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SOME APPLICATIONS OF FOURIER TRANSFORM TO NEW FRACTIONAL DERIVATIVE WITH NON – SINGULAR KERNEL

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Abstract: *In this paper, we obtained the relation between Fourier Transform of new fractional derivatives with nonsingular kernels of a function in terms of Kummers Hyper geometric functions as an application of it, we have solve the example of fractional order partial differential equation.*

Keywords: *Fourier Transformation, Kummers Hyper geometric function, Mittag - Leffler Function,fractional derivatives.*

Mathematical Subject Classification (2010): *26A33, 42A38, 33C45, 33E12*

1. INTRODUCTION

Though the idea of fractional transformations, fractional derivatives have long back history the idea of fractional partial differential equation in different spaces has been rediscovered mainly in quantum mechanics, fluid dynamics, and stochastic processes. In recentyears many linear boundary value and initial value problems in applied mathematics, mathematical physics and engineering science are effectively solved by Laplace, Fourier and other Transforms. It has been studied by many researchers and contributed.The partial differential equations has applications in the field of mathematics as well in real life situations, such as Abel's integral equation, visco - elasticity, capacitor theory, conductance of biological systems [1, 11].

The term fractional derivative has long back history from 16th century [11] as development in the field goes on increasing much faster in the last century, there are several different approaches for fractional derivative definitions [11] like Caputo, Riemann - Liouville definition being used, then to find numerical solutions to these fractional order equations many different numerical methods are also used [1]. These fractional differential equations have manyof its applications in Fractals, Bio - Mathematics [1, 11], Recently Abdon Atangana and Dumitru Baleanu [2] gave new definition of fractional derivativewith Non local and Non Singular kernel with exponential function.

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The main aim of our paper is to derive the properties of new fractional derivative with its applications, by using Fourier transform method which has not been done yet before. The paper mainly divided into two parts, in the first part we derive the properties of new fractional derivatives. In the second part, we provide an application of it along with the discussion of the obtained result. The conclusion part ends our manuscript.

In the following we present some basic definitions needed in proving the main results.

Definition 1: Atangana - Baleanu Riemann fractional derivative

Consider a function $f \in H^1\,(a, b)$, $b > a$, $\alpha \in [0, 1]$ which is of exponential order then the new ABR fractional derivative [6] of f(t) is defined as,

$${}^{ABR}_{\alpha}D^{\alpha}_{t}\left(f(t)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}[\int_{\alpha}^{t} f(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)]b > a, \alpha \in [0,1]$$ and $B(\alpha)$ is normalization function obeying $B(0) = B(1) = 1$. (1)

Definition 2: Atangana - Baleanu Caputo fractional derivative

Consider a function $f \in H^1\,(a, b)$, $b>a$, $\alpha[0, 1]$ which is of exponential order then the new ABC fractional derivative [6] of $f(t)$ is defined as,

$${}^{ABR}_{\alpha}D^{\alpha}_{t}\left(f(t)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}[\int_{\alpha}^{t} f'(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)dx]b > a, \alpha \in [0,1]$$ and $B(\alpha)$ is normalization function obeying $B(0) = B(1) = 1$. (2)

Definition 3: Mittag - Leffler function

The Mittag - Leffler function [4] is an entire function defined by theSeries

$$E_{\alpha}(z) \equiv \Sigma_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{3}$$

Definition 4: Hyper geometric function

The Kummers hyper geometric function of the first kind ${}_1M_1\,(a; b; z)$ is a degenerate form of the hyper geometric function ${}_2M_1\,(a; b; c; z)$ which arises as a solutions of confluent hyper geometric differential Equation it has hyper geometric series representation given by

$${}_1M_1(a;b;z) = M\,(a;b;z) = \Sigma_{k=0}^{\infty}\frac{a^{(k)}}{b^{(k)}\Gamma(k+1)}z^k \tag{4}$$

it also have the integral representation as [4, 14]

$$\int_0^x x^{s-1} e^{-x}\, dx = \frac{x^s}{s} M(s, s+1, -x) \tag{5}$$

Definition 5: Fourier Transform

Let $f(t)$ be real valued function which is in $L^1(R)$ then the Fourier Transform of it is denoted by $\widehat{f(\xi)}$ and is given by $\widehat{f(\xi)} = \int_{-\infty}^{\infty} f(t) e^{-it\xi} dt$ (6)

2. MAIN RESULT

Theorem

Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$ which is of exponential order then the relation between Fourier Transform of ABR and ABC fractional derivative respectively is given

$$\text{by,}\left(\widehat{{}^{ABR}_{0}D^{\alpha}_{t} f(t)}\right)(\xi) = \left(\widehat{{}^{ABC}_{0}D^{\alpha}_{t} f(t)}\right)(\xi),$$

where the Fourier Transform has been calculated in terms of Kummers Hyper geometric functions by means of approximations

Proof

(PART A)

Consider the function $f(t) \in H^1(a, b)$, $b>a$, α, $\alpha \in [0,1]$, $b > a$, $\alpha \in [0,1]$ and $B(\alpha)$ is normalization function obeying $B(0) = B(1) = 1$ which is of exponential order, then from (1) the ABR fractional derivative is given by,

$${}^{ABR}_{a}D^{\alpha}_{t}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\Big[\int_a^t f(x) E_\alpha\Big(-\alpha\frac{(t-x)^\alpha}{1-\alpha}\Big)\, dx\Big]$$

we approximate the right hand side in terms of Kummers Hyper geometric function as follows:

$${}^{ABR}_{0}D^{\alpha}_{t}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\Big[\int_0^t f(x)\Big[\sum_{k=0}^{\infty}\Big(\frac{\Big(-\left(\frac{\alpha}{1-\alpha}\right)^{1/\alpha}(t-x)\Big)^{\alpha k}}{\Gamma(\alpha k+1)}\Big)\Big]dx\Big]$$

By neglecting the higher order terms as the series is convergent, we get

$${}^{ABR}_{0}D^{\alpha}_{t}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\sum_{k=0}^{N}\left[\int_0^t f(x)\left(\frac{\alpha}{1-\alpha}\right)^k\left(\frac{(-t+x)^{\alpha k}}{\Gamma(\alpha k+1)}\right)dx\right]$$

$$ {}^{ABR}_{0}D_t^{\alpha}(f(t)) = B(\alpha)\frac{d}{dt}\sum_{k=0}^{N}\frac{\alpha^k}{(1-\alpha)^{k+1}}\left[\int_0^t f(x)\left(\frac{(-t+x)^{\alpha k}}{\Gamma(\alpha k+1)}\right)dx\right] \quad (7) $$

we define a new function as;

$$ \text{Define } h(x-t) = \begin{cases} 0 & otherwise \\ (x-t)^{\alpha k}, & 0 < x < t \end{cases} \quad (8) $$

By using the convolution property the equation becomes (7) and (8) together gives us

$$ {}^{ABR}_{0}D_t^{\alpha}(f(t)) = B(\alpha)\sum_{k=0}^{N}\frac{\alpha^k}{(1-\alpha)^{k+1}}\frac{d}{dt}(f*h)(x) $$

Applying Fourier Transform (6) on both sides of the above equation,

$$ \Longrightarrow $$

$$ \left(\widehat{{}^{ABR}_{0}D_t^{\alpha}f}(t)\right)(\xi) = B(\alpha)\sum_{k=0}^{N}\frac{(i\xi)}{\Gamma(\alpha k+1)}\frac{\alpha^k}{(1-\alpha)^{k+1}}\hat{\hat{f}}(\xi)\int_0^t t^{\alpha k}e^{-it\xi}dt $$

$$ \left(\widehat{{}^{ABR}_{0}D_t^{\alpha}f}(t)\right)(\xi) = B(\alpha)\sum_{k=0}^{N}\frac{(i\xi)^{1-\alpha k}}{\Gamma(\alpha k+1)}\frac{\alpha^k}{(1-\alpha)^{k+1}}\hat{\hat{f}}(\xi)\int_0^t (i\xi t)^{\alpha k}e^{-it\xi}dt $$

By applying the definition (5) to the above equation,

$$ \left(\widehat{{}^{ABR}_{0}D_t^{\alpha}f}(t)\right)(\xi) = B(\alpha)\hat{f}(\xi)\sum_{k=0}^{N}\frac{\alpha^k(i\xi)^{1-\alpha k}}{(1-\alpha)^{k+1}}\frac{(i\xi t)^{\alpha k+1}}{\Gamma(\alpha k+2)}M(\alpha k+1,\alpha k+2,-t),\ \xi \neq 0 \quad (A) $$

(PART B)

Consider the function $f(t) \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$, $b > a$ $\alpha \in [0,1]$ and $B(\alpha)$ is normalization function obeying $B(0) = B(1) = 1$ which is of exponential order, then from (2) the ABC fractional derivative is given by,

$$ {}^{ABR}_{a}D_t^{\alpha}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\left[\int_a^t f'(x)E_\alpha\left(-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right)dx\right] $$

we approximate the right hand side in terms of Kummers Hyper geometric function as follows:

$$ {}^{ABC}_{0}D_t^{\alpha}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\left[\int_0^t f'(x)\left[\sum_{k=0}^{\infty}\left(\frac{\left(-\left(\frac{\alpha}{1-\alpha}\right)^{1/\alpha}(t-x)\right)^{\alpha k}}{\Gamma(\alpha k+1)}\right)\right]dx\right] $$

By neglecting the higher order terms as the series is convergent,

$$ {}^{ABC}_{\ 0}D_t^{\alpha}\big(f(t)\big) = \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{N}\left[\int_0^t f'(x)\left(\frac{\alpha}{1-\alpha}\right)^k\left(\frac{(-t+x)^{\alpha k}}{\Gamma(\alpha k+1)}\right)dx\right] $$

$$ {}^{ABC}_{\ 0}D_t^{\alpha}(f(t)) = B(\alpha)\textstyle\sum_{k=0}^{N}\frac{\alpha^k}{(1-\alpha)^{k+1}}\left[\int_0^t f'(x)\left(\frac{(-t+x)^{\alpha k}}{\Gamma(\alpha k+1)}\right)dx\right] \quad (9) $$

we define a new function as;

$$ \text{Define } h(x-t) = \begin{cases} 0 & otherwise \\ (x-t)^{\alpha k}, & 0 < x < t \end{cases} \qquad (9) $$

By using the convolution property the equation becomes (9) and (10) together gives us

$$ {}^{ABC}_{\ 0}D_t^{\alpha}(f(t)) = B(\alpha)\sum_{k=0}^{N}\frac{\alpha^k}{(1-\alpha)^{k+1}}(f' * h)(x) $$

applying Fourier Transform (6) on both sides of the above equation,

$$ \Rightarrow $$

$$ \left(\widehat{{}^{ABC}_{\ 0}D_t^{\alpha}f(t)}\right)(\xi) = B(\alpha)\sum_{k=0}^{N}\frac{(i\xi)}{\Gamma(\alpha k+1)}\frac{\alpha^k}{(1-\alpha)^{k+1}}\hat{\hat{f}}(\xi)\int_0^t t^{\alpha k}e^{-it\xi}dt $$

$$ \left(\widehat{{}^{ABC}_{\ 0}D_t^{\alpha}f(t)}\right)(\xi) = B(\alpha)\sum_{k=0}^{N}\frac{(i\xi)^{1-\alpha k}}{\Gamma(\alpha k+1)}\frac{\alpha^k}{(1-\alpha)^{k+1}}\hat{\hat{f}}(\xi)\int_0^t (i\xi t)^{\alpha k}e^{-it\xi}dt $$

By applying the definition (5) to the above equation it gives us

$$ \left(\widehat{{}^{ABC}_{\ 0}D_t^{\alpha}f(t)}\right)(\xi) = B(\alpha)\hat{f}(\xi)\textstyle\sum_{k=0}^{N}\frac{\alpha^k(i\xi)^{1-\alpha k}}{(1-\alpha)^{k+1}}\frac{(i\xi t)^{\alpha k+1}}{\Gamma(\alpha k+2)}M(\alpha k+1,\alpha k+2,-t), \xi \neq 0 \quad (\mathbf{B}) $$

Hence from (A) and (B)

$$ \left(\widehat{{}^{ABR}_{\ 0}D_t^{\alpha}f(t)}\right)(\xi) = \left(\widehat{{}^{ABC}_{\ 0}D_t^{\alpha}f(t)}\right)(\xi) $$

where the Fourier Transform has been calculated in terms of Kummers Hyper geometric functions by means of approximations.

The ABR and ABC fractional derivative definition has been recently developed and its relation by using Laplace Transform has been obtained, but the relation between them by using Fourier Transform has not been obtained yet. Here we found this relation for general case also as an application the fractional partial differential equation by using this new definition has been solved for particular case, which is very much effective and simple.

3. APPLICATION

As an application to the above method of finding the value of ABR and ABC fractional derivative by using Fourier Transform method, we have solve the problem of ABR time fractional partialdifferential equation for particular case.

Consider the time fractional partial differential equation satisfying the initial and boundary conditions as follows:

$$u_x(x, t) = u_t^{\alpha}(x, t), \ -\infty < x < \infty, t$$

subject to B.C.: u_x, $u \to 0$ as $|x| \to \infty$ and I.C.: $u(0, t) = 1$

To solve this fractional partial differential equation we will use the definition [1]

$$u_x(x,t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}[\int_0^t u(x,\tau)E_\alpha(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha)d\tau]$$

$$u_x(x,t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}[\int_0^t f(x)[\sum_{k=0}^{\infty}(\frac{(-\left(\frac{\alpha}{1-\alpha}\right)^{1/\alpha}(t-x))^{\alpha k}}{\Gamma(\alpha k+1)})]dx]$$

By neglecting the higher order terms as the series is convergent,

$$u_x(x,t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\sum_{k=0}^{N}\left[\int_0^t f(x)\left(\frac{\alpha}{1-\alpha}\right)^k\left(\frac{(-t+x)^{\alpha k}}{\Gamma(\alpha k+1)}\right)dx\right]$$

$$u_x(x,t) = B(\alpha)\frac{d}{dt}\sum_{k=0}^{N}\frac{\alpha^k}{(1-\alpha)^{k+1}}\left[\int_0^t f(x)\left(\frac{(-t+x)^{\alpha k}}{\Gamma(\alpha k+1)}\right)dx\right] \tag{10}$$

Now we define a new function as;

$$\text{Define } h(x-t) = \begin{cases} 0 & otherwise \\ (x-t)^{\alpha k}, & 0 < x < t \end{cases} \tag{11}$$

By using the convolution property the equation becomes (11) and (12) together gives us

$$u_x(x,t) = B(\alpha)\sum_{k=0}^{N}\frac{\alpha^k}{(1-\alpha)^{k+1}}\frac{d}{dt}(f*h)(x)$$

By applying Fourier Transform on both side of the above equation and by using the

Property of Convolution and FT of function [6],

$$\Rightarrow$$

$$U_x(x,\xi) = B(\alpha)\sum_{k=0}^{N}\frac{(i\xi)}{\Gamma(\alpha k+1)}\frac{\alpha^k}{(1-\alpha)^{k+1}}U(x,\xi)\int_0^t t^{\alpha k}\,e^{-it\xi}dt$$

$$\left(\widehat{{}^{ABR}_{0}D^{\alpha}_{t}f(t)}\right)(\xi) = B(\alpha)U(x,\xi)\sum_{k=0}^{N}\frac{\alpha^k(i\xi)^{1-\alpha k}}{(1-\alpha)^{k+1}}\frac{(i\xi t)^{\alpha k+1}}{\Gamma(\alpha k+2)}M(\alpha k+1,\alpha k+2,-t)$$

Now put $N = 2$ and $\alpha = \frac{1}{2}$ in above equation,

$$U_x(x,\xi) = U(x,\xi)B(\frac{1}{2})(\frac{(i\xi)^2}{\Gamma(2)}tM(1,2,-t) + \frac{2(i\xi)^2t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{4}\right)}M\left(\frac{3}{2},\frac{5}{2},-t\right) + \frac{2(i\xi)^2}{\Gamma(3)}t^2M(2,3,-t)$$

With $u(0, t) = 1 \Rightarrow U(0, \xi) = \delta(\xi)$

Let, $A_1 = -tB\left(\frac{1}{2}\right)\frac{M(1,2,-t)}{\Gamma(2)}, A_2 = -2t^{\frac{3}{2}}B\left(\frac{1}{2}\right)\frac{M\left(\frac{3}{2},\frac{5}{2},-t\right)}{\Gamma\left(\frac{5}{4}\right)}\ and A_3 = -2t^2B\left(\frac{1}{2}\right)\frac{M(2,3,-t)}{\Gamma(3)}$

So that the above equation becomes

$U_x(x, \xi) = U(x, \xi)\,[A_1 + A_2 + A_3]\,\xi^2$ whose solution is given by

$$U(x,\xi) = ce^{(A_1+A_2+A_3)\xi^2 x}$$

By using the given initial condition $U(0, \xi) = \delta(\xi) \Rightarrow c = \delta(\xi)$

$$\Rightarrow U(x, \xi) = \delta(\xi)\,e^{Kx}, \text{ where } K = (A_1 + A_2 + A_3)\,\xi^2$$

We know that $e^X = 1 + X + X^2 + X^3 + \ldots$

$\Rightarrow U(x, \xi) = \delta(\xi)[1 + Kx + (Kx)^2 + \ldots]$, Neglecting the higher order terms it gives $U(x, \xi) = \delta(\xi)[1 + Kx]$

$$\Rightarrow U(x, \xi) = \delta(\xi) + (A_1 + A_2 + A_3)\,x\delta(\xi)\,\xi^2$$

$$\Rightarrow U(x, \xi) = H(x, t)\,\delta(\xi)\,\xi^2 \tag{12}$$

Where, $H(x, t) = 1 + (A_1 + A_2 + A_3)x$

Applying inverse Fourier Transform and applying again convolution property, the required approximate analytic solution is given by

$$u(x,t) = \frac{1}{2\pi}H(x,t)\breve{\xi}^2$$

Which is the required solution of the given time fractional ABR partial fractional Differential Equation.

4. CONCLUSION

The aim of this manuscript was to suggest the new approach for solving the fractional partial differential equation with the new fractional derivative definition to find the solution.we use the relation of Mittag - Leffler functions and Kummers Hyper geometric functions byusing integral Transform in this method as value of N increases we get better approximations.

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