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PERPETUAL INTEGRAL FUNCTIONALS OF BROWNIAN MOTION AND BLOWUP OF SEMILINEAR SYSTEMS OF SPDES

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Abstract. We calculate the probability density of perpetual integral functionals of the form $\int_0^\infty e^{-((\sigma W_s-\mu s)+x)}\mathbb{1}_{\{(\sigma W_s-\mu s)+x\geq 0\}}ds$, $x\geq 0$, where $\sigma>0$ and $\mu>0$ are constants and $\{W_t,\ t\geq 0\}$ is a one-dimensional Brownian motion; we achieve this by a direct computation of the potential measure of Brownian motion with drift. By means of the functionals above we obtain bounds for the blowup times of systems of the form $du_i(t,x)=[-(-\Delta)^{\alpha/2}u_i(t,x)+G_i(u_{3-i}(t,x))]\,dt+\kappa_i u_i(t,x)\,dW_t$ on a bounded domain with Dirichlet boundary condition and nonnegative initial values, where $0<\alpha\leq 2$, $\kappa_i\geq 0$ is constant and $G_i(z)\geq z^{1+\beta_i}$ for $z\geq 0$ with $\beta_i>0$, $i=1,2$.

1. Introduction

Let $D\subset\mathbb{R}^d$ be a bounded smooth domain, and let $\kappa_1,\kappa_2\in\mathbb{R}$, be given constants. Denote by $\{W_t,\ t\geq 0\}$ a one-dimensional standard Brownian motion defined in some probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let $f_1,f_2\in C^2(D)$ be two positive functions. In [6] lower and upper bounds for the explosion time of positive solutions of the semilinear system of SPDEs

$$\begin{aligned} du_1(t,x) &= [(\Delta+V_1)u_1(t,x)+u_2^p(t,x)]\,dt+\kappa_1u_1(t,x)\,dW_t,\\ du_2(t,x) &= [(\Delta+V_2)u_2(t,x)+u_1^q(t,x)]\,dt+\kappa_2u_2(t,x)\,dW_t,\\ u_i(0,x) &= f_i(x)\geq 0,\quad x\in D,\\ u_i(t,x) &= 0,\quad t\geq 0,\quad x\in\mathbb{R}^d\setminus D,\quad i=1,2, \end{aligned}\tag{1.1}$$

were obtained in the case $V_i=\lambda_1+\kappa_i^2/2$, $i=1,2$, where $\lambda_1>0$ is the first eigenvalue of the Laplacian on D and $p\geq q>1$. It was shown that there exist random times ϱ_{**}, ϱ^{**} such that $\varrho_{**}\leq\varrho\leq\varrho^{**}$, where ϱ is the explosion time of (1.1) and the laws of ϱ_{**} and ϱ^{**} are given, respectively, in terms of exponential functionals of the forms

$$\int_0^t\left(e^{aW_r}\wedge e^{bW_r}\right)dr\quad\text{and}\quad\int_0^t\left(e^{aW_r}\vee e^{bW_r}\right)dr,\quad t\geq 0,\tag{1.2}$$

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for certain real constants a, b. Our aim in this article is to obtain lower and upper bounds for the explosion time of positive solutions of the system of SPDEs

$$\begin{aligned} du_1(t,x) &= \left[\Delta_\alpha u_1(t,x) + G_1(u_2(t,x))\right] dt + \kappa_1 u_1(t,x)\, dW_t, \\ du_2(t,x) &= \left[\Delta_\alpha u_2(t,x) + G_2(u_1(t,x))\right] dt + \kappa_2 u_2(t,x)\, dW_t, \\ u_i(0,x) &= f_i(x) \geq 0, \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1,2. \end{aligned} \tag{1.3}$$

Here, Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, where $0 < \alpha \leq 2$, and G_i is a locally Lipschitz positive function such that

$$G_i(z) \geq z^{1+\beta_i}, \quad z \geq 0, \tag{1.4}$$

with $\beta_i > 0$, $i = 1, 2$. We assume (1.4) in Section 3.1 only; it is replaced by (3.14) in Section 3.2. We refer to [7] for definitions of blow-up times, and for types of solutions of SPDEs. Equations and systems of the above kind arise as mathematical models describing processes of diffusion of heat and burning in two-component continuous media, where the functions u_1, u_2 are treated as temperatures of interacting components in a combustible mixture. Hence, it is natural and relevant to investigate properties of positive solutions of such equations. Since we do not assume G_i to be Lipschitz, $i = 1, 2$, blowup of the solution of (1.3) in finite time cannot be left out. One of the main contributions of this work is to show that there are random times τ_{**} and τ^{**} such that $\tau_{**} \leq \tau \leq \tau^{**}$, where τ is the explosion time of (1.3). In this case, the distributions of the random times τ_{**} and τ^{**} are given in terms of functionals of the form

$$\int_0^t \left(e^{aW_s} \wedge e^{bW_s}\right) e^{-Ms} dr \quad \text{and} \quad \int_0^t \left(e^{aW_s} \vee e^{bW_s}\right) e^{-\mu s} ds \tag{1.5}$$

for some positive constants a, b, M and μ, which depend on the parameters of the system (1.3). Notice that the functionals (1.2) are a special case of (1.5), hence the present paper can be considered as a generalization and an extension of [6]. Although the laws of the functionals (1.5) are not given explicitly in this paper, we find random times $\tau_{\prime\prime}$ and τ'' such that $\tau_{\prime\prime} \leq \tau_{**}$ and $\tau^{**} \leq \tau''$. The random times τ'' and $\tau_{\prime\prime}$ are given in terms of random functionals of the form

$$F_1(t) = \int_0^t e^{-(\sigma W_s - \mu s)} \mathbb{1}_{\{\sigma W_s - \mu s \geq 0\}} dr \quad \text{and} \quad F_2(t) = \int_0^t e^{\sigma W_s - \mu s} ds, \quad t \geq 0,$$

respectively, where σ and μ are certain constants. The function F_2 is known as *Dufresne's functional* and the distribution of its perpetual version $F_2(\infty)$ was computed in [8] for $\mu > 0$. The density function of $F_2(t)$ for $0 \leq t < \infty$ was obtained by M. Yor using techniques based on hitting times of Bessel processes; see [17], [3] and [13]. The function F_1 is known as *one-sided Dufresne's functional*. We believe that the law of its perpetual version could be obtained by the method of hitting times of Bessel processes as in the case of F_2, or else using the method of Pintoux and Privault [12]. In the present work we calculate the probability density function of $F_1(\infty)$ by a straight analytical approach based on the explicit computation of the potential measure of the process $X_t = \sigma W_t - \mu t$, $t \geq 0$. This allows us to obtain a related integral equation for the function

$$H(x,z) = \mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-(X_s+x)}\mathbb{1}_{\{X_s+x\geq 0\}}ds\right)\right], \quad x \geq 0, \quad z > 0,$$

(which gives as a special case the Laplace transform of $F_1(\infty)$), and upon solving it we obtain an explicit expression for H. By inverting the transform H we get the distribution of the perpetual functional $F_1(\infty)$ which is needed further to obtain a lower bound for the probability of explosion in finite time. This is the subject of Section 2. With the aim of getting suitable sub- and supersolutions of (1.3) –from which we will obtain upper and lower bounds for τ–, in Section 3 we transform system (1.3) into a related system of random partial differential equations. This procedure is similar to the one performed in [6] and is inspired in a classical result of Doss [5]. In Section 3 we also obtain upper and lower bounds for the explosion time τ. In Section 4 we give explicit non-trivial bounds for the probability of explosion in finite time of positive solutions of system (1.3), under the assumptions that $\beta_1 = \beta_2$ and the initial values are of the form $f_i(x) = L_i\psi(x)$, $x \in D$, with $L_i > 0$, $i = 1, 2$, where ψ is the eigenfunction corresponding to the first eigenvalue of Δ_α on D. Such bounds depend on the functionals we found in Section 3.

2. An Exponential Functional of Brownian Motion

Let $\{W_t,\ t \geq 0\}$ be a one-dimensional standard Brownian motion. Let σ and μ be positive constants. It is well known (see e.g. [8]) that Dufresne's functional $\int_0^\infty e^{\sigma W_s - \mu s}ds$ has the following distribution for all $c \geq 0$:

$$\mathbb{P}\left(\int_0^\infty e^{\sigma W_s-\mu s}ds > c\right) = \frac{\gamma\left(\frac{2\mu}{\sigma^2}, \frac{2}{\sigma^2 c}\right)}{\Gamma\left(\frac{2\mu}{\sigma^2}\right)}, \tag{2.1}$$

where $\gamma(a,x) = \int_0^x e^{-s}s^{a-1}ds$ and $\Gamma(a) = \gamma(a,\infty)$ for all $a > 0$ and $x \geq 0$.

Let $X_t = \sigma W_t - \mu t$, $t \geq 0$. The motivation of this section is to study, from an analytical point of view, some distributional properties of the exponential functional

$$\int_0^\infty e^{-(X_t+x)}\mathbb{1}_{\{X_t+x\geq 0\}}dt, \quad x \geq 0.$$

This kind of functionals, also named one-sided variants of Dufresne's functional, emerges for instance in the problem of explosion in finite time of systems of SPDEs. In particular we calculate explicitly its Laplace transform and its distribution at $x = 0$. Recall (see [2]) that the potential measure of the process $\{X_t,\ t \geq 0\}$ is the Borel measure U defined by

$$U(B) = \int_0^\infty \mathbb{P}(X_t \in B)\,dt, \quad B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ stands for the Borel σ-algebra in $\mathbb{R} = (-\infty,\infty)$.

Lemma 2.1. *The measure U is absolutely continuous with respect to the Lebesgue measure, and the density function of U is given by*

$$u(x) = \frac{1}{\mu}\mathbb{1}_{(-\infty,0)}(x) + \frac{1}{\mu}e^{-\frac{2\mu}{\sigma^2}x}\mathbb{1}_{[0,\infty)}(x), \quad x \in \mathbb{R}. \tag{2.2}$$

Proof. First note that the transition probability of $\{X_t,\ t\geq 0\}$ is given by

$$p(t,x)=\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{(x+\mu t)^2}{2\sigma^2 t}\right],\quad x\in\mathbb{R},\quad t>0.$$

From [14, page 242] we know that

$$\begin{aligned}u(x)&=\int_0^\infty p(t,x)\,dt\\&=\sqrt{\frac{2}{\pi\sigma^2\mu}}\int_0^\infty \exp\left[-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{x}{s}+\sqrt{\frac{\mu}{2\sigma^2}}s\right)^2\right]ds,\quad x\in\mathbb{R},\end{aligned}$$

where we have used the change of variables $s=\sqrt{\mu t}$ to obtain the second equality. Now we note that for all $s>0$,

$$\begin{aligned}&\sqrt{\frac{2}{\pi\sigma^2\mu}}e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{x}{s}+\sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\\&=\frac{e^{-\frac{\mu}{\sigma^2}(|x|+x)}}{2\mu}\left[-\frac{2}{\sqrt{\pi}}e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}-\sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2}-\sqrt{\frac{\mu}{2\sigma^2}}\right)\right.\\&\quad\left.+e^{\frac{2\mu}{\sigma^2}|x|}\frac{2}{\sqrt{\pi}}e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}+\sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2}+\sqrt{\frac{\mu}{2\sigma^2}}\right)\right].\end{aligned}\tag{2.3}$$

Integrating both sides of (2.3) with respect to s, we get for all $x\in\mathbb{R}$,

$$\begin{aligned}u(x)=\frac{e^{-\frac{\mu}{\sigma^2}(|x|+x)}}{2\mu}&\left[-\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}-\sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2}-\sqrt{\frac{\mu}{2\sigma^2}}\right)ds\right.\\&\left.+e^{\frac{2\mu}{\sigma^2}|x|}\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}+\sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2}+\sqrt{\frac{\mu}{2\sigma^2}}\right)ds\right].\end{aligned}$$

Performing the change of variables

$$t=\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}-\sqrt{\frac{\mu}{2\sigma^2}}s\quad\text{and}\quad t=\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}+\sqrt{\frac{\mu}{2\sigma^2}}s$$

in the integrals of the right hand side renders

$$\begin{aligned}u(x)=\frac{e^{-\frac{\mu}{\sigma^2}(|x|+x)}}{2\mu}&\left[-\mathrm{erf}\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}-\sqrt{\frac{\mu}{2\sigma^2}}s\right)\right.\\&\left.+\ e^{\frac{2\mu}{\sigma^2}|x|}\mathrm{erf}\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s}+\sqrt{\frac{\mu}{2\sigma^2}}s\right)\right]\Bigg|_0^\infty,\end{aligned}$$

where

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds,\quad z\in\mathbb{R},$$

is the error function. Since $\mathrm{erf}(\infty)=1$ and $\mathrm{erf}(-\infty)=-1$, it follows that

$$u(x)=\frac{e^{-\frac{2\mu}{\sigma^2}x}}{2\mu}\left[-\mathrm{erf}(-\infty)+e^{\frac{2\mu}{\sigma^2}x}\mathrm{erf}(\infty)+\mathrm{erf}(\infty)-e^{\frac{2\mu}{\sigma^2}x}\mathrm{erf}(\infty)\right]=\frac{1}{\mu}e^{-\frac{2\mu}{\sigma^2}x},$$

for all $x\geq 0$. Similarly, if $x<0$ we conclude that $u(x)=1/\mu$ and the result follows. ◻

Define

$$H(x,z)=\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-(X_s+x)}\mathbb{1}_{\{X_s+x\geq 0\}}\,ds\right)\right]$$

for all $x\geq 0$, $z\in\mathbb{C}$.

Lemma 2.2. *For all $x\geq 0$ and $z\in\mathbb{C}$, $H(x,z)$ satisfies the integral equation*

$$H(x,z)=1-\mu^{-1}ze^{\frac{2\mu}{\sigma^2}x}\int_x^\infty e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u}H(u,z)\,du-\mu^{-1}z\int_0^x e^{-u}H(u,z)\,du. \tag{2.4}$$

Proof. For simplicity of notation, for any fixed $z\in\mathbb{C}$, let $f_z(x)=-ze^{-x}\mathbb{1}_{\{x\geq 0\}}$, $x\in\mathbb{R}$. Define the function

$$v_t(x,z)=\mathbb{E}\left[\exp\left(\int_0^t f_z(X_s+x)ds\right)\right],\quad t\geq 0,\quad x\geq 0.$$

Using that

$$\begin{aligned}&\int_0^t \exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)ds\\ &=-\int_0^t\frac{d}{ds}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)\right]ds=\exp\left(\int_0^t f_z(X_u+x)du\right)-1,\end{aligned}$$

from the Dominated Convergence Theorem we get

$$\begin{aligned}v_t(x,z) &= 1+\mathbb{E}\left[\int_0^t \exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)ds\right]\\ &= 1+\int_0^t\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)\right]ds. \qquad (2.5)\end{aligned}$$

Since $f_z(X_s+x)$ is measurable with respect to $\sigma(X_r,0\leq r\leq s)$, $0\leq s\leq t$, then

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)\right]\\ &=\mathbb{E}\left[f_z(X_s+x)\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)\middle|\sigma(X_r,0\leq r\leq s)\right]\right]. \qquad (2.6)\end{aligned}$$

Due to the independence of increments property of $\{X_t,t\geq 0\}$ we get

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)\middle|\sigma(X_r,0\leq r\leq s)\right]\\ &=\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u-X_s+X_s+x)du\right)\middle|\sigma(X_r,0\leq r\leq s)\right]\\ &=h(X_s+x), \qquad (2.7)\end{aligned}$$

where the function h is defined by

$$h(y)=\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u-X_s+y)du\right)\right].$$

Due to stationarity of increments of $\{X_t, t \geq 0\}$, we obtain that

$$h(y) = \mathbb{E}\left[\exp\left(\int_0^{t-s} f_z(X_u + y)du\right)\right] = v_{t-s}(y,z). \tag{2.8}$$

Plugging (2.6), (2.7) and (2.8) into (2.5) we finally get

$$\begin{aligned} v_t(x,z) &= 1 + \int_0^t \mathbb{E}\left[f_z(X_s + x)v_{t-s}(X_s + x, z)\right] ds \\ &= 1 + \mathbb{E}\left[\int_0^t f_z(X_s + x)v_{t-s}(X_s + x, z)ds\right] \\ &= 1 - z\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\left(\int_0^t v_{t-s}(x+y,z)\,\mathbb{P}(X_s \in dy)\,ds\right). \end{aligned}$$

Since the improper integral $\int_0^\infty e^{-(X_s+x)}\mathbb{1}_{\{X_s+x\geq 0\}}\,ds$ is a.s. finite due to [9, Theorem 1.4], using dominated convergence we get

$$v_t(x,z) \to H(x,z) \quad \text{as} \quad t \to \infty.$$

The fact that

$$0 \leq \left|v_{t-s}(x+y,z)\,\mathbb{1}_{[0,t]}(s)\right| \leq 1$$

for all $s \geq 0$ implies, for all $x \geq 0$,

$$\begin{aligned} &\left|\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\left(\int_0^\infty v_{t-s}(x+y,z)\,\mathbb{1}_{[0,t]}(s)\,\mathbb{P}(X_s \in dy)\,ds\right)\right| \\ &\leq \int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\,U(dy) = \frac{1-e^{-x}}{\mu} + \frac{\sigma^2 e^{-x}}{\mu\sigma^2 + 2\mu^2} \leq \frac{1}{\mu} + \frac{\sigma^2}{\mu\sigma^2 + 2\mu^2}. \end{aligned}$$

Using again dominated convergence we get that for every $z \in \mathbb{C}$ and every $x \geq 0$ the function $H(\cdot, z)$ satisfies the integral equation

$$H(x,z) = 1 - z\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\,H(x+y,z)\,U(dy). \tag{2.9}$$

From (2.2) and (2.9) it follows that for every $z \in \mathbb{C}$ and every $x \geq 0$

$$\begin{aligned} H(x,z) &= 1 - z\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\,H(x+y,z)\,\mu^{-1}\mathbb{1}_{(-\infty,0)}(y)\,dy \\ &\quad - z\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\,H(x+y,z)\,\mu^{-1}e^{-\frac{2\mu}{\sigma^2}y}\mathbb{1}_{[0,\infty)}(y)\,dy \\ &= 1 - \mu^{-1}z\int_{-x}^0 e^{-(x+y)}H(x+y,z)\,dy \\ &\quad - \mu^{-1}z\int_0^\infty e^{-(x+y)}H(x+y,z)\,e^{-\frac{2\mu}{\sigma^2}y}dy \\ &= 1 - \mu^{-1}ze^{\frac{2\mu}{\sigma^2}x}\int_x^\infty e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u}H(u,z)\,du - \mu^{-1}z\int_0^x e^{-u}H(u,z)\,du. \end{aligned}$$

□

Theorem 2.3. *Let $\theta \in \mathbb{C}$ be such that $|\theta| < 1$, and let*

$$I(x,u) = e^{\frac{2\mu}{\sigma^2}x} e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} \mathbb{1}_{[x,\infty)}(u) + e^{-u}\mathbb{1}_{[0,x)}(u), \quad x, \quad u \geq 0.$$

Then the integral equation

$$g(x) = 1 - \theta \int_0^\infty I(x,u)\, g(u)\, du \tag{2.10}$$

possesses a unique solution

$$g(x) = \sum_{n\geq 0} (-\theta)^n \psi_n(x) \in C_b\left(\mathbb{R}^+\right),$$

where

$$\psi_0(x) = 1, \quad \psi_{n+1}(x) = \int_0^\infty I(x,u)\, \psi_n(u)\, du, \quad n \geq 0, \quad x \geq 0.$$

Proof. Consider the Banach space $(C_b(\mathbb{R}^+), \|\cdot\|_\infty)$. We have that

$$\int_0^\infty I(x,u)\, du = 1 - \frac{2\mu}{\sigma^2 + 2\mu} e^{-x},$$

which implies that the function

$$h(x) := 1 - \theta \int_0^\infty I(x,u)\, g(u)\, du, \quad x \in \mathbb{R}^+,$$

satisfies $h \in C_b(\mathbb{R}^+)$ for all $g \in C_b(\mathbb{R}^+)$. Now we prove that the operator $T : C_b(\mathbb{R}^+) \to C_b(\mathbb{R}^+)$, defined by $T(g) = h$, is a contraction mapping. In fact, for $g_1, g_2 \in C_b(\mathbb{R}^+)$,

$$\begin{aligned} \|T(g_1) - T(g_2)\|_\infty &= |\theta| \left\| \int_0^\infty I(\cdot,u)\, g_1(u)\, du - \int_0^\infty I(\cdot,u)\, g_2(u)\, du \right\|_\infty \\ &\leq |\theta| \left\| \int_0^\infty I(\cdot,u)\, du \right\|_\infty \|g_1 - g_2\|_\infty = |\theta| \, \|g_1 - g_2\|_\infty, \end{aligned}$$

i.e., T is a contraction mapping. From the Banach fixed point theorem it follows that (2.10) has a unique solution. To prove the power series representation of g first we note that $\|\psi_n\|_\infty \leq 1$ for all $n \geq 0$, which can be easily proved by induction. Then under the assumption $|\theta| \in [0,1)$, the series

$$\sum_{n\geq 0} (-\theta)^n \psi_n$$

is absolutely and uniformly convergent. By Fubini's theorem we finally get that

$$\begin{aligned} &1 - \theta \int_0^\infty I(x,u) \sum_{n\geq 0} (-\theta)^n \psi_n(u)\, du = 1 + \sum_{n\geq 0} (-\theta)^{n+1} \int_0^\infty I(x,u)\, \psi_n(u)\, du \\ &= 1 + \sum_{n\geq 0} (-\theta)^{n+1} \psi_{n+1}(x) = \sum_{n\geq 0} (-\theta)^n \psi_n(x). \end{aligned}$$

Therefore

$$g(x) = \sum_{n\geq 0} (-\theta)^n \psi_n(x)$$

for all $x \geq 0$. □

From Lemma 2.2 and Theorem 2.3 we deduce one of the main results of this section.

Theorem 2.4. *For all $x \geq 0$ and all $z \in \mathbb{C}$ such that $|z|\,\mu^{-1} < 1$, the function $H\,(x,z)$ is the unique solution of the integral equation*

$$F\,(x,z) = 1 - \mu^{-1} z e^{\frac{2\mu}{\sigma^2} x} \int_x^\infty e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} F\,(u,z)\,du - \mu^{-1} z \int_0^x e^{-u} F\,(u,z)\,du. \tag{2.11}$$

In order to get a closed expression for H, we proceed by induction over $n \geq 0$ to prove that

$$\psi_{n+1}\,(x) = \sum_{k=1}^{n} B_k \psi_{n+1-k}\,(x) + B_{n+1} \left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+1+\frac{2\mu}{\sigma^2}} e^{-(n+1)x}\right), \tag{2.12}$$

where

$$B_k := \frac{(-1)^{k-1}\,\Gamma\left(\frac{2\mu}{\sigma^2}\right)\left(\frac{2\mu}{\sigma^2}\right)^k}{k!\,\Gamma\left(k+\frac{2\mu}{\sigma^2}\right)}, \quad k \in \mathbb{N}.$$

For $n = 0$, under the convention $\sum_{k=1}^{0} \equiv 0$ and the fact that $B_1 = 1$, we get

$$\psi_1\,(x) = \int_0^\infty \left(e^{\frac{2\mu}{\sigma^2}x} e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} \mathbb{1}_{[x,\infty)}\,(u) + e^{-u} \mathbb{1}_{[0,x)}\,(u)\right) du = 1 - \frac{\frac{2\mu}{\sigma^2}}{1+\frac{2\mu}{\sigma^2}} e^{-x},$$

which shows that (2.12) holds for $n = 0$. Assume that (2.12) is true for some $n \geq 0$. Then

$$\begin{aligned}
\psi_{n+1}\,(x) &= \int_0^\infty I\,(x,u)\,\psi_n\,(u)\,du \\
&= \int_0^\infty I\,(x,u) \left(\sum_{k=1}^{n-1} B_k \psi_{n-k}\,(u) + B_n \left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} e^{-nu}\right)\right) du \\
&= \sum_{k=1}^{n-1} B_k \int_0^\infty I\,(x,u)\,\psi_{n-k}\,(u)\,du \\
&\quad + B_n \left(\int_0^\infty I\,(x,u)\,du - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} \int_0^\infty I\,(x,u)\,e^{-nu} du\right) \\
&= \sum_{k=1}^{n-1} B_k \psi_{n+1-k}\,(x) + B_n \psi_1\,(x) \\
&\quad - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} B_n \left(\frac{1}{n+1} - \frac{\frac{2\mu}{\sigma^2}}{(n+1)\left(n+1+\frac{2\mu}{\sigma^2}\right)} e^{-(n+1)x}\right) \\
&= \sum_{k=1}^{n} B_k \psi_{n+1-k}\,(x) + B_{n+1} \left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+1+\frac{2\mu}{\sigma^2}} e^{-(n+1)x}\right),
\end{aligned}$$

where in the second equality we have used the induction hypothesis, the definition of ψ_n for the fourth one and the fact that

$$B_{n+1} = -\frac{\frac{2\mu}{\sigma^2}}{(n+1)\left(n+\frac{2\mu}{\sigma^2}\right)} B_n$$

for the last equality. This proves (2.12). Moreover, notice that

$$\begin{aligned}
&\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n \psi_n(x) \\
&= \sum_{n\geq 1}\left(-\mu^{-1}z\right)^n \left(\sum_{k=1}^{n-1} B_k \psi_{n-k}(x) \right.\\
&\quad \left. + B_n\left(1-\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}e^{-nx}\right)\right) \\
&= \sum_{k\geq 1} B_k \sum_{n\geq k+1}\left(-\mu^{-1}z\right)^n \psi_{n-k}(x) \\
&\quad + \sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n\left(1-\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}e^{-nx}\right) \\
&= \sum_{k\geq 1} B_k \sum_{j\geq 1}\left(-\mu^{-1}z\right)^{j+k} \psi_j(x) \\
&\quad + \sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n\left(1-\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}e^{-nx}\right) \\
&= \sum_{k\geq 1} B_k \left(-\mu^{-1}z\right)^k \sum_{j\geq 1}\left(-\mu^{-1}z\right)^j \psi_j(x) \\
&\quad + \sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n\left(1-\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}e^{-nx}\right),
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
&\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n \psi_n(x) \\
&= \frac{\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n\left(1-\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}e^{-nx}\right)}{1-\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n},
\end{aligned}$$

and therefore we get

$$H(x,z) = \frac{1-\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n\left(\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}\right)e^{-nx}}{1-\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n}. \tag{2.13}$$

From the definition of B_n,

$$\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n\left(\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}\right)e^{-nx} \tag{2.14}$$

$$= \sum_{n\geq 1}\left(-\mu^{-1}ze^{-x}\right)^n \frac{(-1)^{n-1}\Gamma\left(\frac{2\mu}{\sigma^2}\right)\left(\frac{2\mu}{\sigma^2}\right)^n}{n!\Gamma\left(n+\frac{2\mu}{\sigma^2}\right)}\left(\frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}}\right)$$

$$= 1-\frac{2\mu}{\sigma^2}\Gamma\left(\frac{2\mu}{\sigma^2}\right)\sum_{n\geq 0}\frac{\left(\frac{2z}{\sigma^2}e^{-x}\right)^n}{n!\Gamma\left(n+1+\frac{2\mu}{\sigma^2}\right)}$$

$$= 1-\frac{2\mu}{\sigma^2}\left(\frac{2z}{\sigma^2}e^{-x}\right)^{-\frac{\mu}{\sigma^2}}\Gamma\left(\frac{2\mu}{\sigma^2}\right)\sum_{n\geq 0}\frac{\left(\frac{2\left(\frac{2z}{\sigma^2}e^{-x}\right)^{1/2}}{2}\right)^{2n+\frac{2\mu}{\sigma^2}}}{n!\Gamma\left(n+1+\frac{2\mu}{\sigma^2}\right)}$$

$$= 1-\frac{2\mu}{\sigma^2}\left(\frac{2z}{\sigma^2}e^{-x}\right)^{-\frac{\mu}{\sigma^2}}\Gamma\left(\frac{2\mu}{\sigma^2}\right)I_{\frac{2\mu}{\sigma^2}}\left(2\left(\frac{2z}{\sigma^2}e^{-x}\right)^{1/2}\right), \tag{2.15}$$

where

$$I_\nu(z) := \sum_{k\geq 0}\frac{\left(\frac{z}{2}\right)^{2k+\nu}}{k!\Gamma(k+1+\nu)}, \quad z\in\mathbb{C},$$

is the modified Bessel function of the first kind of order $\nu\in\mathbb{R}$. Similarly, it can be shown that

$$\sum_{n\geq 1}\left(-\mu^{-1}z\right)^n B_n = 1-\Gamma\left(\frac{2\mu}{\sigma^2}\right)\left(\frac{2z}{\sigma^2}\right)^{\frac{1}{2}-\frac{\mu}{\sigma^2}} I_{\frac{2\mu}{\sigma^2}-1}\left(2\left(\frac{2z}{\sigma^2}\right)^{1/2}\right). \tag{2.16}$$

Plugging (2.15) and (2.16) into (2.13) we get

$$H(x,z) = 2\mu\sigma^{-1}(2z)^{-1/2}e^{\frac{\mu}{\sigma^2}x}\frac{I_{\frac{2\mu}{\sigma^2}}\left(2\sigma^{-1}(2z)^{1/2}e^{-\frac{x}{2}}\right)}{I_{\frac{2\mu}{\sigma^2}-1}\left(2\sigma^{-1}(2z)^{1/2}\right)}, \tag{2.17}$$

for all $x\geq 0$ and $z\in\mathbb{C}$ such that $|z|\mu^{-1}<1$. In particular we obtain

Theorem 2.5. *The equality*

$$\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt\right)\right] = \frac{4\mu I_{\frac{2\mu}{\sigma^2}}\left(\frac{\sqrt{8z}}{\sigma}\right)}{\sigma\sqrt{8z}I_{\frac{2\mu}{\sigma^2}-1}\left(\frac{\sqrt{8z}}{\sigma}\right)} \tag{2.18}$$

holds for every $z\in\mathbb{C}$ such that $|z|\mu^{-1}<1$.

Let F be the distribution function of the random variable

$$\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}\,dt.$$

Let $\left\{j_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1}$ be the increasing sequence of all positive zeros of the Bessel function of the first kind of order $\frac{2\mu}{\sigma^2}-1>-1$, and let

$$J_{\frac{2\mu}{\sigma^2}-1}(z):=\sum_{m\geq 0}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\frac{2\mu}{\sigma^2}-1}}{m!\Gamma\left(m+\frac{2\mu}{\sigma^2}\right)},\quad z\in\mathbb{C}.$$

From the fact that

$$\frac{J_{\frac{2\mu}{\sigma^2}}(z)}{J_{\frac{2\mu}{\sigma^2}-1}(z)}=-2z\sum_{n\geq 1}\left(z^2-j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)^{-1},\qquad z\in\mathbb{C}\setminus\left\{\pm j_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1},$$

(see [10, formula 7.9(3)]) and the relation $J_\nu(zi)=i^\nu I_\nu(z)$, which holds for all $\nu,z\in\mathbb{R}$, it follows that

$$z^{-1/2}\frac{I_{\frac{2\mu}{\sigma^2}}\left(z^{1/2}\right)}{I_{\frac{2\mu}{\sigma^2}-1}\left(z^{1/2}\right)}=2\sum_{n\geq 1}\frac{1}{z+j^2_{\frac{2\mu}{\sigma^2}-1,n}},\qquad z\in\mathbb{C}\setminus\left\{-j^2_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1}. \tag{2.19}$$

Notice that the function

$$z\mapsto z^{-1/2}\frac{I_{2\mu/\sigma^2}(z^{1/2})}{I_{2\mu/\sigma^2-1}(z^{1/2})},\quad z\in\mathbb{C},$$

has no poles in the region

$$\{w\in\mathbb{C}:\mathrm{Re}\ w>0,\ |w|<\mu\}.$$

Using an analytic continuation argument we conclude that

$$\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt\right)\right]=\frac{4\mu I_{\frac{2\mu}{\sigma^2}}\left(\frac{\sqrt{8z}}{\sigma}\right)}{\sigma\sqrt{8z}I_{\frac{2\mu}{\sigma^2}-1}\left(\frac{\sqrt{8z}}{\sigma}\right)},$$

for all $z\in\{w\in\mathbb{C}:\mathrm{Re}\ w>0\}$. In particular we get that the Laplace transform of the random variable

$$\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt$$

is given, for all $z\geq 0$, by

$$\begin{aligned}\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt\right)\right]&=\frac{8\mu}{\sigma^2}\sum_{n\geq 1}\frac{1}{\frac{8z}{\sigma^2}+j^2_{\frac{2\mu}{\sigma^2}-1,n}}\\&=\frac{8\mu}{\sigma^2}\sum_{n\geq 1}\int_0^\infty e^{-zy}\left[\frac{\sigma^2}{8}e^{-\left(\frac{\sigma^2}{8}j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)y}\right]dy\\&=\int_0^\infty e^{-zy}\left[\mu\sum_{n\geq 1}e^{-\left(\frac{\sigma^2}{8}j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)y}\right]dy,\end{aligned}$$

where we used the fact that

$$\sum_{n\geq 1}\frac{1}{j^2_{\nu,n}}=\frac{1}{4(\nu+1)}$$

for any $\nu > -1$ (see [4, formula (32)]). In this way we have proved the following result.

Theorem 2.6. *F is absolutely continuous with respect to the Lebesgue measure. Furthermore, if $y \geq 0$ then*

$$F\left(dy\right) = \mu \left(\sum_{n \geq 1} \exp \left\{ - \left(\frac{\sigma^2}{8} j^2_{\frac{2\mu}{\sigma^2}-1,n} \right) y \right\} \right) dy. \tag{2.20}$$

3. Bounds for the Explosion Time

In this section we obtain upper and lower bounds for the explosion time of the semilinear system (1.3). For this, we first construct a suitable subsolution of (1.3) by means of the change of variables

$$v_i\left(t,x\right) := \exp\left\{-\kappa_i W_t\right\} u_i\left(t,x\right), \quad t \geq 0, \quad x \in D, \quad i = 1,2,$$

which transforms a weak solution (u_1, u_2) of (1.3) into a weak solution of a system of random parabolic PDEs. Proceeding as in [6] (see also [5]) one can see that the function $(v_1(t,x), v_2(t,x))$ is a weak solution of the system of RPDEs

$$\begin{aligned}
\frac{\partial}{\partial t} v_1\left(t,x\right) &= \left(\Delta_\alpha v_1\left(t,x\right) - \frac{\kappa_1^2}{2} v_1\left(t,x\right) \right) + e^{-\kappa_1 W_t} G_1\left(e^{\kappa_2 W_t} v_2\left(t,x\right)\right), \\
\frac{\partial}{\partial t} v_2\left(t,x\right) &= \left(\Delta_\alpha v_2\left(t,x\right) - \frac{\kappa_2^2}{2} v_2\left(t,x\right) \right) + e^{-\kappa_2 W_t} G_2\left(e^{\kappa_1 W_t} v_1\left(t,x\right)\right), \\
v_i\left(0,x\right) &= f_i\left(x\right) \geq 0, \quad x \in D, \\
v_i\left(t,x\right) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1,2,
\end{aligned} \tag{3.1}$$

with the same assumptions as in (1.3). Notice that $v_i(t,\cdot)$ is non-negative on D for each $t \geq 0$ and $i = 1,2$, which follows from the Feynman-Kac representation of (3.1); see e.g. [1]. Hence

$$u_i(t,\cdot) = \exp\{\kappa_i W_t\} v_i(t,\cdot)$$

is also non-negative on D for each $t \geq 0$ and $i = 1,2$. Moreover, it is clear that if τ is the blowup time of system (1.3), then τ is also the blowup time of system (3.1). Let λ and ψ be, respectively, the first eigenvalue and eigenfunction of Δ_α in D, with ψ normalized so that $\int_D \psi\left(x\right) dx = 1$.

3.1. An upper bound for the explosion time.

In order to get an upper bound for the explosion time τ, we first show that the function

$$t \mapsto \int_D v\left(t,x\right) \psi\left(x\right) dx, \quad t > 0,$$

satisfies the differential inequality

$$\begin{aligned}
\frac{d}{dt} \int_D v_i\left(t,x\right) \psi\left(x\right) dx \geq &- \left(\lambda + \frac{\kappa_i^2}{2} \right) \int_D v_i\left(t,x\right) \psi\left(x\right) dx \\
&+ e^{((1+\beta_i)\kappa_{3-i} - \kappa_i) W_t} \left(\int_D v_{3-i}\left(t,x\right) \psi\left(x\right) dx \right)^{1+\beta_i},
\end{aligned} \tag{3.2}$$

for $i=1,2$ and $t>0$. In fact, since $v_i(t,x)$ is a weak solution of (3.1) and

$$G_i(z) \geq z^{1+\beta_i}, \quad z \geq 0,$$

then in particular we have

$$\begin{aligned}\int_D v_i(t,x)\psi(x)\,dx \geq \int_D f_i(x)\psi(x)\,dx &+ \int_0^t \int_D v_i(s,x)\,\Delta_\alpha\psi(x)\;dx\,ds \\ &- \frac{\kappa_i^2}{2}\int_0^t \int_D v_i(s,x)\,\psi(x)\;dx\,ds \\ &+ \int_0^t \int_D e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s} v_{3-i}^{1+\beta_i}(s,x)\,\psi(x)\;dx\,ds.\end{aligned} \tag{3.3}$$

Since v_i and ψ are non-negative in D, by Hölder's inequality we get that

$$\begin{aligned}\int_D v_{3-i}(s,x)\,\psi(x)\,dx &= \int_D v_{3-i}(s,x)\,\psi^{\frac{1}{1+\beta_i}}(x)\,\psi^{\frac{\beta_i}{1+\beta_i}}(x)\,dx \\ &\leq \left(\int_D v_{3-i}^{1+\beta_i}(s,x)\,\psi(x)\,dx\right)^{\frac{1}{1+\beta_i}}.\end{aligned} \tag{3.4}$$

Using the fact that

$$\Delta_\alpha\psi(x) = -\lambda\psi(x) \quad \text{on} \quad D,$$

we finally obtain

$$\begin{aligned}\int_D v_i(t,x)\,\psi(x)\,dx \geq \int_D f_i(x)\,\psi(x)\,dx &- \left(\lambda + \frac{\kappa_i^2}{2}\right)\int_0^t \int_D v_i(s,x)\,\psi(x)\,dxds \\ &+ \int_0^t e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s}\left(\int_D v_{3-i}(s,x)\,\psi(x)\,dx\right)^{1+\beta_i} ds,\end{aligned}$$

whose differential form is (3.2). Using now (3.2) and a comparison theorem (see e.g. [16, Lemma 1.2]), we deduce that the function h_i determined by the equation

$$\begin{aligned}\frac{d}{dt}h_i(t) &= -\left(\lambda + \frac{\kappa_i^2}{2}\right)h_i(t) + e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t} h_{3-i}^{1+\beta_i}(t), \\ h_i(0) &= \int_D f_i(x)\,\psi(x)\,dx,\end{aligned}$$

is a subsolution of v_i, $i=1,2$. We define

$$m = \lambda + \max_{i=1,2}\left\{\frac{\kappa_i^2}{2}\right\}, \quad M_t = \min_{i=1,2}\left\{e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t}\right\}, \quad t \geq 0,$$

and consider the system of random ODEs

$$\frac{d}{dt}z_i(t) = -mz_i(t) + M_t z_{3-i}^{1+\beta_i}(t), \quad z_i(0) = h_i(0), \quad i=1,2.$$

By the transformation

$$y_i(t) := e^{mt}z_i(t), \quad t \geq 0, \quad i=1,2,$$

it follows that

$$\frac{d}{dt}y_i(t) = e^{-m\beta_i t}M_t y_{3-i}^{1+\beta_i}(t), \quad y_i(0) = h_i(0), \quad i=1,2. \tag{3.5}$$

By a comparison argument it follows that

$$h_i(t) \geq z_i(t), \quad t \geq 0, \quad i = 1, 2.$$

For $t \geq 0$ we define

$$E(t) = y_1(t) + y_2(t) \quad \text{with} \quad E(0) = \sum_{i=1}^{2} \int_D f_i(x) \psi(x)\, dx.$$

We present the main result of this section, where

$$A := \min_{i=1,2} \{(1+\beta_i)\kappa_{3-i} - \kappa_i\}.$$

Theorem 3.1. *Assume that $A > 0$ and let τ be the blow-up time of system (1.3).*

(1) *If $\beta_1 = \beta_2$, then $\tau \leq \tau'$, where*

$$\tau' = \inf\left\{t \geq 0 : \int_0^t e^{-(AW_s - m\beta_1 s)} \mathbb{1}_{\{AW_s - m\beta_1 s \geq 0\}} ds \geq 2^{\beta_1} \beta_1^{-1} (E(0))^{-\beta_1}\right\}. \tag{3.6}$$

(2) *Suppose $\beta_1 > \beta_2 > 0$. Let*

$$\epsilon_0 = \min\left\{1, \left(\frac{h_2(0)}{A_0^{1/(1+\beta_2)}}\right)^{\beta_1-\beta_2}, \left(\frac{2^{-(1+\beta_2)}(E(0))^{1+\beta_2}}{A_0}\right)^{\frac{\beta_1-\beta_2}{1+\beta_2}}\right\},$$

with

$$A_0 = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{-\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}.$$

Assume that

$$2^{-\beta_2}\epsilon_0 (E(0))^{1+\beta_2} - \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0 > 0, \tag{3.7}$$

and let

$$C_0 = 2^{-\beta_2}\epsilon_0 - \frac{\epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0}{(E(0))^{1+\beta_2}}.$$

Then $\tau \leq \tau''$, where

$$\tau'' = \inf\left\{t \geq 0 : \int_0^t e^{-(AW_s - m\beta_2 s)} \mathbb{1}_{\{AW_s - m\beta_2 s \geq 0\}} ds \geq C_0^{-1} \beta_2^{-1} (E(0))^{-\beta_2}\right\}. \tag{3.8}$$

Proof. Recall that

$$x^{1+\beta_1} + y^{1+\beta_1} \geq 2^{-\beta_1}(x+y)^{1+\beta_1}$$

for all $x, y \in [0, \infty)$. Therefore, from (3.5) we get

$$\frac{d}{dt}E(t) \geq 2^{-\beta_1} e^{-m\beta_1 t} M_t E^{1+\beta_1}(t).$$

Using a comparison argument as before, it is clear that I is a subsolution of E, where

$$\frac{d}{dt}I(t) = 2^{-\beta_1} e^{-m\beta_1 t} M_t I^{1+\beta_1}(t), \quad I(0) = E(0).$$

The solution of this equation is given by

$$I(t)=\left(I^{-\beta_1}(0)-2^{-\beta_1}\beta_1\int_0^t e^{-m\beta_1 s}M_s ds\right)^{-\frac{1}{\beta_1}},\quad t\in[0,\tau^*),$$

with

$$\tau^*:=\inf\left\{t\geq 0:\int_0^t e^{-m\beta_1 s}M_s ds\geq 2^{\beta_1}\beta_1^{-1}I^{-\beta_1}(0)\right\}. \tag{3.9}$$

The inequality $\tau\leq\tau^*$ is clear since I is a subsolution of v_1+v_2. There remains to show the inequality $\tau^*\leq\tau'$, where τ' is defined in (3.6). This follows easily from the fact that

$$e^{-m\beta_1 s}M_s\geq e^{-m\beta_1 s}e^{AW_s}\mathbb{1}_{\{W_s\geq 0\}}$$

and

$$\{AW_s-m\beta_1\geq 0\}\subseteq\{W_s\geq 0\}$$

for all $s\geq 0$. We conclude that

$$\int_0^t e^{-m\beta_1 s}M_s ds\geq\int_0^t e^{-(AW_s-m\beta_1 s)}\mathbb{1}_{\{AW_s-m\beta_1 s\geq 0\}}ds,$$

and the assertion follows. Therefore $\tau\leq\tau'$.

We now prove part (2) of the theorem. According to Young's inequality,

$$xy\leq\frac{\delta^{-p}x^p}{p}+\frac{\delta^q y^q}{q} \tag{3.10}$$

for all $x,y\in[0,\infty)$, $\delta>0$ and $p,q\in(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. Taking A_0 as in the statement and setting

$$x=\epsilon,\quad y=y_2^{1+\beta_2}(t),\quad \delta=\left(\frac{1+\beta_1}{1+\beta_2}\right)^{\frac{1+\beta_2}{1+\beta_1}}\quad\text{and}\quad q=\frac{1+\beta_1}{1+\beta_2}$$

in (3.10), it follows that for all $\epsilon>0$,

$$y_2^{1+\beta_1}(t)\geq\epsilon y_2^{1+\beta_2}(t)-\epsilon^{\frac{1+\beta_1}{\beta_1-\beta_2}}A_0,\quad t\geq 0.$$

Using (3.5) we get

$$\frac{d}{dt}E(t)\geq e^{-m\beta_1 t}M_t\left(y_1^{1+\beta_2}(t)+\epsilon y_2^{1+\beta_2}(t)-\epsilon^{\frac{1+\beta_1}{\beta_1-\beta_2}}A_0\right). \tag{3.11}$$

Suppose $\epsilon\in(0,1]$. Using Jensen's inequality we conclude that

$$\begin{aligned} y_1^{1+\beta_2}(t)+\epsilon y_2^{1+\beta_2}(t) &\geq 2^{-\beta_2}\left[y_1(t)+\epsilon^{\frac{1}{1+\beta_2}}y_2(t)\right]^{1+\beta_2} \\ &\geq 2^{-\beta_2}\epsilon\left[y_1(t)+y_2(t)\right]^{1+\beta_2}=2^{-\beta_2}\epsilon E^{1+\beta_2}(t), \end{aligned}$$

hence

$$\frac{d}{dt}E(t)\geq e^{-m\beta_1 t}M_t\left(2^{-\beta_2}\epsilon E^{1+\beta_2}(t)-\epsilon^{\frac{1+\beta_1}{\beta_1-\beta_2}}A_0\right).$$

Take ϵ_0 as in the statement. We claim that

$$E(t)\geq E(0)>0 \text{ for all } t\geq 0.$$

In fact, let J be the solution of the differential equation

$$J'(t)=e^{-m\beta_1 t}M_t f(J(t)),\quad J(0)=E(0),$$

where

$$f(x) := 2^{-\beta_2}\epsilon_0 x^{1+\beta_2} - \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0, \quad x \geq 0.$$

By comparison $E(t) \geq J(t)$ for all $t \geq 0$, and therefore it suffices to show that

$$J(t) \geq E(0), \quad t \geq 0.$$

Notice that f is increasing and has only one zero at

$$x_0 = (2^{\beta_2}\epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0)^{\frac{1}{1+\beta_2}} > 0,$$

with $x_0 < E(0)$ due to (3.7). Let

$$T = \inf\{t > 0 : J(t) < E(0)\}.$$

Then $T > 0$ because J is strictly increasing around 0, and $J(t) \geq E(0)$ for all $t \in (0,T)$. Suppose that $T < \infty$. Being J continuous on $[0,T]$ and differentiable on $(0,T)$, Rolle's theorem yields that $J'(c) = 0$ for some $c \in (0,T)$. Hence $J(c) = x_0$ which implies that $x_0 \geq E(0)$. This contradiction says that $T = \infty$ and

$$E(t) \geq E(0) \text{ for all } t \geq 0,$$

which proves the claim. Therefore,

$$\frac{d}{dt}E(t) \geq e^{-m\beta_1 t} M_t E^{1+\beta_2}(t)\left[2^{-\beta_2}\epsilon_0 - \frac{\epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0}{(E(0))^{1+\beta_2}}\right].$$

Let C_0 be as in the statement and let I be the solution of the equation

$$\frac{d}{dt}I(t) = e^{-m\beta_1 t} M_t I^{1+\beta_2}(t)\, C_0, \quad t \in [0,\tau^{**}); \quad I(0) = E(0),$$

where τ^{**} will be defined below. Then $I(t) \leq E(t)$. The expression for I is given in this case by

$$I(t) = \left(I^{-\beta_2}(0) - C_0\beta_2 \int_0^t e^{-m\beta_1 s} M_s ds\right)^{-\frac{1}{\beta_2}},$$

for all $t \in [0,\tau^{**})$, with τ^{**} given by

$$\tau^{**} = \inf\left\{t \geq 0 : \int_0^t e^{-m\beta_1 s} M_s ds \geq C_0^{-1}\beta_2^{-1} I^{-\beta_2}(0)\right\}. \tag{3.12}$$

Taking τ'' as in (3.8) and proceeding as in the proof of Part 1, we get $\tau \leq \tau^{**} \leq \tau''$. □

Remark 3.2. When $\kappa_1 = \kappa_2 = 0$ and $\beta_1 = \beta_2 > 0$, from the inequality $\tau \leq \tau^*$ it follows that

$$\begin{aligned}\mathbb{P}(\tau < \infty) &\geq \mathbb{P}\left(\frac{1}{\lambda} > 2^{\beta_1} I^{-\beta_1}(0)\right) \\ &\geq \mathbb{P}\left(\frac{1}{\lambda} > \left(\min_{i=1,2}\left\{\int_D f_i(x)\,\psi(x)\,dx\right\}\right)^{-\beta_1}\right) \\ &= \begin{cases} 1 & \text{if } \lambda^{\frac{1}{\beta_1}} < \min_{i=1,2}\left\{\int_D f_i(x)\,\psi(x)\,dx\right\}, \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

which is the deterministic result given in [11].

3.2. A lower bound for the explosion time. Suppose that $\{Y_t,\ t \geq 0\}$ is a spherically symmetric α-stable process with infinitesimal generator Δ_α. Let

$$\tau^D := \inf\{t > 0 :\ Y_t \notin D\}$$

and consider the killed process $\left\{Y_t^D,\ t \geq 0\right\}$ given by

$$Y_t^D = \begin{cases} Y_t & \text{if } t < \tau^D \\ \partial & \text{if } t \geq \tau^D,\end{cases}$$

where ∂ is a cemetery point. Let $T \geq 0$ be a random time. Recall that a pair of $\mathcal{F}_t$-adapted random fields

$$(v_1(t,x), v_2(t,x)), \quad x \in D, \quad t \geq 0, \quad i = 1,2,$$

is a mild solution of (3.1) in the interval $[0,T]$ if

$$v_i(t,x) = e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\kappa_i W_r} e^{-\frac{\kappa_i^2}{2}(t-r)} P_{t-r}^D\left[G_i\left(e^{\kappa_{3-i}W_r} v_{3-i}(r,x)\right)\right] dr, \tag{3.13}$$

$\mathbb{P}$-a.s. for all $t \in (0,T]$, $i = 1,2$, where $\left\{P_t^D,\ t \geq 0\right\}$ is the semigroup of the process $\left\{Y_t^D, t \geq 0\right\}$. In what follows we will assume that G_i is a locally Lipschitz positive function such that

$$G_i(z) \leq z^{1+\beta_i}, \quad z \geq 0, \quad i = 1,2. \tag{3.14}$$

Moreover, we set

$$A = \min_{i=1,2}\{(1+\beta_i)\kappa_{3-i} - \kappa_i\} \quad \text{and} \quad B = \max_{i=1,2}\{(1+\beta_i)\kappa_{3-i} - \kappa_i\}.$$

Theorem 3.3. *Let* $\beta = \max_{i=1,2}\{\beta_i\}$ *and*

$$\phi(t) = e^{-(\kappa_1 \wedge \kappa_2)^2 t/2} \max_{i=1,2}\left\{\sup_{s \in [0,t]} \left\|P_s^D f_i\right\|_\infty\right\}, \quad t \geq 0.$$

Assume that $A > 0$. *Then*

$$v_i(t,x) \leq \phi(t) B(t)$$

for all $0 \leq t < \tau_*$, $x \in D$ *and* $i = 1,2$, *where*

$$B(t) = \left(1 - \beta \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} dr\right)^{-\frac{1}{\beta}}.$$

and

$$\tau_* = \inf\left\{t \geq 0 : \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} dr \geq \frac{1}{\beta}\right\}. \tag{3.15}$$

Proof. Notice that $B(0) = 1$ and

$$\frac{d}{dt}B(t) = \left(e^{AW_t} \vee e^{BW_t}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(t)\right\} B^{1+\beta}(t), \quad t > 0,$$

hence

$$B(t) = 1 + \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} B^{1+\beta}(r)\, dr.$$

Now let $V : [0,\infty) \times D \to \mathbb{R}$ be a non-negative continuous function such that

$$V(t,\cdot) \in C_0(D), \quad t \geq 0,$$

and satisfying

$$V(t,x) \leq \phi(t) B(t), \quad t \in [0,\tau_*), \quad x \in D. \tag{3.16}$$

Define the operator F_i by

$$F_i(V(t,x)) := e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\kappa_i W_r} e^{-\frac{\kappa_i^2}{2}(t-r)} P_{t-r}^D \left[G_i\left(e^{\kappa_{3-i}W_r} V(r,x)\right)\right] dr,$$

for $i = 1,2$. Using (3.14) and that the semigroup $\left\{P_t^D, t \geq 0\right\}$ preserves positivity we get

$$F_i(V(t,x)) \leq \phi(t) + \int_0^t e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} e^{-\frac{(\kappa_1 \wedge \kappa_2)^2}{2}(t-r)} P_{t-r}^D \left[V^{1+\beta_i}(r,x)\right] dr$$

$$\leq \phi(t) + \int_0^t e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} e^{-\frac{(\kappa_1 \wedge \kappa_2)^2}{2}(t-r)} \phi^{1+\beta_i}(r) B^{1+\beta_i}(r)\, dr,$$

where we have used (3.16) to obtain the last inequality. Notice that if $t \in [0,\tau_*)$ and $r \in [0,t]$ then

$$e^{-\frac{(\kappa_1 \wedge \kappa_2)^2}{2}(t-r)} \phi(r) \leq \phi(t),$$

and since $B(t) \geq 1$,

$$B^{1+\beta_i}(r) \leq B^{1+\beta}(r), \quad 0 \leq r \leq t.$$

Therefore, for all $t \in [0,\tau_*)$ and $x \in D$,

$$\begin{aligned} F_i(V(t,x)) &\leq \phi(t)\left[1 + \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} B^{1+\beta}(r)\, dr\right] \\ &= \phi(t)B(t). \end{aligned}$$

Now we will define increasing sequences which will converge to the mild solution of (3.1). Let

$$v_{1,0}(t,x) = e^{-\frac{\kappa_1^2}{2}t} P_t^D f_1(x), \quad v_{2,0}(t,x) = e^{-\frac{\kappa_2^2}{2}t} P_t^D f_2(x), \quad (t,x) \in [0,\tau_*) \times D,$$

and for any $n \geq 0$ define

$$v_{1,n+1}(t,x) = F_1(v_{2,n}(t,x)), \quad v_{2,n+1}(t,x) = F_2(v_{1,n}(t,x)),$$

for $(t,x) \in [0,\tau_*) \times D$. To prove that $(v_{1,n}(t,x))_{n\geq 0}$ and $(v_{2,n}(t,x))_{n\geq 0}$ are increasing for all $t \in [0,\tau_*)$ and $x \in D$, note that

$$\begin{aligned} v_{i,}(t,x) &\leq e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\kappa_i W_r} e^{-\frac{\kappa_i^2}{2}(t-r)} P_{t-r}^D \left[G_i\left(e^{\kappa_{3-i} W_r} v_{3-i,0}(r,x)\right)\right] dr \\ &= v_i^1(t,x), \quad i=1,2. \end{aligned}$$

Suppose that, for some $n \geq 0$,

$$v_{i,n} \geq v_{i,n-1}, \quad i = 1,2.$$

Then

$$v_{i,n+1}(t,x) = F_i\left(v_{3-i,n}(t,x)\right) \geq F_i\left(v_{3-i,n-1}(t,x)\right) = v_{i,n}(t,x)$$

for all $(t,x) \in [0,\tau_*) \times D$, where we have used the monotonicity of F_i, $i=1,2$. By induction, this shows that both sequences $(v_{1,n}(t,x))_{n\geq 0}$ and $(v_{2,n}(t,x))_{n\geq 0}$ are increasing. Therefore the limits

$$v_1(t,x) := \lim_{n\to\infty} v_1^n(t,x) \quad \text{and} \quad v_2(t,x) := \lim_{n\to\infty} v_2^n(t,x)$$

exist for all $t \in [0,\tau_*)$ and $x \in D$. From the Monotone Convergence Theorem we conclude that

$$v_i(t,x) = F_i v_{3-i}(t,x), \quad i=1,2,$$

for all $t \in [0,\tau_*)$ and $x \in D$. Moreover,

$$v_i(t,x) \leq \phi(t) B(t), \quad i = 1,2,$$

for all $t \in [0,\tau_*)$ and $x \in D$, and the result follows. □

Corollary 3.4. *Under the assumptions of Theorem 3.3, if*

$$\beta \int_0^\infty \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2} \left\{\phi^{\beta_i}(r)\right\} dr \leq 1,$$

then the mild solution of (3.1) is global.

4. Bounds for the Probability of Explosion in Finite Time

Throughout this section we make the following assumptions:

(1) $\beta_1 = \beta_2 > 0$,
(2) the initial values in (1.3) are of the form

$$f_i(x) = L_i \psi(x), \quad x \in D, \quad i = 1,2,$$

where L_1 and L_2 are positive constants,
(3) $G(z) = z^{1+\beta_1}$, $z \geq 0$.

As above we denote

$$A = \min_{i=1,2}\left\{(1+\beta_1)\kappa_{3-i} - \kappa_i\right\}, \quad B = \max_{i=1,2}\left\{(1+\beta_1)\kappa_{3-i} - \kappa_i\right\},$$

and assume that $A > 0$. We also abbreviate $\Lambda := \frac{(\kappa_1 \wedge \kappa_2)^2}{2}$.

4.1. An upper bound for the probability of blowup in finite time. Consider the random variable τ_{**} defined by

$$\tau_{**} := \inf \left\{ t \geq 0 : \int_0^t \left(e^{AW_r} \vee e^{BW_r} \right) e^{-\Lambda \beta_1 r} dr \geq \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2} \left\{ \frac{1}{L_i^{\beta_1}} \right\} \right\}.$$

It is easy to see that $\tau_{**} \leq \tau_*$. Furthermore, noticing that

$$\begin{aligned} &\int_0^t \left(e^{AW_r} \vee e^{BW_r} \right) e^{-\Lambda \beta_1 r} dr \\ &= \int_0^t e^{AW_r - \Lambda\beta_1 r} \mathbb{1}_{\{W_r < 0\}} dr + \int_0^t e^{BW_r - \Lambda \beta_1 r} \mathbb{1}_{\{W_r \geq 0\}} dr \\ &\leq \int_0^\infty e^{-\Lambda\beta_1 r} dr + \int_0^t e^{BW_r - \Lambda\beta_1 r} dr = \frac{1}{\Lambda\beta_1} + \int_0^t e^{BW_r - \Lambda\beta_1 r} dr, \end{aligned}$$

it follows that

$$\tau_{\prime\prime} := \inf \left\{ t \geq 0 : \frac{1}{\Lambda\beta_1} + \int_0^t e^{BW_r - \Lambda\beta_1 r} dr \geq \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2} \left\{ \frac{1}{L_i^{\beta_1}} \right\} \right\} \tag{4.1}$$

satisfies $\tau_{\prime\prime} \leq \tau_{**}$ as long as $A > 0$.

Theorem 4.1. *Assume that*

$$\frac{\|\psi\|_\infty^{\beta_1}}{\Lambda} < \min_{i=1,2} \left\{ \frac{1}{L_i^{\beta_1}} \right\}.$$

Then

$$\mathbb{P}(\tau < \infty) \leq \frac{\gamma \left(\frac{2\Lambda\beta_1}{B^2}, \frac{2}{B^2 \left(\frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2} \left\{ \frac{1}{L_i^{\beta_1}} \right\} - \frac{1}{\Lambda\beta_1} \right)} \right)}{\Gamma\left(\frac{2\Lambda\beta_1}{B^2} \right)}. \tag{4.2}$$

Proof. From the relation $\tau_{\prime\prime} \leq \tau$ and the continuity of paths of Brownian motion, it follows that

$$\begin{aligned} \mathbb{P}(\tau < \infty) &\leq \mathbb{P}(\tau_{\prime\prime} < \infty) \\ &= 1 - \mathbb{P}\left(\int_0^\infty e^{BW_r - \Lambda\beta_1 r} dr \leq \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2} \left\{ \frac{1}{L_i^{\beta_1}} \right\} - \frac{1}{\Lambda\beta_1} \right) \\ &= \mathbb{P}\left(\int_0^\infty e^{BW_r - \Lambda\beta_1 r} dr > \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2} \left\{ \frac{1}{L_i^{\beta_1}} \right\} - \frac{1}{\Lambda\beta_1} \right). \end{aligned}$$

The result follows from (2.1). □

Remark 4.2. Notice that $\mathbb{P}(\tau < \infty) < \delta$ for any given $\delta > 0$ provided that the positive constants L_1, L_2 are sufficiently small, i.e., for sufficiently small initial conditions, the system (1.3) explodes in finite time with small probability.

4.2. Lower bound for the probability of explosion in finite time.

Theorem 4.3. *If* $m = \lambda + \frac{1}{2}(\kappa_1 \vee \kappa_2)^2$ *then*

$$\mathbb{P}(\tau < \infty) \geq \frac{8m\beta_1}{A^2} \sum_{n\geq 1} \frac{\exp\left\{-\frac{A^2 2^{\beta_1}}{8\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right\}}{j^2_{\frac{2m\beta_1}{A^2}-1,n}}. \tag{4.3}$$

Proof. From the relation $\tau \leq \tau'$, the continuity of paths of Brownian motion and Theorem 2.6, it follows that

$$\begin{aligned}
\mathbb{P}(\tau < \infty) &\geq \mathbb{P}(\tau' < \infty) \\
&= \mathbb{P}\left(\int_0^\infty e^{-(AW_s - m\beta_1 s)} \mathbb{1}_{\{AW_s - m\beta_1 s\}} ds \geq \frac{2^{\beta_1}}{\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}}\right) \\
&= \int_{\frac{2^{\beta_1}}{\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}}}^\infty m\beta_1 \sum_{n\geq 1} \exp\left\{-\left(\frac{A^2}{8} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right) y\right\} dy. \\
&= \frac{8m\beta_1}{A^2} \sum_{n\geq 1} \frac{\exp\left\{-\frac{A^2 2^{\beta_1}}{8\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right\}}{j^2_{\frac{2m\beta_1}{A^2}-1,n}},
\end{aligned}$$

where we used the Monotone Convergence Theorem to obtain the last equality. □

Remark 4.4. Notice that for sufficiently large L_1 and L_2, the relation

$$\frac{8m\beta_1}{A^2} \sum_{n\geq 1} \frac{\exp\left\{-\frac{A^2 2^{\beta_1}}{8\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right\}}{j^2_{\frac{2m\beta_1}{A^2}-1,n}} \sim 1 - \frac{\sqrt{8} m \beta_1^{1/2} 2^{\beta_1/2}}{A\pi (L_1+L_2)^{\beta_1/2} \|\psi\|_2^{\beta_1}}$$

holds; see [4, formula (39)]. Therefore $\mathbb{P}(\tau < \infty) > 1 - \epsilon$ for any given $\epsilon > 0$ provided that the positive constants L_1, L_2 are sufficiently large, i.e., for sufficiently large initial conditions, the solution of system (1.3) explodes in finite time with high probability.

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