_1 + k_2)\, k_2 R_m(\chi, \varepsilon, k_2) \tag{5.4A}$$

$$\Pi_m^v(\chi, \varepsilon, k_1) = -\frac{2}{\pi} k_1 \sqrt{\frac{\varepsilon}{\chi}}\, \delta(\varepsilon - \chi) - k_1^2 R_m(\chi, \varepsilon, k_1) \tag{5.4B}$$

where

$$R_m(\chi, \varepsilon, k_i) = \frac{2}{\pi} \sqrt{\chi \varepsilon}\, P.V.\int_0^\infty J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon) \frac{\lambda}{\lambda - k_i}\, d\lambda \tag{5.5}$$

by considering (A.6) for components $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the function R_m has the following form

$$R_m(\chi, \varepsilon, k_i) = \frac{4}{\pi^2} T_m(\chi, \varepsilon, k_i) \tag{5.6}$$

where

$$T_m(\chi, \varepsilon, k_i) = P.V.\int_0^\infty \begin{pmatrix} \cos \lambda\chi \\ \sin \lambda\chi \end{pmatrix} \begin{pmatrix} \cos \lambda\varepsilon \\ \sin \lambda\varepsilon \end{pmatrix} \frac{d\lambda}{\lambda - k_i} \tag{5.7}$$

The function T_m is symmetric with respect to the parameter χ, ε and can be treated as Hilbert transform. By considering (A.8) the function T_m for $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ can be expressed as follows,

$$T_m(x, \varepsilon) = -\frac{1}{2} \sum_{j=0}^{1} (-1)^{(j+m)m} \left\{ \sin\left(k_i \left| E_j \right|\right) \left[\pi + si\left(k_i \left| E_j \right|\right) \right] + \cos(k_i E_j)\, ci\,(k_i E_j) \right\} \tag{5.8}$$

where

$$E_j = \chi + (-1)^j\, \varepsilon$$

The functions Π_m^v, Π_m^p, K_m can be re-written for $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as following

$$\Pi_m^v(\chi, \varepsilon, k_2) = -\frac{2}{\pi} (k_1 + k_2) \sqrt{\frac{3}{\chi}}\, \delta(\varepsilon - \chi) - \frac{4}{\pi^2} k_2 (k_1 + k_2)\, T_m(\chi, \varepsilon, k_2) \tag{5.9A}$$

$$\Pi_m^v(\chi, \varepsilon, k_1) = -\frac{2}{\pi} k_1 \sqrt{\frac{\varepsilon}{\chi}}\, \delta(\varepsilon - \chi) - \frac{4}{\pi^2} k_1^2\, T_m(\chi, \varepsilon, k_1) \tag{5.9B}$$

$$\Pi_m^p = -4ik_2^2 \sqrt{\chi\varepsilon}\, (k_1[1 - H(k_2 - a)] + k_2 H(k_2 - a))\, J_{m-1/2}(k_2\chi)\, j_{m-1/2}(k_2\varepsilon) \tag{5.10A}$$

$$\Pi_m^p = -2ik_1^3 \sqrt{\chi\varepsilon}\, J_{m-1/2}(k_1\chi)\, j_{m-1/2}(k_1\varepsilon) \tag{5.10B}$$

$$K_m(\xi,\chi) = -\frac{2}{\pi}(k_1+k_2)\,G_m(\xi,\chi) - \frac{4}{\pi^2}\,k_2(k_1+k_2)\int_0^b G_m(\xi,\varepsilon)\,T_m(\chi,\varepsilon,k)\,d\varepsilon$$

$$-\frac{8}{\pi}k_2 i[k_1(1-H(k_2-a))+k_2H(k_2-a)]\begin{pmatrix}\cos k_2\chi\\ \sin k_2\chi\end{pmatrix}\int_0^b G_m(\xi,\varepsilon)\begin{pmatrix}\cos k_2\varepsilon\\ \sin k_2\varepsilon\end{pmatrix}d\varepsilon \quad (5.11A$$

$$K_m(\xi,\chi) = -\frac{2}{\pi}\,k_1 G_m(\xi,\chi) - \frac{4}{\pi^2}\,k_1^2\int_0^a G_m(\xi,\varepsilon)\,T_m(\chi,\varepsilon,k_1)\,d\varepsilon$$

$$-\frac{4}{\pi}\,k_1^2\begin{pmatrix}\cos k_1\chi\\ \sin k_1\chi\end{pmatrix}\int_0^a G_m(\xi,\varepsilon)\begin{pmatrix}\cos k_1\varepsilon\\ \sin k_1\varepsilon\end{pmatrix}d\varepsilon \qquad (5.11\text{B})$$

Now, we want to obtain K_m for $m \geq 2$ substituting (A.7) into (5.5) the function R_m can be expressed in the following form

$$R_m(\chi,\varepsilon,k_i) = \frac{2}{\pi^2}\left(\sqrt{\chi\varepsilon}\right)^2\,P.V.\int_0^\infty \frac{\lambda\sin\lambda\zeta}{\lambda-k_i}\,d\lambda\int_{-1}^{1}P_{m-1}(\tau)\,\frac{d\tau}{\zeta} \qquad (5.12)$$

where

$$\zeta = \sqrt{\chi^2+\varepsilon^2-2\chi\varepsilon\tau}$$

calculating (5.12), R_m is reduced into the following relation

$$R_m(\chi,\varepsilon,k_i) = \overline{R}_m(\chi,\varepsilon) + k_2\hat{R}_m(\chi,\varepsilon,k_i) \qquad (5.13)$$

where

$$\overline{R}_m(\chi,\varepsilon) = \frac{2}{\pi\sqrt{\pi}}\,\frac{(\xi\varepsilon)^m}{(\chi+\varepsilon)^{2m}}\,\frac{\Gamma(m)}{\Gamma(m+1/2)}\,F\left[m,m;2m;\frac{4\chi\varepsilon}{(\chi+\varepsilon)^2}\right] \qquad (5.14)$$

$$\hat{R}_m(\chi,\varepsilon,k_1) = \frac{2}{\pi^2}\,\chi\varepsilon\int_{-1}^{1}T_d(k_1\zeta)\,P_{m-1}(\tau)\,\frac{d\tau}{\zeta} \qquad (5.15)$$

and

$$T_d(k_i\zeta) = \{\cos k_i\zeta\,[\pi + si(k_i\zeta)] - \sin(k_i\zeta)\,ci\,(k_i\zeta)\}$$

F is the Hyper – geometric function and P_{m-1} is the Lergendre polynomials of the first kind. In this case for $m \geq 2$, (5.4) and (4.4) lead to the following relations

$$\Pi_m^v(\chi,\varepsilon) = -\frac{2}{\pi}(k_1+k_2)\sqrt{\frac{\varepsilon}{\chi}}\,\delta(\varepsilon-\chi) - k_2(k_1+k_2)\,\overline{R}_m(\chi,\varepsilon)$$

$$-\,k_2^2(k_1+k_2)\,\hat{R}_m(\chi,\varepsilon,k_2) \qquad (5.16\text{A})$$

$$\Pi_m^{\nu}(\chi,\,\varepsilon,\,k_1) = -\frac{2}{\pi}\,k_1\sqrt{\frac{\varepsilon}{\chi}}\,\delta(\varepsilon-\chi) - k_1^2\,\overline{R}_m(\chi,\varepsilon) - k_1^3\,\hat{R}_m(\chi,\varepsilon,k_2) \tag{5.17B}$$

$$K_m(\xi,\,\chi) = -\frac{2}{\pi}(k_1+k_2)G_m(\xi,\chi) - k_2(k_2+k_1)\int_0^b G_m(\xi,\varepsilon)\,\overline{R}_m(\chi,\varepsilon)\,d\varepsilon$$

$$-k_2^2\,(k_1+k_2)\int_0^b G_m(\xi,\varepsilon)\hat{R}_m(\chi,\varepsilon,k)\,d\varepsilon - k_2^2 i\,[k_1(1-H(k_2-a))$$

$$+\,k_2H(k_2-a)]\sqrt{\chi}\,J_{m-1/2}(k\chi)\int_0^b G_m(\xi,\varepsilon)\,J_{m-1/2}(k\varepsilon)\sqrt{\varepsilon}\,d\varepsilon \tag{5.17A}$$

$$K_m(\xi,\,\chi) = -\frac{2}{\pi}\,k_1G_m(\xi,\chi) - k_1^2\int_0^a G_m(\xi,\varepsilon)\,\overline{R}_m(\chi,\varepsilon)\,d\varepsilon$$

$$-\,k_1^3\int_0^a G_m(\xi,\varepsilon)\,\hat{R}_m(\chi,\varepsilon,k_1)\,d\varepsilon$$

$$-\,2k_1^3\,i\sqrt{\chi}\,J_{m-1/2}(k_1\chi)\int_0^a G_m(\xi,\varepsilon)\,J_{m-1/2}(k_1\varepsilon)\sqrt{\varepsilon}\,d\varepsilon \tag{5.17B}$$

Approximation of the Kernel in Long Wave Region

Now, we need to obtain an asymptotic solution of integral equation (3.10) in the region $k_i \ll 1$. Upon substituting (A.8) into (5.8) the function T_m for $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ can be expressed as a power-logarithmic series

$$T_m = \sum_{j=0}^{1}(-1)^{(j+m)m}\sum_{n=0}^{\infty} k_i^n\,\left|E_j\right|^n\left[\left(\hat{B}_n + \overline{B}_n\,\ln k_i\right) + \overline{B}_n\,\ln\left|E_j\right|\right] \tag{5.18}$$

where $\hat{B}_n\overline{B}_n$ are as following

$$\hat{B}_{0-5} = -0{,}2886;\ -1{,}5718;\ -0{,}2307;\ 0{,}2618;\ -0{,}04675;\ -0{,}0262$$

$$\overline{B}_n = \begin{cases} \dfrac{(-1)^{m+1}}{2(n)!}, & n = 2m \\ 0 & n = 2m+1 \end{cases} \qquad m = 0, 1, 2, \ldots \tag{5.19}$$

Also, (5.18) can be represented as:

$$T_m = \sum_{n=0}^{\infty}\sum_{j=0}^{1}\sum_{\tau=0}^{n} k_i^n(-1)^{(j+m)m}\,c_n^{\tau}(-1)^{j\tau}\chi^{n-\tau}\,\varepsilon^{\tau}\left[\left(\hat{B}_n + \overline{B}_n\,\ln k_i\right) + \overline{B}_n\,\ln\left|E_j\right|\right] \tag{5.20}$$

where c_n^{τ} are binomial coefficients.

Substituting (5.20) into (5.11) K_m for $m=\begin{pmatrix}0\\1\end{pmatrix}$ leads to the following

$$K_m(\xi,\chi) = -\frac{2}{\pi}(k_1-k_2)G_m(\xi,\chi) - \frac{4}{\pi^2}k_2(k_1+k_2)\sum_{n=0}^{\infty} k_2^n \sum_{j=0}^{1}(-1)^{(j+m)m}\sum_{\tau=0}^{\infty}(-1)^{j\tau}c_n^\tau \chi^{n-\tau}$$

$$\times\left\{(\hat{B}_n + B_n \ln k_2)\, G_m^\tau(\xi) + \overline{B}_n\, G_m^{j,\tau}(\xi,\chi)\right\} - -\frac{8}{\pi}k_2[k_1(1-H(k_2-a))$$

$$+k_2 H(k_2-a)]i\sum_{n,l=0}^{\infty}(-1)^{n+l}k_2^{2(n+l)}\chi^{2l}\begin{bmatrix}\dfrac{1}{(2n)!(2l)!}G_0^{2n}(\xi)\\ \dfrac{k_2^2\xi}{(2n+1)!(2l+1)!}G_1^{2n+1}(\xi)\end{bmatrix} \qquad (5.21A)$$

$$K_m(\xi,\chi) = -\frac{2}{\pi}k_1 G_m(\xi,\chi) - \frac{4}{\pi^2}k_1^2\sum_{n=0}^{\infty}k_1^n\sum_{j=0}^{1}(-1)^{(j+m)m}\sum_{\tau=0}^{\infty}(-1)^{j\tau}c_n^\tau\chi^{n-\tau}$$

$$\times\left\{(\hat{B}_n + \overline{B}_n \ln k_1)\, G_m^\tau(\xi) + \overline{B}_n\, G_m^{j,\tau}(\xi,\chi)\right\}$$

$$-\frac{4}{\pi}k_1^2\, i\sum_{n,l=0}^{\infty}(-1)^{n+l}k_1^{2(n+l)}\chi^{2l}\begin{bmatrix}\dfrac{1}{(2n)!(2l)!}G_0^{2n}(\xi)\\ \dfrac{1}{(2n+1)!(2l+1)!}G_1^{2n+1}(\xi)\end{bmatrix} \qquad (5.21B)$$

where

$$G_m^j(\xi) = \int_0^{r_i} G_m(\xi,\varepsilon)\,\varepsilon^j d\varepsilon$$

$$G_m^{j,\tau}(\xi,\chi) = \int_0^{r_i} G_m^{j,\tau}(\xi,\varepsilon)\, lm\left|E_j\right|\varepsilon^\tau d\varepsilon$$

In order to obtain K_m for $m \geq 2$, by substituting (A.8) into we get

$$T_d(k_i\zeta) = \sum_{n=0}^{\infty}(k_i\zeta)^n\left[\hat{D}_n + \overline{D}_n \ln(k_i\zeta)\right] \qquad (5.22)$$

$\hat{D}_n, \overline{D}_n$, are known numerical coefficients of the expansion, and substituting (5.22) into (5.15) the function $\hat{R}_m$ can be obtained.

$$\hat{R}_m(\chi,\varepsilon,k_i) = \frac{2}{\pi^2}(\chi\varepsilon)\sum_{n=0}^{\infty}k_i^n\left[(\hat{D}_n + \overline{D}_n \ln k_i)\,\hat{\Psi}_m^n(\chi,\varepsilon) + \overline{D}_n\,\ddot{\Psi}_m^n(\chi,\varepsilon)\right] \qquad (5.23)$$

where

$$\hat{\Psi}_m^n(\chi,\varepsilon) = \int_{-1}^{1} \zeta^{n-1} P_{m-1}(\tau)\, d\tau$$

$$\ddot{\Psi}_m^n(\chi,\varepsilon) = \int_{-1}^{1} \zeta^{n-1} \ln\zeta\, P_{m-1}(\tau)\, d\tau$$

Also, substitution of (5.23) into (5.17) and using (A.9) and introducing transformation of G_m, we get K_m for $m \geq 2$

$$\begin{aligned} K_m(\xi,\chi) = & -\frac{2}{\pi}(k_1+k_2)G_m(\xi,\chi) - k_2(k_2+k_1)\bar{G}_m(\xi,\chi) \\ & -\frac{2}{\pi^2}k_2^2(k_1-k_2)\chi \sum_{n=0}^{\infty} k_2^n \left[(\hat{D}_n + \bar{D}_n \ln k_2)\hat{G}_m^n(\xi,\chi) + \bar{D}_n\, \ddot{G}_m^n(\chi,\varepsilon) \right] \\ & -4\pi i k_2^2\,[k_1(1-H(k_2-a)) + k_2 H(k_2-a)] \\ & \sum_{n,l=0}^{\infty} \frac{(-1)^{n+l} k_1^{2m+2(n+l)-1} \chi^{m+2n-1/2} G_m^{m+2l-1/2}(\xi)}{2^{2m+2(n+l)-1} n!\, l\Gamma(n+m+1/2)\Gamma(l+n+1/2)} \end{aligned} \qquad (5.24\text{A})$$

$$\begin{aligned} K_m(\xi,\chi) = & -\frac{2}{\pi}k_1 G_m(\xi,\chi) - k_1^2\bar{G}_m(\xi,\chi) \\ & -\frac{2}{\pi^2}k_1^3\chi \sum_{n=0}^{\infty} k_1^n \left[(\hat{D}_n + \bar{D}_n \ln k_1)\,\hat{G}_m^n(\xi,\chi) + \bar{D}_n\, \ddot{G}_m^n(\chi,\varepsilon) \right] \\ & -\pi i k_1^3 \sum_{n,l=0}^{\infty} \frac{(-1)^{n+l} k_1^{2m+2(n+l)-1} \chi^{m+2n-1/2} G_m^{m+2l-1/2}(\xi)}{2^{2m+2(n+l)-1} n!\, l\Gamma(n+m+1/2)\Gamma(l+n+1/2)} \end{aligned} \qquad (5.24\text{B})$$

that

$$\bar{G}_m = \int_0^{r_i} G_m(\xi,\varepsilon)\, \bar{R}_m(\chi,\varepsilon)\, d\varepsilon$$

$$\hat{G}_m^n(\xi,\chi) = \int_0^{r_i} G_m(\xi,\varepsilon)\, \hat{\Psi}_m^n(\chi,\varepsilon)\varepsilon\, d\varepsilon$$

$$\ddot{G}_m^n(\xi,\chi) = \int_0^{r_i} G_m(\xi,\varepsilon)\, \ddot{\Psi}_m^n(\chi,\varepsilon)\varepsilon\, d\varepsilon$$

Relations (5.21) and (5.24) indicate K_m can be expressed as a power-logarithmic series with respect to the wave number k_2.

$$K_m(\xi, \chi) = k_i \sum_{n=0}^{\infty} [M_m^n(\xi, \chi) + N_m^n(\xi, \chi) \ln k_i]\, k_i^n \tag{5.25}$$

where M_m^n, N_m^n are known function that can be determined from (5.21) and (5.24). Also for we have $N_m^n = 0$, $n = 1, 3, 5, ...$

6. SHALLOW WATER APPROXIMATIN

In this case, we have $k_2 d$, $k_1 d \ll 1$ and the functions $s(\lambda)$ and $\lambda s(\lambda)$ have the following representation

$$s(\lambda) = \frac{k_1^2[1 - H(\lambda - a)] + k_2^2 H(\lambda - a)}{k_2^2 - \lambda^2} \tag{6.1A}$$

$$\lambda s(\lambda) = \frac{k_1^2[1 - H(\lambda - a)] + k_2^2 H(\lambda - a)}{2}\left(\frac{1}{\lambda - k_2} + \frac{1}{\lambda - k_2}\right) \tag{6.1B}$$

$$s(\lambda) = \frac{k_1^2}{k_1^2 - \lambda^2} \tag{6.2A}$$

$$\lambda s(\lambda) = \frac{k_1^2}{2}\left(\frac{1}{\lambda - k_2} + \frac{1}{\lambda + k_2}\right) \tag{6.2B}$$

for Π_m^v, (4.7), in this case we have

$$\Pi_m^v = -\frac{1}{\pi}\sqrt{\chi\varepsilon}\,(k_1^2 + k_2^2)\, P.V. \int_0^\infty \left(\frac{1}{\lambda - k_2} + \frac{1}{\lambda - k_2}\right) J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon)\, \lambda d\lambda$$

$$= -(k_1^2 + k_2^2)\, R_m \tag{6.3A}$$

$$\Pi_m^v = -k_1^2 R_m \tag{6.3B}$$

where

$$R_m(\chi, \varepsilon, k_i) = \frac{2}{\pi}\sqrt{\chi\varepsilon}\, P.V. \int_0^\infty J_{m-1/2}(\lambda\chi)\, J_{m-1/2}(\lambda\varepsilon)\, \frac{\lambda^2}{\lambda^2 - k_i^2}\, d\lambda \tag{6.4}$$

for $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, R_m has the following form

$$R_m = \frac{4}{\pi^2}\, P.V. \int_0^\infty \begin{pmatrix} \cos\lambda\chi \\ \sin\lambda\chi \end{pmatrix} \begin{pmatrix} \cos\lambda\varepsilon \\ \sin\lambda\varepsilon \end{pmatrix} \frac{\lambda}{\lambda^2 - k_i^2}\, d\lambda \tag{6.5}$$

by considering

$$\frac{\lambda}{\lambda^2 - k_i^2} = \frac{1}{2}\left(\frac{1}{\lambda - k_i} + \frac{1}{\lambda + k_i}\right)$$

relation (6.5) can be expressed as Hilbert and Siltjes transform of cos λE_j. as a result for R_m we have

$$R_m = -\frac{2}{\pi^2}\sum_{j=0}^{1}(-1)^{(j+m)m}\left\{\frac{\pi}{2}\sin k_i\left|E_j\right| + \left[\sin k_i\left|E_j\right| si\left(k_i\left|E_j\right|\right) + \cos k_i\left|E_j\right| ci\left(k_i\left|E_j\right|\right)\right]\right\} \quad (6.6)$$

In this case K_m, (4.4) has the following form

$$K_m(\xi, \chi) = -(k_1^2 + k_2^2)\int_0^b G_m(\xi, \varepsilon)\, R_m\,(\chi, \varepsilon, k)\, d\varepsilon$$

$$-\frac{8}{\pi}k_2 i\frac{k_1^2[1-H(k_2-a)]+k_2^2H(k_2-a)}{k_2+\dfrac{k_2}{d}(1-k_2^2d^2)}\begin{pmatrix}\cos k_2\chi\\ \sin k_2\chi\end{pmatrix}\int_0^b G_m(\xi,\varepsilon)\begin{pmatrix}\cos k_2\varepsilon\\ \sin k_2\varepsilon\end{pmatrix}d\varepsilon \quad (6.7A)$$

$$K_m(\xi, \chi) = -k_1^2\int_0^a G_m(\xi,\varepsilon)R_m(\chi,\varepsilon,k_1)\,d\varepsilon - \frac{2}{\pi}k_1^2 i\begin{pmatrix}\cos k_1\chi\\ \sin k_1\chi\end{pmatrix}\int_0^a G_m(\xi,\varepsilon)\begin{pmatrix}\cos k_1\varepsilon\\ \sin k_1\varepsilon\end{pmatrix}d\varepsilon \quad (6.7B)$$

At this point, we need to obtain K_m for $m \geq 2$, upon substituting (A.7) into (6.3) we get

$$R_m = \bar{R}_m + k_i\hat{R}_m \quad (6.8)$$

where

$$\bar{R}_m(\chi, \varepsilon) = -\frac{2}{\pi\sqrt{\pi}}\frac{(\chi\varepsilon)^m}{(\chi+\varepsilon)^{2m}}\frac{\Gamma(m)}{\Gamma(m+1/2)}F\left[m, m; 2m; \frac{4\chi\varepsilon}{(\chi+\varepsilon)^2}\right] \quad (6.9)$$

$$\hat{R}_m = \frac{2\chi\varepsilon}{\pi^2}\int_{-1}^{1}T_i(k_i\zeta)P_{m-1}(\tau)\frac{d\tau}{\zeta} \quad (6.10)$$

and

$$T_s(k_i\zeta) = -\frac{\pi}{2}\cos(k_i\zeta) - \cos(k_i\zeta)\, si\,(k_i\zeta) + \sin(k_i\zeta)\, ci\,(k_i\zeta)$$

in this case for $m \geq 2$, and (4.4) leads to

$$K_m(\xi, \chi) = -(k_1^2 + k_2^2)\int_0^b G_m(\xi, \varepsilon)\, \bar{R}_m\,(\chi, \varepsilon)\, d\varepsilon - k_2(k_1^2 + k_2^2)\int_0^b G_m(\xi, \varepsilon)\, \hat{R}_m\,(\chi, \varepsilon, k)\, d\varepsilon$$

$$-4ik_2^2\,\frac{k_1^2[1-H(k_2-a)]+k_2^2 H(k_2-a)}{k_2+\dfrac{k_2}{d}(1-k_2^2 d^2)}\sqrt{\chi}\,J_{m-1/2}(k\chi)\int_0^b G_m(\xi,\varepsilon) J_{m-1/2}(k\varepsilon)\sqrt{\varepsilon}\,d\varepsilon \tag{6.11A}$$

$$K_m(\xi, \chi) = -k_1^2\int_0^a G_m(\xi, \varepsilon)\, \bar{R}_m\,(\chi, \varepsilon)\, d\varepsilon - k_1^3\int_0^a G_m(\xi, \varepsilon)\, \hat{R}_m\,(\chi, \varepsilon, k_1)\, d\varepsilon$$

$$-ik_1^3\sqrt{\chi}\,J_{m-1/2}(k_1\chi)\int_0^a G_m(\xi, \varepsilon)\, J_{m-1/2}\,(k_1\varepsilon)\sqrt{\varepsilon}\,d\varepsilon \tag{6.11B}$$

7. CONCLUDING REMARKS

In this article, we have presented an extension of dual integral equation for the Dock – Problem in presence of cracked – barrier near the plate. For solving, we reduce dual integral equation into Fredholm's integral equation and approximate the kernel of integral equation for deep (shallow) water and long wave region. Also, we can extend this problem for two, three barriers with the specified boundary conditions.

APPENDIX

1. A Fourier series with Bessel function components

$$e^{ikr\cos\theta} = \sum_{m=0}^{\infty} f^m J_m(kr)\cos m\theta$$

$$f^m = \{1,\, m = 0; \quad 2i^m,\, m \geq 1\}$$

2. An integral representation for Bessel functions

$$\lambda J_{m-\frac{1}{2}}(\lambda\xi) = \xi^{-m-\frac{1}{2}}\,\frac{d}{d\xi}\left[\xi^{m+\frac{1}{2}} J_{m+\frac{1}{2}}(\lambda\xi)\right]$$

3. A integral relation for a multiplication of the Bessel function [1, 9]

$$\int_0^\infty J_{m+\frac{1}{2}}(\lambda\xi) J_m(\lambda r)\sqrt{\lambda}\,d\lambda = \sqrt{\frac{2}{\pi}}\,\frac{r^m H(\xi - r)}{\xi^{m+\frac{1}{2}}\sqrt{\xi^2 - r^2}}$$

4. Integral representation for Bessel functions

$$J_m(\lambda r) = \sqrt{\frac{2\lambda}{\pi}}\, r^{-m} \int_0^{r_i} \frac{\xi^{m+\frac{1}{2}}}{\sqrt{r^2 - \xi^2}}\, J_{m-\frac{1}{2}}(\lambda\xi)\, d\xi$$

5. $$J_{\pm\frac{1}{2}}(x) = \begin{pmatrix} \sqrt{\frac{2}{\pi x}}\sin(x) \\ \sqrt{\frac{2}{\pi x}}\cos(x) \end{pmatrix}$$

6. A representation of the Dirac delta function through Hankel transform [1, 9]

$$\int_0^{\infty} J_m(\lambda\chi)\, J_m(\lambda\varepsilon)\, \lambda d\lambda = \frac{1}{\chi}\,\delta(\varepsilon - \chi)$$

7. An integral representation for multiplication of Bessel function

$$J_{m-\frac{1}{2}}(\lambda\chi)\, J_{m-\frac{1}{2}}(\lambda\varepsilon) = \frac{1}{\pi}\sqrt{\chi\varepsilon} \int_{-1}^{1} \frac{\sin\lambda\zeta}{\zeta}\, P_{m-1}(\tau)\, d\tau \qquad \zeta = \sqrt{\chi^2 + \varepsilon^2 - 2\chi\varepsilon\tau}$$

8. Series and integral representations for sine and cosine integral

$$ci(x) = \gamma + \ln x + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!(2j)} \qquad \begin{pmatrix} ci\,(x) \\ si\,(x) \end{pmatrix} = \left(\int_0^{\infty} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \frac{dt}{t} \right)$$

$$si(x) = -\frac{\pi}{2} + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!(2j+1)}$$

9. Series representation for Bessel function

$$J_m(k_2\varepsilon) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+m+1)n!} \left(\frac{k_2\varepsilon}{2}\right)^{2n+m}$$

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A. Aghili & A. Ansari
Department of Mathematics, Faculty of Sciences
Guilan University, P.O. Box 1841, Rasht, Iran
E-mails: armanaghili@yahoo.com, & *alireza_1038@yahoo.com*