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Some New Results in Physics

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Abstract: *In this report, we provide some of the latest results that we have derived.*

1. PHOTOELECTRIC EFFECT IN 2D & 3D CRYSTALS

1.1. 2D crystals

Let us consider a photoelectric arrangement for some 2D crystal like graphene, where a source emits photons and falls on the crystal sheet. We consider two plates separated by a distance d, where the first plate is a thin sheet of graphene and the second plate collects the photoelectrons emitted from the first plate. If the applied potential is given by ϕ then the generated electric field is

$$|E| = \frac{\phi}{d}$$

Again, if j is the current density, i the current and σ_p is the conductivity due to the photoelectric phenomenon then the conductivity through a circular area of radius r is given by

$$\sigma_p = \frac{J}{|E|} = \frac{i}{|E|\pi r^2}$$

So using all the known relations, we have

$$\sigma_p = \frac{id}{\pi r^2 \phi} \tag{1}$$

Now, we know that in case of graphene we have a minimal conductivity which amounts to

$$\sigma_{\min} = \sigma \frac{e^2}{h}$$

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where, $\alpha = \frac{4}{\pi}$ (theoretical value) or, $\alpha = 4$ (experimentally observed) [1]. Thus, when light illuminates the first plate of graphene sheet we will have photoelectric conductivity combined with the minimum conductivity of graphene given as

$$\sigma = \sigma_p + \sigma_{\min}$$

$$\sigma = \frac{\alpha e^2}{h} + \frac{id}{\pi r^2 \phi} \tag{2}$$

Again, we also know that the current will be related to the electric charge as

$$i = \frac{e}{\tau}$$

where, $\tau = t_0 - t$. Here, t is the time taken by a photon to come from the source and fall on a particular electron thereby exciting it. Also, consistent with the wavelength, the incident photon frequency (f) will be given by

$$f = \frac{1}{t}$$

On the other hand, t_0 ($= t + \delta t$) is the time elapsed from the emission of the photon to the emission of the excited electron from the surface of the graphene sheet, such that the threshold frequency (f_0) is given by

$$f_0 = \frac{1}{t_0}$$

Essentially, a small amount of time δt is elapsed for the electron to get excited and finally be emitted. Thus, we have the current as

$$i = \frac{e}{t_0 - t}$$

Now, while considering n number of charge carriers or electrons at the surface of the graphene sheet we can argue that the time interval δt will be the same for all electrons. Consequently, we can write the net current simply as

$$I = \frac{ne}{t_0 - t}$$

Again, since

$$t_0 = \frac{1}{f_0}, t = \frac{1}{f}$$

we have

$$I = \frac{ef\,f_0}{f - f_0}$$

Again, we know by law of the photoelectric effect we have

$$e\phi = h\ (f - f_0) \tag{4}$$

where, ϕ is the stopping potential, h is the Planck's constant. Thus the total conductivity for the graphene sheet is given by

$$\sigma = \frac{\alpha e^2}{h} + \frac{I\,de}{\pi r^2 h(f - f_0)}$$

Using the expression of the current from relation (3), we obtain

$$\sigma = \frac{\alpha e^2}{h} + \frac{ned}{\pi r^2 h(f - f_0)} \times \frac{e\,f\,f_0}{f - f_0}$$

Thus, we have finally

$$\sigma = [\alpha + \frac{ndf\,f_0}{\pi r^2 (f - f_0)^2}]\frac{e^2}{h} \tag{5}$$

As we can see, the conductivity of the graphene sheet subjected to the photoelectric phenomenon has increased. In other words, we can also argue that if we use a graphene sheet instead of any other conventional metal, then the conductivity is increased on account of the intrinsic minimal conductivity of graphene [2, 3, 4].

It is noteworthy that the second contributing term in the conductivity in (5) depends on quite a few factors other than the frequency of the photon. Particularly, the second term increases as the carrier concentration and/or distance between the two plates is increased; and also, it increases as the radius of the circular area is decreased. On the other hand the first term in (5) gives a constant contribution to the total conductivity. Taking these into considerations we might be able to amplify the consequences of photoelectricity and it's uses.

Now, let us delve into finding an expression for the total current. Again, considering the fact that there are no external contributions to the electric field, we have

$$J_{total} = \sigma \times |E| = \sigma \times \frac{\phi}{d}$$

Using the value of the total conductivity from equation (2) we have

$$J = \frac{\alpha e^2 \phi}{hd} + \frac{i}{\pi r^2}$$

So, using the relation for photoelectric effect (4), the current density is finally given by

$$J = \frac{\alpha e}{d}(f - f_0) + \frac{i}{\pi r^2} \tag{6}$$

Therefore, we have the total current ($J \times \pi r^2$) as

$$I = \frac{\alpha e \pi r^2}{d}(f - f_0) + i \tag{7}$$

where, the first term is the contribution due to the intrinsic minimal conductivity present in graphene. Let us denote this current as

$$i_{\text{int}\,rinsic} = \frac{\alpha e \pi r^2}{d}(f - f_0) \tag{8}$$

Incidentally, we can see that

$$\frac{I}{i} = 1 + \frac{i_{\text{int}\,rinsic}}{i}$$

which means that the ratio of the total current to the current excluding graphene's intrinsic current is always greater than 1. Essentially, we have found that the intrinsic conductivity of graphene adds to the phenomena of photoelectricity when it is employed.

Now, from relation (8) we can see that the current owing to the 2D crystal's intrinsic properties, increases as the difference between the incident photon frequency and the threshold frequency increases; the current also increases as the radius of the circular area is increased and/or the distance between the two plates is decreased. One can adjust the values of the parameters to get a significant amount of total current (I) Interestingly, this result might be extended for various other 2D crystal structures too. We shall also try to remark upon similar results in case of 3D crystals, in the subsequent section.

1.2. 3D crystals

In a recent paper [5], it has been have shown that in case of 3D crystals one can obtain a minimal conductivity, using Kubo's formula [4, 6]. The derived relation is given by

$$\sigma_{\min} = \frac{\pi e^2}{8h}\frac{\Omega^2}{\omega} \tag{9}$$

where, ω is the frequency of an electron and Ω is the imaginary part of the self energy. Essentially, this intrinsic minimal conductivity can be attributed to the *zitterbewegung* effects that play a central role below the Compton scale. This is mainly due to the fluctuations of the electron. Replacing the first term in relation (2) with min as given above, we obtain

$$\sigma = \frac{\beta e^2}{h}\frac{\Omega^2}{\omega} + \frac{id}{\pi r^2 \phi}$$

where, $\beta = \frac{\pi}{8}$. Thus, relation (5) is modified as

$$\sigma = \left[\beta\frac{\Omega^2}{\omega} + \frac{ndf\,f_0}{\pi r^2 (f - f_0)^2}\right]\frac{e^2}{h} \tag{10}$$

and therefore resorting to the same methodology as in the previous section, the current is finally given by

$$I = \frac{\beta e \pi r^2}{d}\frac{\Omega^2}{\omega}(f - f_0) + i \tag{11}$$

Thus, we have an extra contribution in case of 3D crystals too.

1.3. Discussions

In the current paper, we have investigated the photoelectric effect on the pedestal of 2D and 3D crystal structures. Knowing the threshold frequencies of 2D and 3D structures one can investigate the notion innovated in this paper. Essentially, this might augment the applications of photoelectric effect and also this will be yet another fruitful application of 2D and 3D crystal structures, like graphene.

We would like to point out another interesting aspect. The stark distinction of quantum mechanics and classical mechanics is due to the role of the observer or observing apparatus. What appears smooth at one level of perception, may turn out to be very irregular on a closer examination [7, 8, 9]. At scales larger than the Compton wavelength but smaller than the De Broglie wavelength, the quantum paths have a fractal dimension 2 of Brownian paths. So, if we consider 1D or 2D graphene samples, then their fractal dimension becomes higher. Essentially, the area of the sample increases. Therefore, the intrinsic current (iintrinsic) which emerges from our approach, increases. Thus, we will have a greater amount of current from the sample under consideration. Let us elaborate this quantitatively. We star by considering a solar panel of fractal surface, having n corrugations of length l. Then the total area of the surface will be given by

$$A = \pi\left(\sqrt{n}l\right)^2$$

Thus, from (12) we obtain for the fractal case

$$I = \frac{\alpha n e \pi l^2}{d}(f - f_0) + i \tag{12}$$

where, n is the new factor that arises due to the fractal nature of the surface of the solar panels. It is obvious that the more corrugations we have, the more current we shall obtain.

2. QUANTUM STATISTICS AND COMPLEX ENERGY

2.1. Energy in the complex domain

We commence with the inception of an early work by Valluri *et al.* [10] where it was shown that the quantity M is related to the Fermi Dirac distribution as

$$M = a\int_0^\infty F(E)dE \tag{13}$$

where,

$$F(E) = \frac{E^{f(r)}}{e^{\beta E+\alpha}+1} \tag{14}$$

where, $\alpha = -\beta\mu$ (μ, being the total chemical potential) and the function $f(r)$ is real valued and continuous such that, $f: \mathbf{R} \mapsto x \in \mathbf{R} \mid x \geq 0$, and $a \in \mathbf{R}$. Considering E to be a continuous variable and keeping α, r and T fixed it has been shown that the extremum in E on the interval $(0, \infty)$ is given as

$$E = \beta[f(r) + W_j\{f(r)e^{-\alpha-f(r)}\}] \tag{15}$$

where, $\beta = \dfrac{1}{kT}$ (k being the Boltzman constant) and the subscript j denotes the branches of the Lambert W function and

$$W_j\{f(r)e^{-\alpha-f(r)}\} = \beta E - f(r) \tag{16}$$

as shown in the paper by Valluri [10]. We would like to further study the equations (14) and (15). Firstly, we examine relation (15). We consider r to be a continuous variable instead of as a fixed parameter and find the extremum in r. Differentiating (15) in terms of r we have

$$\frac{dE}{dr} = \beta\left[\frac{df}{dr} + \frac{dz}{dr} \times \frac{d}{dz}W_j(z)\right] \tag{17}$$

where, $z = f(r)e^{-\alpha-f(r)}$. Now, let $f(r)$ be a standard *logistic function* such that

$$f(r) = \frac{1}{1+e^{-r}} = \frac{e^r}{1+e^r} \tag{18a}$$

$$\frac{df}{dr} = f'(r) = f(r)[1-f(r)] \tag{18b}$$

$$\frac{dz}{dr} = f(r)[1-f(r)]^2 e^{-\alpha-f(r)} \tag{18c}$$

Again, we know that

$$\frac{dW(z)}{dz} = \frac{1}{z + e^{W(z)}} \tag{19}$$

Using all of the equations (6) and (19) in equation (17) we have

$$\frac{dE}{dr} = \beta[f(r)\{1 - f(r)\} + f(r)\{1 - f(r)\}^2 e^{-\alpha - f(r)} \times \frac{1}{f(r)e^{-\alpha - f(r)} + e^{W\{f(r)e^{-\alpha - f(r)}\}}}]$$

Setting, $\frac{dE}{dr} = 0,$ we have

$$1 + \{1 - f(r)\}e^{-\alpha - f(r)} \times \frac{1}{f(r)e^{-\alpha - f(r)} + e^{W\{f(r)e^{-\alpha - f(r)}\}}} = 0$$

Rearranging the terms we have

$$e^{W\{f(r)e^{-\alpha - f(r)}\} + \alpha + f(r)} + 1 = 0 \tag{20}$$

Now, on the first look this doesn't seem to be a feasible relation. But, we know the *Euler's identity* as

$$e^{\pm i(2n+1)\pi} + 1 = 0$$

where n is odd integer ($n \in \mathbf{z}$). Thus using this relation we write

$$W\{f(r)e^{-\alpha - f(r)}\} + \alpha + f(r) = \pm i(2n+1)\pi$$

Using this relation in equation 5 we obtain

$$E = \beta[\pm i(2n+1)\pi - \alpha] \tag{21}$$

As one can see, E has been extended to the complex domain. Maximizing the energy in terms of r has given us an analytical feature. We can still have interesting applications. For example, let us consider the complex conjugate of E such that

$$E^* = -\beta[\mp i(2n+1)\pi + \alpha]$$

Then, we can write

$$|E|^2 = EE^* = (\beta)^2[\alpha^2 + (2n + 1)^2 \pi^2]$$

where, $|E|$ denotes the modulus of E. This finally yields

$$|E| = \pm\beta\sqrt{\alpha^2 + (2n+1)^2\pi^2} \tag{22}$$

where, $\alpha = -\mu\beta$. The signs on the right hand side obviously denote the positive and negative energies respectively. This result might have newimplications, since the energy has been extended to the complex domain. It would be interesting to see that if E can be treated as a holomorphic function in the complex domain. Resorting to a similar methodology as above, one can obtain similar results for Bose-Enstein statistics. This could provide a benchmark for novel explorations in the fields of quantum mechanics and quantum statistics.

2.2. Bosonization

In this section, we shall endeavour to find a condition for which *bosonization* effect can take place, in terms of the Lambert W function. Bosonization has been dicussed over the previous decades by several authors [11, 12, 13, 14, 56, 57]. Essentially, the interactions between electrons are modelled as bosonic interactions [11, 12]. Sidharth [17, 18], has considered this topic, though in a separate manner than the aforementioned references.

As an inception, let us first consider a *logistic function* $g(\delta)$ such that

$$g(\delta) = \frac{e^{\delta}}{e^{\delta} + 1} \tag{23}$$

where, $\delta = \beta E + \alpha$ and $\frac{d\delta}{dE} = \beta$. Using this relation one can express (14) as

$$F(E) = \frac{g(\delta)}{e^{\delta}} E^{f(r)} \tag{24}$$

Now, keeping the parameters α and T fixed we differentiate the above relation with respect to E such that

$$\frac{dF}{dE} = f(r)E^{f(r)-1}\frac{g(\delta)}{e^{\delta}} + E^{f(r)}\frac{1}{e^{\delta}}g'(\delta) - \frac{E^{f(r)}g(\delta)}{e^{\delta}}\beta$$

Maximizing $F(E)$ with respect to the energy E (setting $\frac{dF}{dE} = 0$) and then simplifying the terms we obtain

$$\frac{dg(\delta)}{dE} + g(\delta)[\frac{f(r)}{E} - \beta] = 0 \tag{25}$$

We simply integrate the equation to get

$$\ln g = \int_0^E [\beta - \frac{f(r)}{E}] dE$$

Considering the function $f(r)$ to be independent of E we derive

$$\ln g = \beta E - \ln E^{f(r)}$$

Or, more precisely,

$$g(\delta) E^{f(r)} = e^{\beta E}$$

Here, using the definition of the logistic function from (23) and simplifying, we get

$$E^{f(r)} - e^{-\alpha} = e^{\beta E}$$

Multiplying both sides by (βE) and then exploiting the definition of the Lambert-W function, we derive

$$W_{jf}[(\beta E)(E^{f(r)} - e^{-\alpha})] = \beta E \tag{26}$$

where, W_{jf} is the Lambert W function for the Fermi-Dirac (FD) statistics. Now, in a similar manner one can consider the logistic function

$$g(\delta) = \frac{e^{\delta}}{e^{\delta} - 1} \tag{27}$$

where, $\delta = \beta E + \alpha$ and $\frac{d\delta}{dE} = \beta$. Thereby, the Bose-Einstein (BE) statistics

$$B(E) = \frac{E^{b(r)}}{e^{\beta E + \alpha} - 1} \tag{28}$$

can be rewritten as

$$B(E) = \frac{g(\delta)}{e^{\delta}} E^{b(r)} \tag{29}$$

where, the function $b(r)$ is real valued and continuous such that, $b : \mathbf{R} \mapsto x \in \mathbf{R} \mid x \geq 0$, just as in case of the function $f(r)$. Now, following the same methodology and using (27) and (29) one can derive the following result for Bose-Einstein statistics

$$W_{jb}[(\beta E)(E^{b(r)} + e^{-\alpha})] = \beta E \tag{30}$$

where, W_{jb} is the Lambert W function for the Bose-Einstein (BE) distribution. Now, suppose there is a Boson-Fermion transmutation and the parameters E and T, that is the energy and the temperature are the same for both distributions then looking at (26) and (30), we must have

$$W_{jf}[(\beta E)(E^{f(r)} - e^{-\alpha})] = W_{jb}[(\beta E)(E^{b(r)} + e^{-\alpha})] \tag{31}$$

From this, one can conclude that for a Fermion-Boson transmutation one must satisfy the relation (31). Essentially, in the study of *bosonization* where interacting fermions are transformed into interacting or non-interacting bosons, the condition (31) might provide a new pedestal. This result could be applied in case of two dimensional crystal structures like graphene to see if such a transmutation is possible.

The prime results of this paper are given by the equations (21), (22) and (31). The first two represent the energy values in two different forms and might prove to be substantial for future research. The relation (31) is novel and interesting in the sense that it can delineate the affine connection between fermions and bosons.

2.3. Conclusions

1) It may be noted that the boson-fermion transmutation has been shown to be possible in the case of mono-energetic beams [17]. This transmutation is a consequence of the mono-energetic nature of the beam of particles which is possible in the case of nanotubes. However, since, in this paper we have derived a condition which when met, bosonization effects can take place, it could pave a new way to investigate further on this topic and open a new field for future research.

2) Also, in a different context, it would be worthwhile to mention for future research that the complex domain where the energy has been extended might be intricately related to the noncommutative nature of spacetime. Alongside other researchers [19, 20, 21, 22], Sidharth and Das [23, 24, 25, 26, 27, 28, 29] have done plethora of works in the field of noncommutative geometry. Very recently [30], it was substantiated that from the basic considerations of noncommutativity one can extract the rudimentary features of special relativity.

It would be interesting to see in the future whether the noncommutative nature of spacetime plays any significant role in the complex domain, for the quantum statistical situations that were discussed in the current paper.

3. SPACETIME GEOMETRY AND THE VELOCITY OF LIGHT

3.1. First Approach

Megidish *et al.* [31, 32] have demonstrated how the time at which quantum measurements are taken and their order, has no effect on the outcome of a quantum mechanical experiment, by entangling two photons that exist at separate times. At first one photon pair, namely (1-2), are created and right away photon 1 is measured. Photon 2 is delayed until a second pair, namely (3-4), is created and photons 2 and 3 are projected onto a Bell basis. When photon 1 is measured in a certain basis, it doesn't 'know' that photon 4 is going to be created, and in which basis it will be measured. Nevertheless, photons 1 and 4 exhibit quantum correlations despite the fact that they never coexisted.

In view of the aforementioned experiment that shows that two photons that are not related in any manner exhibit correlations, let us consider a system comprised of those four photons. The second and third interact and are excluded from our considerations. In accordance with a previous paper [33], we consider particularly the system constituted by the first and fourth photon, namely, A and B respectively. Considering the system set being a *coherent space*, we can assign coordinates to the particles and the fixed point P (reference point) shall serve as the reference point having zeroth coordinates in the coordinate system. Suppose, during the close contact of 'A' and 'B' their positions with respect to P are respectively $\mathbf{r}_1$ and $\mathbf{r}_2$. If the position of 'A' with respect to 'B' is $\mathbf{r}$ and that of 'B' with respect to 'A' is $-\mathbf{r}$ then

$$\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_2 \tag{32}$$

Therefore, the single-particle wavefunctions for 'A' and 'B' will be respectively given as

$$\psi_A(\mathbf{r}_1) = \psi_A(\mathbf{r} + \mathbf{r}_2) \tag{33}$$

$$\psi_B(\mathbf{r}_2) = \psi_B(\mathbf{r}_1 - \mathbf{r}) \tag{34}$$

Now, a pair of identical particles is described by a singlet wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$ which is symmetric with respect to interchange of coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$. This leads to correlations which are known as *pair correlations* and is explained by the *pair correlation function*. However, that is not the subject of this article.

Now, we consider a time-dependent wavefunction. Thus, the wavefunction will be given by [34]

$$\psi(r,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} a(p)e^{i/\hbar(pr-Et)}dp$$

and the Fourier transform is given by

$$a(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \psi(x,t)e^{i/\hbar(pr-Et)}dr$$

In such a case, using equation (32), the wavefunction for the photon A can also be written as

$$\psi_A(r_1, t_1) = \psi_A(r + r_2, t_1)$$

Now, let us consider the detection of the photons *A* and *B*. Considering them to be detected with respect two different frames of reference the time recorded with respect to the fourth photon (relative to the first photon) will be given by the inverse Lorentz transformation as

$$t_1 = t_2 + \frac{vr_2}{c^2}$$

Thus, the wavefunction for the photon A gets modified to

$$\psi_A(r_1,t_1) = \psi_A(r + r_2, t_2 + \frac{r_2}{c}) \tag{35}$$

since $v = c$. Thus, we essentially have the wavefunction as

$$\psi_A(r_1,t_1) = \psi_A(r + r_2, t_2 + \frac{r_2}{c}) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} a(p_1)e^{i/\hbar[p(r+r_2)-E(t_2+\frac{r_2}{c})]}dp_1 \tag{36}$$

and

$$a(p_1) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \psi_A(r + r_2, t_2 + \frac{r_2}{c})\,e^{i/\hbar[p(r+r_2)-E(t_2+\frac{r_2}{c})]}d(r + r_2) \tag{37}$$

As we can see, with respect to the reference point *P*, the wavefunction has an explicit correlation with the fourth photon via the space-time coordinates, or more precisely the spacetime geometry. This will be discussed elaborately, later, in terms of noncommutative geometry. The last relation can be looked upon as a influence of the future (photon B) on the past (photon A). In a similar manner, one can show for the photon B

$$\psi_B(r_2, t_2) = \psi_B(r_1 - r, t_1 - \frac{r_1}{c}) \tag{38}$$

Again, with $v = c$, we would obtain the following two relations

$$\psi_B(r_2, t_2) = \psi_B(r_1 - r, t_1 - \frac{r_1}{c}) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} a(p_2) e^{i/\hbar[p(r_1 - r) - E(t_1 - \frac{r_1}{c})]} dp_2 \tag{39}$$

and

$$a(p_2) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi_B(r_1 - r, t_1 - \frac{r_1}{c}) e^{i/\hbar[p(r_1 - r) - E(t_1 - \frac{r_1}{c})]} d(r_1 - r) \tag{40}$$

This means that the past has an influence on the future via the same methodology as before. By two different approaches, this possibility of leap over time will be discussed in the subsequent sections.

3.2. Second Approach

Now, it is known that [35, 36] if $x_1 - x_2$ is spacelike $\left(|x_1 - x_2| > |x_1^0 - x_2^0|\right)$, then there is a finite and nonzero probability that a particle reaches point x_2 starting from point x_1. More precisely, this can be discerned as, the probability is nonnegligible as long as

$$(x_1 - x_2)^2 - (x_1^0 - x_2^0)^2 \leq \frac{\hbar^2}{m^2}$$

where, natural units have been considered, with $c = 1$. Now, if we write

$$x_1 - x_2 = \Delta x$$

and

$$x_1^0 - x_2^0 = \Delta t$$

then we will get

$$\frac{\Delta x^2}{\Delta t^2} - 1 \leq \frac{\hbar^2}{m^2 \Delta t^2}$$

Writing, $v = \frac{\Delta x}{\Delta t}$ and taking the square root on both sides we have

$$v \leq \sqrt{1 + \frac{\hbar^2}{m^2 \Delta t^2}} \tag{41}$$

Now, this equation is very important when we consider a light signal, which is apparently a collection of photons. Since, the light quanta is conventionally considered as massless particles we find that the right hand side of the above equation goes to infinity. This means that

$$v \leq \infty$$

Hence, the upper limit of the velocity of a photon is infinity, which means that a photon can attain velocities that are superluminal, since there is no finite upper bound. At least, there is a nonzero probability of a superluminal velocity to exist. Essentially, the maximum velocity in the universe is not limited to c, the velocity of light.

In a recentwork, [29] the authors of the current paper had used Snyder's noncommutative relations [37]

$$[x_\mu, x_v] = \rho\beta(l^2)$$

and the noncommutativity of the differentials of infinitesimal lengths

$$[dx_\mu, dx_v] = \xi\beta(l^2) \tag{42}$$

to derive the noncommutative relations for momenta

$$[p_\mu, p_v] = \eta\theta(l^2)$$

where, $\beta(l^2) = \beta_{\mu v}(l^2)$, $\theta(l^2) = \theta_{\mu v}(l^2)$ are real, nonsingular and antisymmetric square matrices. Now, as we mentioned in the preceding section, we shall use this known noncommutativity of the momenta to delve into the geometry of spacetime from which this superluminal nature emanates. Let us consider any arbitrary particle with velocity v. Then, in light of our previous works [5, 38] by the authors of the current paper, one can start with the relation

$$[p_x, p_y] = \eta\theta(l^2)$$

where,

$$p_x = P_x\sigma_x$$

and

$$p_y = P_y \sigma_y$$

where, P_x and P_y are the scalar values of the momenta and the σ's are the Pauli matrices. Then, considering the commutation relation of the Pauli matrices

$$[\sigma_x, \sigma_y] = 2i\epsilon_{xyz}\sigma_z$$

one can obtain

$$P_a P_b = \epsilon f(l^2)$$

which yields finally the following fundamental relation for noncommutativity

$$v^2 = \frac{f(l^2)}{m^2} \tag{43}$$

by resorting to the noncommutative nature of spacetime. So, let us once again consider relation (41). We can simplify it to obtain

$$v^2 \leq \frac{\hbar^2 + m^2 \Delta t^2}{m^2 \Delta t^2}$$

which finally gives

$$v^2 \leq \frac{h(l^2)}{m^2} \tag{44}$$

where, $h(l^2) = \frac{\hbar^2 + m^2 \Delta t^2}{\Delta t^2}, l$ being the Compton length (which in case of the photon will simply be the inverse of the wavenumber). Looking at the equations (43) and (44) we can conclude that equation (43) is the epitomization of noncommutativity. Therefore, the underlying geometry behind the superluminal nature of a particle is simply the noncommutative geometry. Essentially, this can yield a great deal of new results in gravity theories and regarding the concept of superluminosity.

3.3. Third Approach

Let us begin with the inception of an experimental setup where there are two points A and B. The system is such that two photons (1 and 2) are emitted from the two points simultaneously in the direction of each other. Suppose, the distance from A to B is given by $\vec{r}(= \hat{i}x + \hat{j}y + \hat{k}z)$ then the distance from B to A will be given by $-\vec{r}$. Now, if the time taken by photon '1' to reach B is t, then it will be the same for the photon '2', namely t. This is the basic premise for the system.

So, the scalar value of the velocity of the first and second photons directed towards each other, is given by

$$c = \frac{r}{t}$$

where, c is the velocity of light (or, photon). Now, let us consider the cross-product of $\vec{r}$ and $-\vec{r}$. It is essentially zero. But, we would like to see what happens if one considers the noncommutative geometry. In that case, we have

$$(\vec{r}) \times (-\vec{r}) = -[\hat{i}(yz - zy) + \hat{j}(zx - xz) + \hat{k}(xy - yx)] \tag{45}$$

Now, since the noncommutative nature of spacetime epitomizes the following relations [37]

$$[x, y] = \theta_{xy}(l^2)$$
$$[y, z] = \theta_{yz}(l^2)$$

and

$$[z, x] = \theta_{zx}(l^2)$$

we can conclude that the cross-product in equation (45) will be non-zero. We may write

$$(\vec{r}) \times (-\vec{r}) = -\Theta_r(l^2) \tag{46}$$

where, $\Theta_r(l^2) = \hat{i}\theta_{yz}(l^2) + \hat{j}\theta_{zx}(l^2) + \hat{k}\theta_{xy}(l^2)$. As we can see, the consideration of noncommutativity has made the cross-product of $\vec{r}$ with itself nonzero. This means that the angle between the starting points of the photons 1 and 2 is nonzero, or more precisely the paths of the two photons are non-parallel. If that is the case, and the photons take the same time to arrive at their respective destinations then one of the paths $\vec{A}B$ or $\vec{B}A$ must be longer than the other. This is because a straight line or a geodesic is the shortest path from one point to another. So, either

$$\vec{A}B > \vec{B}A$$

Or,

$$\vec{B}A > \vec{A}B$$

Whichever be the case, there is one photon that traverses a longer distance within the same time. Apparently, the path gets curved because of the noncommutative nature of spacetime. This means that, that particular photon's velocity must be greater than the other. Hence, we have a velocity v, such that

$$v > c \tag{47}$$

As we can see, in all the three approaches resorted to in this paper, we have been able to find compelling reasons to argue that a velocity greater than that of light is not at all improbable. Superluminosity might be a pragmatic phenomena altogether, based on the considerations of the current paper. This could point to an altogether new domain of physics where noncommutative geometry yields results that are not perceptible by means of normal geometric considerations, be it the Euclidean space or the four-dimensional manifold of Minkowski.

3.4. Zero point field and the zitterbewegung region

The zitterbewegung region below the Compton scale has been the subject since Dirac's relativistic theory of the electron [39, 40]. The authors of the present paper have shown recently [41] that in this region there are several modifications to known laws. However, we would like to delve into a particular aspect of this region, where the fluctuations owing to the zero point field (zpf) comes into play. The region is essentially bounded by the Compton length

$$l = \frac{\hbar}{mc}$$

Sidharth, in one of his early papers [40], started with the Langevin equation

$$m\frac{dv}{dt} = -\alpha v + F(t)$$

where, the frictional force is given by the Stokes' law

$$\alpha = 6\pi\eta a$$

and the symbols have their usual meanings, and then he coupled the zpf to an accelerated charge to obtain the following result

$$l^2 = \frac{\hbar}{m\omega} \tag{48}$$

where, the frequency is $\omega = \frac{mc^2}{\hbar}$. Interestingly, this Compton length yields the mass of a particle (in some cases very small) in a non trivial manner. We can therefore think of a particle confined in the zitterbewegung region. It can be modelled according to the above equation (48), or more simply by a point particle oscillating with the amplitude of the Compton length. Thus, there is a non-zero probability for the particle to spill over outside the light cone ($v = c$).

This substantiates that with a large number of such point particles there is a non-zero probability for some of them exhibiting superluminal behaviour.

Now, the Compton region or the Zitterbewegung region is characterized by the Compton length given as

$$l = \frac{\hbar}{mc}$$

where, m is the mass of the particle under consideration. So, comparing an electron and a neutrino we know that the neutrino mass is

$$m_n \approx \alpha m_e$$

where, $\alpha \sim 10^{-8}$. So, essentially, the Compton length of the neutrino is

$$l_n \approx 10^8 l_e$$

where, l_e is the Compton length of the electron. So, using the electron's Compton length ($l_e \approx 10^{-10} cms$) we have for a neutrino

$$l_n \approx 10^{-2} cms$$

which is quite large. So, for a superluminal neutrino

$$l_n = (c + c')t$$

where, c' is the extra contribution to the neutrino velocity that corresponds with the relation (41) of the third section. Essentially, due to the increase of the spread, the neutrino could be anywhere in this vast region of $10^{-2} cms$. In this case, the neutrino has a finite and non-zero probability to exhibit superluminal nature in anywhere within the region bounded by contour l_n. Again, in case of a photon the mass is zero. So, the Compton length

$$l_\gamma = \infty$$

which means that the photon's Compton region is much larger and consequently the probability for a photon to exhibit superluminal nature is also higher compared to other particles. It is to be mentioned that Franson [42] has given an explanation for the Supernovae 1987A [43] where the researchers had found that neutrinos arrived about four hours prior to the photons. The scientific community has tried to give feasible explanations for this phenomenon, but there is no definitive proof. However, considering the possibility of a superluminal phenomenon Sidharth [46, 47, 48] has argued about this superluminosity and it's possibilities from different perspectives, in accordance with known results [44, 45]. All this points to the fact that superluminal phenomena are not as unlikely as they are deemed.

3.5. Conclusions

In the current paper, we have substantiated that correlations between the past and future, future and past can exist. Based on some experimental results, we have provided a pedestal to further investigate these correlations. Also, we have shown that considering a space-like interval the photon can have a velocity than that of c. This can be considered to be true for any arbitrary massless particles, since the relation (41) will be valid for any massless particle which will yield an infinite upper bound. Nevertheless, the possibility of existence of velocities greater than c was further substantiated by another point of view resorting to noncommutativity applied to the cross-product of the position vector with itself.

It must be borne in mind that in this paper we have present merely the existence of a finite probability for superluminal velocities to exist. Inasmuch as causality or such principles are concerned, there might be some counter phenomenon that maintains the balance in a scenario where superluminal effects prevail. But, such principles and counter phenomena will be the subject of future research.

Essentially, this noncommutative nature of spacetime is the underlying geometry to which the superluminal nature of a photon can be attributed to. From an abstract point of view, one can consider S_m to represent the Minkowski spacetime geometry and S_n to be the noncommutative spacetime geometry. Then, a transformation

$$T : Sm \mapsto S_n$$

can be defined such that the operations of the algebraic structure S_m are no longer preserved. The commutative Minkowski spacetime is transformed to the noncommutative spacetime, where the basic laws have to be slightly modified to adjust with the noncommutative feature. This is antithetical to the area-preserving Baker transformation [49] or the various classes of homomorphisms [50, 51].

4. TRANSFORMATION OF METRIC SPACES IN THE CONTEXT OF GRAVITATIONALWAVES

4.1. Theory

We begin with a metric space which is a pair (X, d), where X is a set and $d(x, y)$ is a metric on X [52]. The set X has the elements

$$x = \{\rho_j\} = \{\rho_1, \rho_2, \ldots\}$$

for all $j = 1, 2, \ldots$. Now, considering an n-dimensional Banach space $B_n^a(\alpha,\beta)$, it's norm would be given by

$$\|x\|_p = \left[\sum_{j=1}^{n} \rho_j^a\right]^{1/a}$$

with a metric defined as

$$d(x,y) = \|x - y\| = \sqrt[a]{|\rho_1 - \eta_1|^a + \cdots + |\rho_n - \eta_n|^a}$$

Here, $[\alpha, \beta]$ is an interval and the subscript a is necessary to remind us that the norm depends on the choice of a. We shall consider the case when $a = 1$ and $a = 2$. Now, a unit square belongs in the Banach space

$$B_2^1(\alpha,\beta)$$

Essentially, in this case, we have the norm

$$\|x\|_1 = \rho_1 + \rho_2 = 1$$

Again, a unit circle belongs in the Banach Space

$$B_2^2(\alpha,\beta)$$

with the following norm

$$\|x\|_2 = \rho_1^2 + \rho_2^2 = 1$$

Similarly, for a unit sphere belonging to the Banach space

$$B_3^2(\alpha,\beta)$$

we will have the norm

$$\|x\|_2 = \rho_1^2 + \rho_2^2 + \rho_3^2 = 1$$

Now, let us digress from this topic and consider two metric spaces, namely, $X(X, d_1)$ and $Y(Y, d_2)$. A mapping

$$M : X \mapsto Y$$

is defined as continuous at a point $x_0 \in X$, if for every $\epsilon > 0$ there is a $\xi > 0$, such that

$$d_2(M_x, M_{x0}) < \epsilon$$

for all x satisfying $d_1(x, x_0) < \xi$. Also, M is said to be continuous if it is continuous at all points of X. So, now we return to our original discussion. We can consider a mapping from the Banach space $B_2^2(\alpha,\beta)$ to the Banach space $B_2^1(\alpha,\beta)$

$$M_1 : B_2^2(\alpha,\beta) \mapsto B_2^1(\alpha,\beta)$$

such that the mapping M_1 is continuous at every point of $B_2^2(\alpha,\beta)$. Essentially, this is a transformation of the unit circle to the unit square. Again, in the same manner we can consider a mapping from the Banach space $B_3^2(\alpha,\beta)$ to the Banach space $B_2^2(\alpha,\beta)$

$$M_2 : B_3^2(\alpha,\beta) \mapsto B_2^2(\alpha,\beta)$$

such that the mapping M_2 is continuous at every point of $B_3^2(\alpha,\beta)$. In this case, the unit sphere is transformed to the unit circle. Now, considering the function composition, M_1 and M_2 can be composed to yield the mapping

$$M_3 = M_1 \circ M_2$$

so that we have

$$M_1 \circ M_2 : B_3^2(\alpha,\beta) \mapsto B_2^1(\alpha,\beta)$$

Here, we have a transformation from the unit sphere to the unit square. This can be generalized such that we get a transformation from the general sphere to the general rectangle or square. One may look upon this methodology as a dimensional transformation. However, this essentially entails that one can make fundamental transformations between metric spaces.

Now, let us consider the Minkowski spacetime where several metrics are considered to define the universe at different epochs and at different places. In such a case where we have a corrugated spacetime, the curved circular or spherical corrugations can be transformed into flat surface and thereby explain the travel of light and other particles in an elegant manner.

Let us further elaborate on this topic. The corrugations can be perceived as a series of hemispheres aligned in an array. Now, with the aforementioned transformation of the norm these hemispheres reduce to a series of flat planes. In other words, let us consider a series of such transformations as

$$T = \sum_{i=1}^{N} [M_3]_i$$

where, the subscript i denotes the i-th hemisphere. Thus, we will obtain a flat Euclidean space. It must be borne in mind that this flat space conceals the change of norm, as described. We can think of an 'experimental' verification of the above idea with the case of a bicycle with square wheels on a series of such hemispherical tubes, as can be witnessed in the museum of mathematics, in New York.

From this point of view, gravitation is more like a change of norm and a transformation of Banach spaces reminiscent of Weyl's gauge invariant theory.

4.2. Conclusion

In the current paper, we have resorted to a toy model and argued that gravitation can be epitomized by considering normed Banach spaces, in the vicinity of a corrugated spacetime. Essentially, we have substantiated that one can consider continuous transformations of such normed Banach spaces and generalize the metric. This might help in future research to simplify conventional line elements and metrics that are used to delineate the universe and celestial objects.

5. UNIDIRECTIONAL BEAMS AND THE VIRIAL THEOREM

5.1. Introduction

In this paper, we examine the behaviour of a beam of monoenergetic particles and explore their peculiar properties, which we shall see below. Our starting point is the quantum virial theorem which we shall see in the next section. In this context, the author Sidharth had derived some interesting results [7, **?**, 17].

Now, it is known that, in classical mechanics, the virial theorem gives a general equation that relates the average over time of the total kinetic energy (T), of a stable system consisting of n particles, bound by potential forces, with that of the total potential energy (V). The quantum mechanical analogue of the aforesaid theoremwas give by Fock [**?**]. However, in this current approach we will endeavour to delineate the monoenergetic nature of particle beams. As mentioned earlier, in his early papers [7, **?**, 17], Sidharth started with the occupation number of a Fermion gas [**?**]

$$\bar{n}_p = \frac{1}{z^{-1}e^{bE_p} + 1}$$

where, the symbols have their usual meanings. Considering a monoenergetic collection of fermions, given by the delta distribution

$$n'_p = \delta(p - p_0)\bar{n}_p$$

he was able to derive the following

$$n = \frac{v}{h^3}\int dpn'_p = \frac{v}{h^3}\int \delta(p - p_0)4\pi p^2 \bar{n}_p \, dp = \frac{4\pi v}{h^3} p_0^2 \frac{1}{z^{-1}e^{bE_p} + 1} \tag{49}$$

which indicates a loss of dimension in the momentum space, owing to the delta function contribution. So, in the next section we will resort to the normal Gaussian distribution and try to elucidate this dimensional loss.

5.2. Theory

Sidharth has previously considered the case of monoenergetic fermions [53, 54, 55], where the fermions exhibit anomalous behaviour. He has argued that when we have a loss of dimension in the momentum space when the beam of particles are considered to be monoenergetic. This is also the very essence of this paper, except we would resort to a different approach and derive the monoenergetic nature of the particles.

We begin with the quantum virial theorem that can be written as

$$2\langle T \rangle = \sum_n \left\langle x_n \frac{dV}{dx_n} \right\rangle \tag{50}$$

where, $T = \Sigma_n \frac{p_n^2}{2m}$ is the kinetic energy, x is the position operator and V is the potential. Also, the angular brackets denote the average value.

Now, let us consider the Gaussian distribution for the kinetic energy (T) as follows

$$g(T \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(T-\mu)^2}{2\sigma^2}} \tag{51}$$

where, T is the random variable, μ is the mean value and σ^2 is the variance, σ being the standard deviation. Now, the standard deviation will be given by

$$\sigma = \sqrt{\langle T^2 \rangle - \langle T \rangle^2} \tag{52}$$

Now, we know that the normal distribution is a bell shaped curved, having a peak and a spread. The peak gives the distribution and the spread is determined by the standard deviation. Now, suppose the kinetic energy is such that

$$\sum_n \left\langle x_n^2 \left(\frac{dV}{dx_n} \right)^2 \right\rangle = \sum_n \left\langle x_n \frac{dV}{dx_n} \right\rangle^2 \tag{53}$$

or, more precisely

$$\langle T^2 \rangle = \langle T \rangle^2$$

Then, we will have from equation (43), the standard deviation as

$$\sigma = 0$$

and consequently the variance

$$\sigma^2 = 0$$

Therefore, the normal distribution in equation (51) becomes

$$g(T) = 0 \tag{54}$$

on account of the exponential term. Therefore, considering a $T - g(T)$ curve where T represents the x-axis and $g(T)$ represents the y-axis, we have a straight line that is parallel to the y-axis and is given by

$$x = T$$

Essentially, the bell shaped curve has been reduced to a straight line, since there is no spread, the standard deviation being zero. Thus we have a sharp peak at $x = T$ since the distribution has dropped. This means that there is a dimensional reduction that is a special case, considering the fact that the condition (53) is met. This is what had been mentioned in the introduction of this paper.

5.3. Boson-Fermion Transmutation

In the previous section we have found that the bell shaped curve for the Gaussian distribution becomes a straight line, with $x = T$. Since, this T represents a straight line and since the spread no longer exists, we can argue that the particles are monoenergetic in nature. Now, in the introduction we had seen that Sidharth had derived the relation (49) as

$$n = \frac{4\pi v}{h^3} p_0^2 \frac{1}{z^{-1} e^{bE_p} + 1}$$

Considering the monoenergetic nature of the particles and resorting to the above relation Sidharth had also derived [53]

$$p_0 = \frac{(4\pi)^{5/2}}{e} \tag{55}$$

which is a threshold momentum value. For momenta greater than that given by the relation (55) the fermions behave as bosons and for momenta less than (44) they retain their usual fermionic behaviour.

In case of the transmutation the occupation number will be given by

$$\bar{n}_p = \frac{1}{z^{-1} e^{bE_p} - 1}$$

Essentially, this implies that the particles exhibit a fermion-boson transmutation [56, 57], owing to the monoenergetic nature of the particles and the dimensional transformation that follows. We have been able to show by means of the virial theorem that the particles' natured gets changed, which can be attributed as a statistical property of the collection of particles, rather than an intrinsic property.

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