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FUZZY NUMBER OPERATION FUZZY FUNCTION WITH DIFFERENTIATION AND INTEGRATION

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Abstract: In this paper we have given the concepet of FUZZY Number and its operations. This paper also deals the Fuzzy function, with intergration and differentiation of Fuzzy function.

Keyword: Fuzzy Number, Triangular Fuzzy Number Trapezoidal Fuzzy Number, Bell shape Fuzzy Number.

INTRODUCTION

4.1 Concept of Fuzzy Number

4.1.1 Interval

When interval is defined on real number $\Re$, this interval is said to be a subset of $\Re$. For instance, if interval is denoted as $A = [a_1, a_3]$ $a_1, a_3 \in \Re$, $a_1 < a_3$, we may regard this as one kind of sets. Expressing the interval as membership function as shown in the following (Fig 4.1) : As shown below

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ 1, & a_1 \le x \le a_3 \\ 0, & x > a_3 \end{cases}$$

If $a_1 = a_3$, this interval indicates a point. That is, $[a_1, a_1] = a_1$

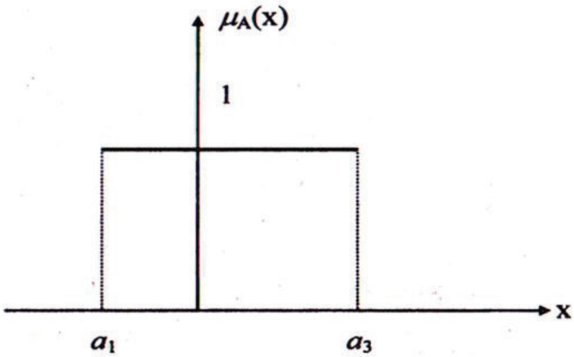


Fig. 4.1. **Interval $A = [a_1, a_3]$**

4.1.2 Fuzzy Number

Fuzzy number is expressed as a fuzzy set defining a fuzzy interval in the real number $\Re$. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally a fuzzy interval is represented by two end points a_1 and a_3 and a peak point a_2 as $[a_1, a_2, a_3]$ (Fig 5.2). The α-cut operation can be also applied to the fuzzy number. If we denote α-cut interval for fuzzy number A as A_α, the obtained interval A_α is defined as

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$

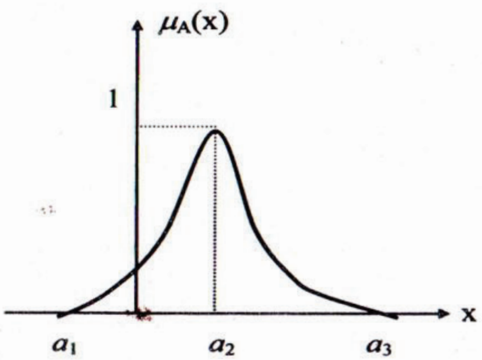


Fig. 4.2 Fuzzy Number $A = [a_1, a_2, a_3]$

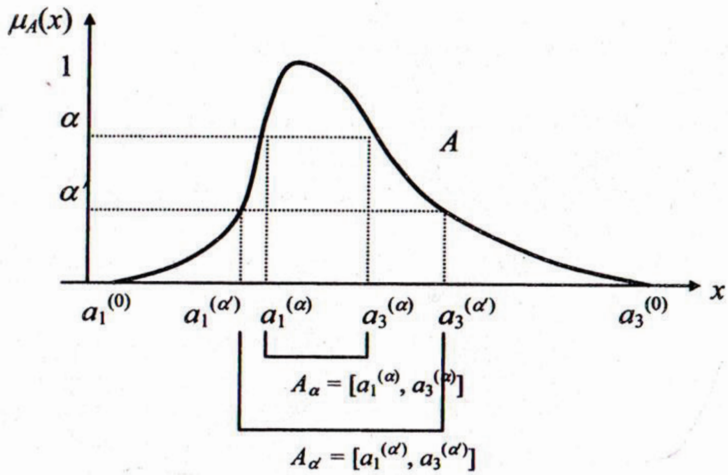


Fig. 4.3 α-cut of fuzzy number $(\alpha' < \alpha) \Rightarrow (A_\alpha \subset A_{\alpha'})$

The convex condition may also be written as,

$$(\alpha' < \alpha) \Rightarrow (A_\alpha \subset A_{\alpha'})$$

4.1.3 Operation of Interval

Operation of fuzzy number can be generalized from that of crisp interval. Let's have a look at the operations of interval.

$\forall a_1, a_3, b_1, b_3 \in \Re$

$$A = [a_1, a_3], B = [b_1, b_3]$$

Assuming A and B as numbers expressed as interval, main operations of interval are

(1) Addition

$$[a_1, a_3]\ (+)\ [b_1, b_3] = [a_1 + b_1, a_3 + b_3]$$

(2) Subtraction

$$[a_1, a_3]\ (-)\ [b_1, b_3] = [a_1 - b_3, a_3 - b_1]$$

(3) Multiplication

$[a_1, a_3]\ (\bullet)\ [b_1, b_3] = [a_1 \bullet b_1 \wedge a_1 \bullet b_3 \wedge a_3 \bullet b_1 \wedge a_3 \bullet b_3, a_1 \bullet b_1 \vee a_1 \bullet b_3 \vee a_3 \bullet b_1 \vee a_3 \bullet b_3]$

(4) Division

$[a_1, a_3]\ (/)\ [b_1, b_3] = [a_1 / b_1 \wedge a_1 / b_3 \wedge a_3 / b_1 \wedge a_3 / b_3, a_1 / b_1 \vee a_1 / b_3 \vee a_3 / b_1 \vee a_3 / b_3]$

excluding the case $b_1 = 0$ or $b_3 = 0$

(5) Inverse interval

$$[a_1, a_3]^{-1} = [1 / a_1 \wedge 1 / a_3, 1 / a_1 \vee 1 / a_3]$$

excluding the case $a_1 = 0$ or $a_3 = 0$

When previous sets A and B is defined in the positive real number $\Re+$, the operations of multiplication, division, and inverse interval are written as,

(3') Multiplication

$$[a_1, a_3]\ (\bullet)\ [b_1, b_3] = [a_1 \bullet b_1, a_3 \bullet b_3]$$

(4') Division

$$[a_1, a_3]\ (/)\ [b_1, b_3] = [a_1 / b_3, a_3 / b_1]$$

(5') Inverse Interval

$$[a_1, a_3]^{-1} = [1 / a_3, 1 / a_1]$$

(6) Minimum

$$[a_1, a_3]\ (\wedge)\ [b_1, b_3] = [a_1 \wedge b_1, a_3 \wedge b_3]$$

(7) Maximum

$$[a_1, a_3]\ (\vee)\ [b_1, b_3] = [a_1 \vee b_1, a_3 \vee b_3]$$

Example 4.1 There are two intervals A and B,

$$A = [3, 5], B = [-2, 7]$$

then following operation might be set.

$$A(+)B = [3-2, \ 5+7] = [1, \ 12]$$

$$A(-)B = [3-7, \ 5-(-2)] = [-4, \ 7]$$

$$\begin{aligned} A(\bullet)B &= [3\bullet(-2)\wedge 3\bullet 7\wedge 5\bullet(-2)\wedge 5\bullet 7, \ \ 3\bullet(-2)\vee\cdots] \\ &= [-10, \ \ 35] \end{aligned}$$

$$\begin{aligned} A(/)B &= [3/(-2)\wedge 3/7\wedge 5/(-2)\wedge 5/7, \ \ 3/(-2)\vee\cdots] \\ &= [-2.5, \ \ 5/7] \end{aligned}$$

$$B^{-1} = [-2,7]^{-1} = \left[\frac{1}{(-2)}\wedge\frac{1}{7}, \ \ \frac{1}{(-2)}\vee\frac{1}{7}\right] = \left[-\frac{1}{2}, \ \ \frac{1}{7}\right] \qquad \square$$

4.2 Operation of Fuzzy Number

4.2.1 Operation of α-cut Interval

We referred to α-cut interval of fuzzy number $A = [a_1, a_3]$ as crisp set

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}], \forall\alpha \in [0, 1], a_1, a_3, a_1^{(\alpha)}, a_3^{(\alpha)} \in \Re$$

so A_α is a crisp interval. As a result, the operations of interval reviewed in the previous section can be applied to the α-cut interval A_α.

If α-cut interval B_α of fuzzy number B is given

$$B = [b_1, b_3], \quad b_1, b_3, \in \Re$$

$$B_\alpha = [b_1^{(\alpha)}, b_3^{(\alpha)}], \forall\alpha \in [0, 1], b_1^{(\alpha)}, b_3^{(\alpha)} \in \Re,$$

operations between A_α and B_α can be described as follows :

$$[a_1^{(\alpha)}, a_3^{(\alpha)}] (+) [b_1^{(\alpha)}, b_3^{(\alpha)}] = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_3^{(\alpha)} + b_3^{(\alpha)}]$$

$$[a_1^{(\alpha)}, a_3^{(\alpha)}] (-) [b_1^{(\alpha)}, b_3^{(\alpha)}] = [a_1^{(\alpha)} - b_3^{(\alpha)}, a_3^{(\alpha)} - b_1^{(\alpha)}]$$

these operations can be also applicable to multiplication and division in the same manner.

4.2.2 Operation of Fuzzy Number

Previous operations of interval are also applicable to fuzzy number. Since outcome of fuzzy number (fuzzy set) is in the shape of fuzzy set, the result is expressed in membership function.

$\forall x, y, z \in \Re$

(1) Addition: $A\,(+)\,B$

$$\mu_{A(+)B}(z) = \bigvee_{z=x+y} (\mu_A(x) \wedge \mu_B(y))$$

(2) Subtraction: $A\,(-)\,B$

$$\mu_{A(-)B}(z) = \bigvee_{z=x-y} (\mu_A(x) \wedge \mu_B(y))$$

(3) Multiplication: $A\,(\bullet)\,B$

$$\mu_{A(\bullet)B}(z) = \bigvee_{z=x\bullet y} (\mu_A(x) \wedge \mu_B(y))$$

(4) Division: $A\,(/)\,B$

$$\mu_{A(/)B}(z) = \bigvee_{z=x/y} (\mu_A(x) \wedge \mu_B(y))$$

(5) Minimum: $A\,(\wedge)\,B$

$$\mu_{A(\wedge)B}(z) = \bigvee_{z=x\wedge y} (\mu_A(x) \wedge \mu_B(y))$$

(6) Maximum: $A\,(\vee)\,B$

$$\mu_{A(\vee)B}(z) = \bigvee_{z=x\vee y} (\mu_A(x) \wedge \mu_B(y))$$

We can multiply a scalar value to the interval. For instance, multiplying $a \in \Re$,

$$a[b_1, b_3] = [a \bullet b_1 \wedge a \bullet b_3, a \bullet b_1 \vee a \bullet b_3]$$

Example 4.2 There is a scalar multiplication to interval. Note the scalar value is negative.

$$\begin{aligned} -4.15\,[-3.55, 0.21] &= [(-4.15) \bullet (-3.55) \wedge (-4.15) \bullet 0.21, (-4.15) \bullet (-3.55) \\ &\quad \vee (-4.15) \bullet 0.21] \\ &= [14.73 \wedge -0.87, 14.73 \vee -0.87] \\ &= [-0.87, 14.73] \quad \square \end{aligned}$$

We can also multiply scalar value to α-cut interval of fuzzy number.

$\forall \alpha \in [0, 1], b_1^{(\alpha)}, b_3^{(\alpha)} \in \Re$

$$a[b_1^{(\alpha)}, b_3^{(\alpha)}] = [a \bullet b_1^{(\alpha)} \wedge a \bullet b_3^{(\alpha)}, a \bullet b_1^{(\alpha)} \vee a \bullet b_3^{(\alpha)}] \quad \square$$

4.2.3 Examples of Fuzzy Number Operation

Example 4.3 Addition A(+)B
For further understanding of fuzzy number operation, let us consider two fuzzy sets A and B. Note that these fuzzy sets are defined on discrete numbers for simplicity.

$$A = \{(2, 1), (3, 0.5)\}, B = \{(3, 1), (4, 0.5)\}$$

First of all, our concern is addition between A and B. To induce $A(+)B$, for all $x \in A, y \in B, z \in A(+)B$, we check each case as follows(Fig 5.4) :

i) for $z < 5$,

$$\mu_{A(+)B}(z) = 0$$

ii) $z = 5$

results from $x + y = 2 + 3$

$$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$$

$$\mu_{A(+)B}(5) = \vee_{5=2+3} (1) = 1$$

iii) $z = 6$

results from $x + y = 3 + 3$ or $x + y = 2 + 4$

$$\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_{A(+)B}(6) = \vee_{\substack{6=3+3 \\ 6=2+4}} (0.5, 0.5) = 0.5$$

iv) $z = 7$

results from $x + y = 3 + 4$

$$\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$$

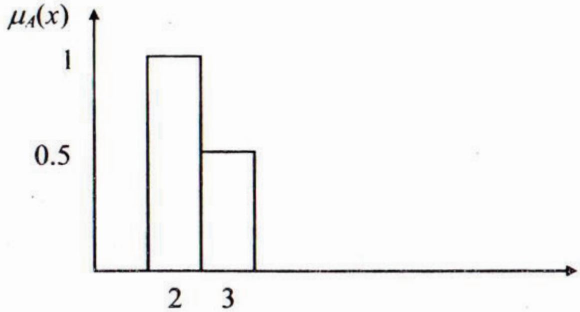
$$\mu_{A(+)B}(7) = \vee_{7=3+4} (0.5) = 0.5$$

v) for $z > 7$

$$\mu_{A(+)B}(z) = 0$$

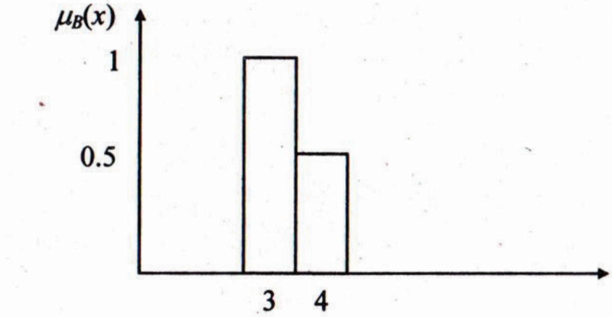
so $A(+)B$ can be written as

$$A(+)B = \{(5, 1), (6, 0.5), (7, 0.5)\} \quad \square$$

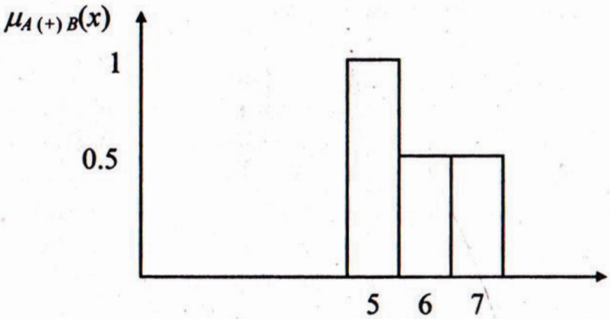


(a) Fuzzy set A

Fig. 4.4 Add operation of fuzzy set



(b) Fuzzy number B



(c) Fuzzy set A (+) B

Fig. 4.4 (cont')

Example 4.5 Subtraction $A(-)B$

Let's manipulate $A(-)B$ between our previously defined fuzzy set A and B. For $x \in A$, $y \in B$, $z \in A(-)B$, fuzzy set $A(-)B$ is defined as follows (Fig5. 5).

i) For $z < -2$,

$$\mu_{A(-)B}(z) = 0$$

ii) $z = -2$

$$\text{results from } x - y = 2 - 4$$
$$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$$
$$\mu_{A(-)B}(-2) = 0.5$$

iii) $z = -1$

$$\text{results from } x - y = 2 - 3 \text{ or } x - y = 3 - 4$$

$$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$$
$$\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$$
$$\mu_{A(-)B}(-1) = \vee_{\substack{-1=2-3 \\ -1=3-4}} (1, 0.5) = 1$$

iv) $z = 0$

results from $x - y = 3 - 3$
$$\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$$
$$\mu_{A(-)B}(0) = 0.5$$

v) For $z \geq 1$

$$\mu_{A(-)B}(z) = 0$$

so $A(-)B$ is expressed as
$$A(-)B = \{(-2, 0.5), (-1, 1), (0, 0.5)\}$$ □

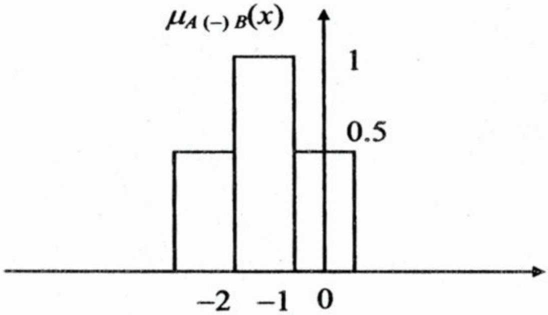


Fig. 4.5 Fuzzy number A (−) B

Example 4.6 Max operation $A(\vee)B$

Let's deal with the operation Max $A(\vee)B$ between A and B

for $x \in A$, $y \in B$, $z \in A(\vee)B$, fuzzy set $A(\vee)B$ is defined by $\mu_{A(\vee)B}(z)$.

i) $z \leq 2$

$$\mu_{A(\vee)B}(z) = 0$$

ii) $z = 3$

from $x \vee y = 2 \vee 3$ and $x \vee y = 3 \vee 3$
$$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$$
$$\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$$
$$\mu_{A(\vee)B}(3) = \vee_{\substack{3=2\vee 3 \\ 3=3\vee 3}} (1, 0.5) = 1$$

iii) $z = 4$

from $x \vee y = 2 \vee 4$ and $x \vee y = 3 \vee 4$
$$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$$

$$\mu_{A(\vee)B}(4) = \bigvee_{\substack{4=2\vee 4 \\ 4=3\vee 4}} (0.5, 0.5) = 0.5$$

v) $z > 5$

impossible $\mu_{A(\vee)B}(z) = 0$

so $A(\vee)B$ is defined to be

$$A(\vee)B = \{(3, 1), (4, 0.5)\} \quad \square$$

so far we have seen the results of operations are fuzzy sets, and thus we come to realize that the extension principle is applied to the operation of fuzzy number.

4.3 Triangular Fuzzy Number

4.3.1 Definition of Triangular Fuzzy Number

Among the various shapes of fuzzy number, triangular fuzzy number(TFN) is the most popular one.

Definition (Triangular fuzzy number) It is a fuzzy number represented with three points as follows :

$$A = (a_1, a_2, a_3)$$

this representation is interpreted as membership functions (Fig5.6).

$$\mu_{(A)}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & x > a_3 \end{cases} \quad \square$$

Now if you get crisp interval by α-cut operation, interval A_α shall be obtained as follows $\forall \alpha \in [0, 1]$

from

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha, \quad \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha$$

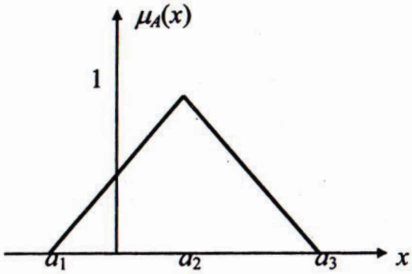


Fig. 4.6 Triangular fuzzy number $A = (a_1, a_2, a_3)$

we get

$$a_1^{(\alpha)} = (a_2 - a_1)\alpha + a_1$$
$$a_3^{(\alpha)} = -(a_3 - a_2)\alpha + a_3$$

thus

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$
$$= [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]$$

Example 4.7 In the case of the triangular fuzzy number A = (–5, –1, 1) (Fig 5.7), the membership function value will be,

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -5 \\ \dfrac{x+5}{4}, & -5 \le x \le -1 \\ \dfrac{1-x}{2}, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

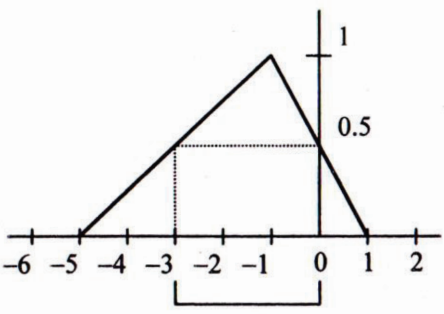


Fig. 4.7. $\alpha = 0.5$ cut of triangular fuzzy number $A = (-5, -1, 1)$

α-cut interval from this fuzzy number is

$$\frac{x+5}{4} = \alpha \quad \Rightarrow \quad x = 4\alpha - 5$$

$$\frac{1-x}{2} = \alpha \quad \Rightarrow \quad x = -2\alpha + 1$$

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}] = [4\alpha - 5, -2\alpha + 1]$$

if $\alpha = 0.5$, substituting 0.5 for α, we get $A_{0.5}$

$$A_{0.5} = [a_1^{(0.5)}, a_3^{(0.5)}] = [-3, 0] \quad \square$$

4.3.2 Operation of Triangular Fuzzy Number

Same important properties of operations on triangular fuzzy number are summarized

(1) The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.

(2) The results from multiplication or division are not triangular fuzzy numbers.

(3) Max or min operation does not give triangular fuzzy number.

but we often assume that the operational results of multiplication or division to be TFNs as approximation values.

1) Operation of triangular fuzzy number

first, consider addition and subtraction. Here we need not use membership function. Suppose triangular fuzzy numbers A and B are defined as,

$$A = (a_1, a_2, a_3), B = (b_1, b_2, b_3)$$

i) Addition

$$\begin{aligned} A(+)B &= (a_1, a_2, a_3)(+)(b_1, b_2, b_3) \quad \text{: triangular fuzzy number} \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \end{aligned}$$

ii) Subtraction

$$\begin{aligned} A(-)B &= (a_1, a_2, a_3)(-)(b_1, b_2, b_3) \quad \text{: triangular fuzzy number} \\ &= (a_1 - b_3, a_2 - b_2, a_3 - b_1) \end{aligned}$$

iii) Symmetric image

$$-(A) = (-a_3, -a_2, -a_1) \quad \text{: triangular fuzzy number}$$

Example 4.8 Let's consider operation of fuzzy number A, B(Fig 5.8).

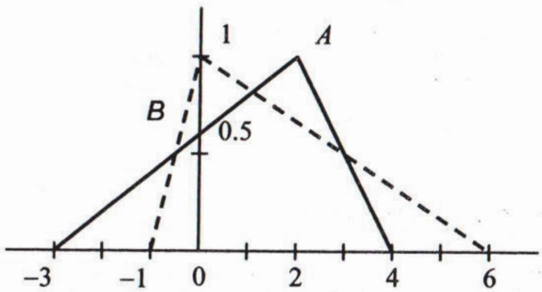
$$A = (-3, 2, 4), B = (-1, 0, 6)$$

$$A\ (+)\ B = (-4, 2, 10)$$

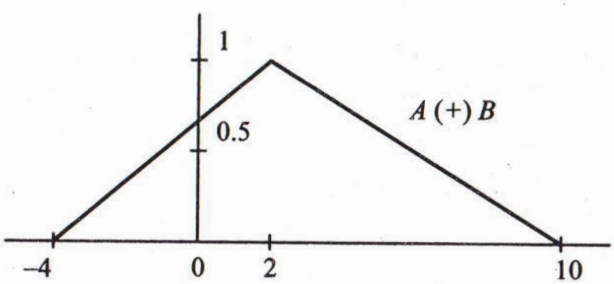
$$A\ (-)\ B = (-9, 2, 5) \quad \square$$

2) Operations with α-cut

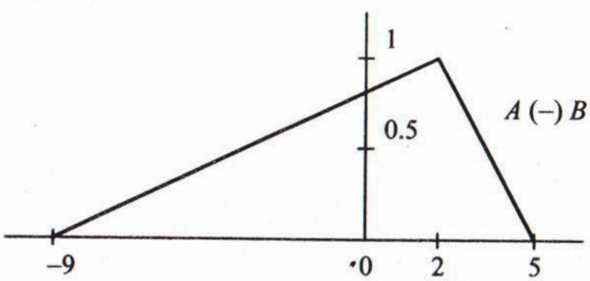
Example 4.9 α-level intervals from α-cut operation in the above two triangular fuzzy numbers A and B are



(a) Triangular fuzzy number A, B



(b) A (+) B of triangular fuzzy numbers



(c) A (–) B triangular fuzzy numbers

Fig. 4.8 A (+) B and A (–) B of triangular fuzzy numbers

$$\begin{aligned} A_\alpha &= [a_1^{(\alpha)}, a_3^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \\ &= [5\alpha - 3, -2\alpha + 4] \\ B_\alpha &= [b_1^{(\alpha)}, b_3^{(\alpha)}] = [(b_2 - b_1)\alpha + b_1, -(b_3 - b_2)\alpha + b_3] \\ &= [\alpha - 1, -6\alpha + 6] \end{aligned}$$

performing the addition of two α-cut intervals A_α and B_α,

$$A_\alpha (+) B_\alpha = [6\alpha - 4, -8\alpha + 10]$$

especially for $\alpha = 0$ and $\alpha = 1$,

$$A_0 (+) B_0 = [-4, 10]$$
$$A_1 (+) B_1 = [2, 2] = 2$$

three points from this procedure coincide with the three points of triangular fuzzy number (-4, 2, 10) from the result $A(+)B$ given in the previous example.

Likewise, after obtaining $A_\alpha(-)B_\alpha$, let's think of the case when $\alpha = 0$ and $\alpha = 1$

$$A_\alpha (-) B_\alpha = [11\alpha - 9, -3\alpha + 5]$$

substituting $\alpha = 0$ and $\alpha = 1$ for this equation,

$$A_0 (-) B_0 = [-9, 5]$$
$$A_1 (-) B_1 = [2, 2] = 2$$

these also coincide with the three points of $A(-)B = (-9, 2, 5)$. □

Consequently, we know that we can perform operations between fuzzy number using α-cut interval.

4.3.3 Operation of General Fuzzy Numbers

Up to now, we have considered the simplified procedure of addition and subtraction using three points of triangular fuzzy number. However, fuzzy numbers may have general form, and thus we have to deal the operations with their membership functions.

Example 4.10 Addition A (+) B

Here we have two triangular fuzzy numbers and will calculate the addition operation using their membership functions

$$A = (-3, 2, 4),\ B = (-1, 0, 6)$$

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -3 \\ \dfrac{x+3}{2+3}, & -3 \le x \le 2 \\ \dfrac{4-x}{4-2}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

$$\mu_{(B)}(y) = \begin{cases} 0, & y < -1 \\ \dfrac{y+1}{0+1}, & -1 \le y \le 0 \\ \dfrac{6-y}{6-0}, & 0 \le y \le 6 \\ 0, & y > 6 \end{cases}$$

for the two fuzzy number $x \in A$ and $y \in B$, $z \in A\,(+)\,B$ shall be obtained by their membership functions.

Let's think when $z = 8$. Addition to make $z = 8$ is possible for following cases :

$$2 + 6,\ 3 + 5,\ 3.5 + 4.5,\ \ldots$$

so

$$\begin{aligned} \mu_{A(+)B} &= \bigvee_{8=x+y} [\mu_A(2) \wedge \mu_B(6),\ \mu_A(3) \wedge \mu_B(5),\ \mu_A(3.5) \wedge \mu_B(4.5), \cdots] \\ &= \vee\ [1 \wedge 0,\ 0.5 \wedge 1/6,\ 0.25 \wedge 0.25, \cdots] \\ &= \vee\ [0, 1/6,\ 0.25, \cdots] \end{aligned}$$

If we go on these kinds of operations for all $z \in A\,(+)\,B$, we come to the following membership functions, and these are identical to the three point expression for triangular fuzzy number $A = (-4, 2, 10)$.

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < -4 \\ \dfrac{z+4}{6}, & -4 \le z \le 2 \\ \dfrac{10-z}{8}, & 2 \le z \le 10 \\ 0, & z > 10 \end{cases}$$

□

There in no simple method using there point expression for multiplication or division operation. So it is necessary to use membership functions.

Example 4.11 Multiplication $A\,(\bullet)\,B$

Let triangular fuzzy numbers A and B be

$$A = (1, 2, 4),\ B = (2, 4, 6)$$

$$\mu_{(A)}(x) = \begin{cases} 0, & x < 1 \\ x - 1, & 1 \le x < 2 \\ -\dfrac{1}{2}x + 2, & 2 \le x < 4 \\ 0, & x \ge 4 \end{cases}$$

$$\mu_{(B)}(y) = \begin{cases} 0, & y < 2 \\ \frac{1}{2}y - 1, & 2 \le y < 4 \\ -\frac{1}{2}y + 3, & 4 \le y < 6 \\ 0, & y \ge 6 \end{cases}$$

calculating multiplication $A\ (\bullet)\ B$ of A and B, $z = x \bullet y = 8$ is possible when $z = 2 \bullet 4$ or $z = 4 \bullet 2$

$$\begin{aligned} \mu_{A(\bullet)B} &= \bigvee_{x \bullet y = 8} [\mu_A(2) \wedge \mu_B(4),\ \mu_A(4) \wedge \mu_B(2), \cdots] \\ &= \vee[1 \wedge 1,\ 0 \wedge 0, \cdots] \\ &= 1 \end{aligned}$$

also when $z = x \bullet y = 12$, $3 \bullet 4$, $4 \bullet 3$, $2.5 \bullet 4.8$, … are possible.

$$\begin{aligned} \mu_{A(\bullet)B} &= \bigvee_{x \bullet y = 12} [\mu_A(3) \wedge \mu_B(4),\ \ \mu_A(4) \wedge \mu_B(3),\ \ \mu_A(2.5) \wedge \mu_B(4.8), \cdots] \\ &= \vee[0.5 \wedge 1,\ \ 0 \wedge 0.5,\ \ 0.75 \wedge 0.6,\ \ \cdots] \\ &= \vee[0.5,\ \ 0,\ \ 0.6,\ \ \cdots] \\ &= 0.6 \end{aligned}$$

From this kind of method, if we come by membership function for all $z \in A\ (\bullet)\ B$, we see fuzzy number as in Fig 5.9. However, since this shape is in curve, it is not a triangular fuzzy number. For convenience, we can express it as a triangular fuzzy number by approximating $A\ (\bullet)\ B$

$$A(\bullet)B \cong (2,\ \ 8,\ \ 24)$$

we can wee that two end points and one peak point are used in this approximation. □

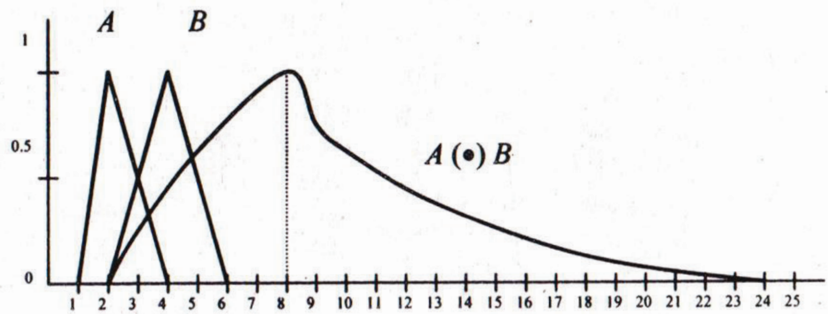


Fig. 4.9. Multiplication $A\ (\bullet)\ B$ of triangular fuzzy number

4.3.4 Approximation of Triangular Fuzzy Number

Since it is possible to express approximated values of multiplication and division as triangular fuzzy numbers, we are now up to the fact that how to get this approximated value easily.

Example 4.12 Approximation of multiplication

First, α-cuts of two fuzzy numbers are our main concern.

$$A = (1, 2, 4), B=(2, 4, 6)$$

$$\begin{aligned} A_\alpha &= [(2-1)\alpha+1, \ -(4-2)\alpha+4] \\ &= [\alpha+1, \ -2\alpha+4] \\ B_\alpha &= [(4-2)\alpha+2, \ -(6-4)\alpha+6] \\ &= [2\alpha+2, \ -2\alpha+6] \end{aligned}$$

For all $\alpha \in [0, 1]$, multiply A_α with B_α which are two crisp intervals. Now in $\alpha \in [0, 1]$, we see that elements of each interval are positive numbers. So multiplication operation of the two intervals is simple.

$$\begin{aligned} A_\alpha(\bullet)B_\alpha &= [\alpha+1, \ -2\alpha+4](\bullet)[2\alpha+2, \ -2\alpha+6] \\ &= [(\alpha+1)(2\alpha+2), \ (-2\alpha+4)(-2\alpha+6)] \\ &= [2\alpha^2+4\alpha+2, \ 4\alpha^2-20\alpha+24] \end{aligned}$$

when $\alpha = 0$,

$$A_0(\bullet)B_0 = [2, \ 24]$$

when $\alpha = 1$,

$$A_0(\bullet)B_1 = [2+4+2, 4-20+24] = [8, 8] = 8$$

we obtain a triangular fuzzy number which is an approximation of A (•) B (Fig **4.9**).

$$A(\bullet)B \cong (2, \ 8, \ 24) \quad \square$$

Example 4.13 Approximation of division

In the similar way, let's express approximated value of A (/) B in a triangular fuzzy number. First, divide interval A_α by B_α. We reconsider the sets A and B in the previous example. For $\alpha \in [0, 1]$, since element in each interval has positive number, we get A_α (/) B_α as follows.

$$A_\alpha(/)B_\alpha = [(\alpha+1)/(-2\alpha+6), \ (-2\alpha+6)/(2\alpha+2)]$$

when $\alpha = 0$,

$$\begin{aligned} A_0(/)B_0 &= [1/6, \ 4/2] \\ &= [0.17, \ 2] \end{aligned}$$

when $\alpha = 1$,

$$\begin{aligned} A_1(/)B_1 &= [(1+1)/(-2+6), \ (-2+4)/(2+2)] \\ &= [2/4, \ 2/4] \\ &= 0.5 \end{aligned}$$

so the approximated value of A (/) B will be

$$A(/)B = (0.17, \ 0.5, \ 2) \quad \square$$

4.4 Other Types of Fuzzy Number

4.4.1 Trapezoidal Fuzzy Number

Another shape of fuzzy number is trapezoidal fuzzy number. This shape is originated from the fact that there are several points whose membership degree is maximum ($\alpha = 1$).

Definition (Trapezoidal fuzzy number) We can define trapezoidal fuzzy number A as

$$A = (a_1, a_2, a_3, a_4)$$

the membership function of this fuzzy number will be interpreted as follows(Fig 4.10).

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4}, & a_3 \le x \le a_4 \\ 0, & x > a_4 \end{cases}$$

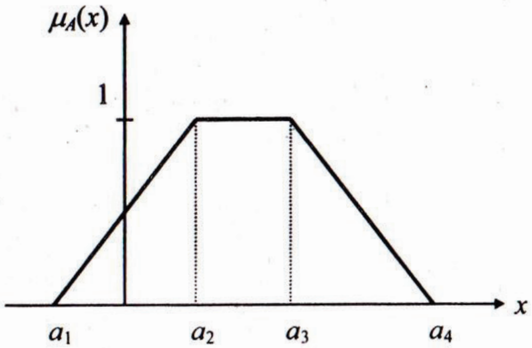


Fig. 4.10. Trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$

α-cut interval for this shape is written below.

$\forall \alpha \in [0, 1]$

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, \quad -(a_4 - a_3)\alpha + a_4]$$

when $a_2 = a_3$, the trapezoidal fuzzy number coincides with triangular one.

4.4.2 Operations of Trapezoidal Fuzzy Number

Let's talk about the operations of trapezoidal fuzzy number as in the triangular fuzzy number,

(1) Addition and subtraction between fuzzy numbers become trapezoidal fuzzy number.

(2) Multiplication, division, and inverse need not be trapezoidal fuzzy number.

(3) Max and Min of fuzzy number is not always in the form of trapezoidal fuzzy number.

But in many cases, the operation results from multiplication or division are approximated trapezoidal shape. As in triangular fuzzy number, addition and subtraction are simply defined, and multiplication and division operations should be done by using membership functions.

(1) Addition

$$A(+)B = (a_1, a_2, a_3, a_4)(+)(b_1, b_2, b_3, b_4)$$
$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

(2) Subtraction

$$A(-)B = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$

Example 4.14 Multiplication

Multiply two trapezoidal fuzzy numbers as following:

$$A = (1, 5, 6, 9)$$
$$B = (2, 3, 5, 8)$$

For exact value of the calculation, the membership functions shall be used and the result is described in (Fig. 5.11) For the approximation of operation results, we use α-cut interval

$$A_\alpha = [4\alpha + 1, -3\alpha + 9]$$
$$B_\alpha = [\alpha + 2, -3\alpha + 8]$$

since, for all $\alpha \in [0, 1]$, each element for each interval is positive, multiplication between α-cut intervals will be

$$A_\alpha(\bullet)B_\alpha = [(4\alpha + 1)(\alpha + 2), \quad (-3\alpha + 9)(-3\alpha + 8)]$$
$$= [4\alpha^2 + 9\alpha + 2, \quad 9\alpha^2 - 51\alpha + 72]$$

if $\alpha = 0$,

$$A_0(\bullet)B_0 = [2, \quad 72]$$

if $\alpha = 1$,

$$A_1(\bullet)B_1 = [4 + 9 + 2, \quad 9 - 51 + 72]$$
$$= [15, \quad 30]$$

so using four points in $\alpha = 0$ and $\alpha = 1$, we can visualize the approximated value as trapezoidal fuzzy number as (Fig. 5.11)

$$A(\bullet)B \cong [2, \ 15, \ 30, \ 72] \quad \square$$

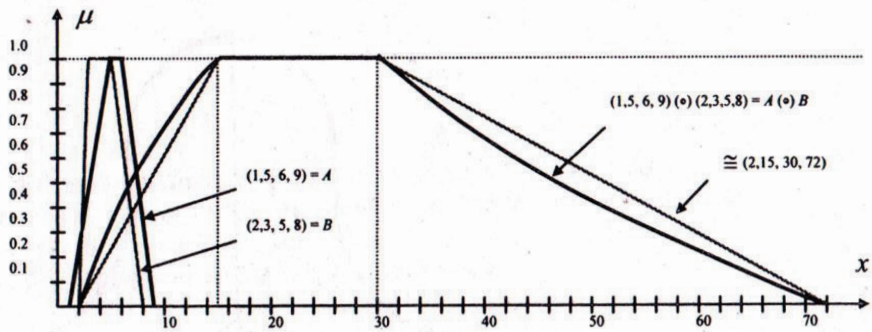


Fig. 4.11. Multiplication of trapezoidal fuzzy number A (•) B

Generalizing trapezoidal fuzzy number, we can get flat fuzzy number. In other words, flat fuzzy number is for fuzzy number A satisfying following

$$\exists m_1, m_2 \in \Re, \qquad m_1 < m_2$$
$$\mu_A(x) = 1, \qquad m_1 \le x \le m_2$$

In this case, not like trapezoidal form, membership function in $x < m_1$ and $x < m_2$ need not be a line as shown in (Fig **4.12.**)

4.4.3 Bell Shape Fuzzy Number

Bell shape fuzzy number is often used in practical applications and its function is defined as follows(Fig **4.13**)

$$\mu_f(x) = \exp\left\{\frac{-(x-m_f)^2}{2\delta_f^2}\right\}$$

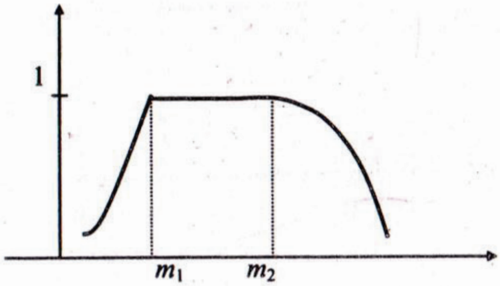


Fig. 4.12 Flat fuzzy number

where μ_f is the mean of the function, δ_f is the standard deviation.

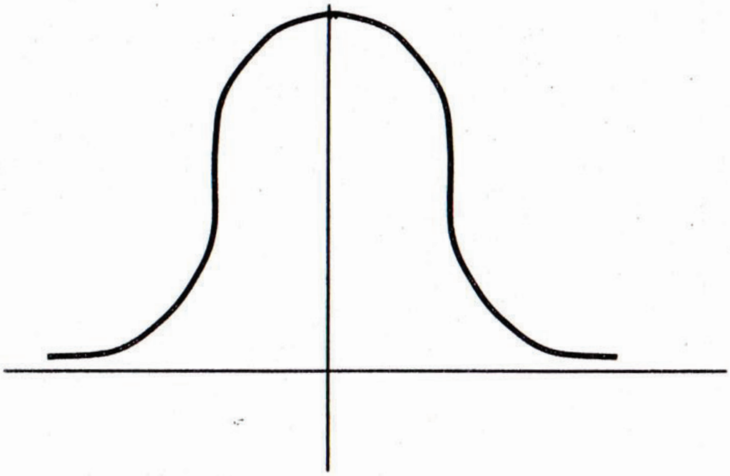


Fig. 4.13. Bell shape fuzzy number

CONCLUSION

Acknowledging the power of abstract approach the twentieth century has been a witness to initiative ideas of unified mathematic for this unification fuzzy mathematies plays an important role.

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