JMM
Journal of Mechanics and MEMS
Special Issue dedicated to Prof. K Y Ye edited by Z M Ye, P Liu and B Sun

ISSN: 0974-8407
© Serials Publications
9(2) 2017: pp. 333-338

BEARING STRESSES IN WEB OF BEAMS SUPPORTING OVERHEAD CRANES

Genshu Tong[1] and Zejun Xuan[2]

ABSTRACT

This paper uses the model of “an infinite beam on a semi-infinite plane” to solve the local bearing stresses in the web of beams supporting overhead cranes. The solutions of an infinite beam on a semi-infinite plane subjected to a concentrated force were already achieved by Soviet scholars in 1930s, but their solutions were very complicated and cumbersome. This paper presents a new and simple method, in which the Fourier transform is applied to the equilibrium equation of the beam and the compatibility condition between the beam and the semi-infinite plane to find the their FT forms. The inverse Fourier transform is then carried out to find the final analytical solution of the problem. Expressions for the beam deflection, bending moment and shear force on the cross-section of the beam are formulated; the stresses in the semi-infinite plane are also presented. Equivalent bearing length is compared with previous results based on FE analysis and approximate model, and good correlation between both is found. Suggestion for modifying the code for design of steel structures is proposed.

Keywords: bearing stress, infinite beam, infinite plane, Fourier Transform

1. INTRODUCTION

Ref. [1] reported a detailed finite element study on the dispersion of bearing stresses in webs of I-shaped crane runway girder under a concentrated wheel load on the rail (Fig. 1). The study showed that the beam end conditions (fixed or pinned), the span of the beam and the depth of the I-shaped girder had little impact on the local bearing stresses in the web of the crane runway girder, indicating that a semi-infinite plane loaded by a concentrated vertical force on the rail might be utilized to analyze the local bearing stresses. However, Ren &Tong [1] solved the problem by treating the beam web as a Winkler foundation on which the rail was sitted. The spring constant of the Winkle foundation was related to the web height. An analytical solution was presented in which the web height is a factor influencing the stress distribution, thus the proposed analytical solution was not in agreement with the FE results. In the final modified solution, the effect of the web height was eliminated to achieve a good corelation with the FEM results.

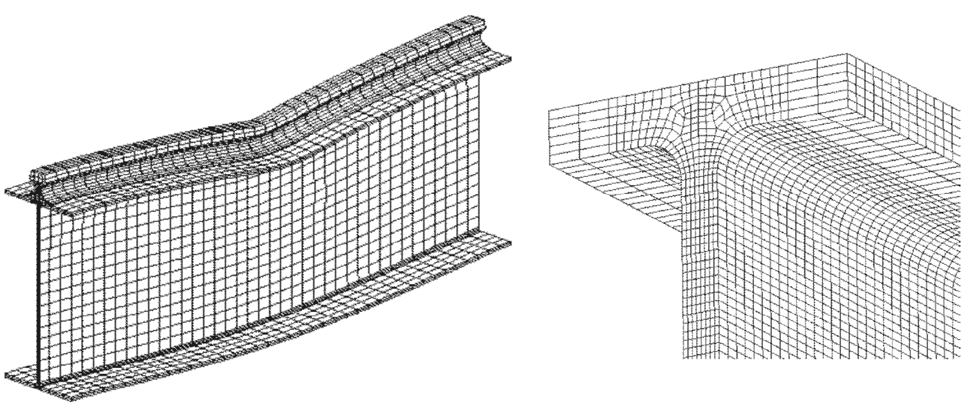

Figure 1: Finite Element Model of I-shape Crane Runway Girder

1. Department of Civil Engineering, Zhejiang University, Hangzhou, Zhejiang 310058, China, *E-mail: tonggs@zju.edu.cn*

From the engineering point of view, the problem has been solved. But the Winkle foundation model, although it provided very useful information, was not be able to model the true behavoir of the web. In the present paper, as suggested by the FE analysis, an infinite beam(representing the rail) on a semi-infinite plane (representing the web) is used to solve for the brearing stresses in the web, when the infinite beam is subjected to a concentrated vertical force.

A semi-infinite plane subjected to a vertical force was solved for stresses and deformations by Flamant, the solution was called as the Flamant solution, and has long been a necessary part in all textbooks on theory of elasticity (Ref. 2,3).

But an infinite beam on a semi-infinite plane, subjected to a concentrated vertical force, was proved to be much complicated, no textbook was found to have somoe introductions and solutions to it, although some tables were presented for design of foundations in civil engineering (Ref.4). According to Ref.4, this problem was studied in as early as 1935 by Gjershevanov N. M., & Macherjet J. A [4]. In Pages 203-204 of Ref. [4], a description was given on how the problem was solved, we cite it here as follows: "In their solutions, they hadn't used the Flamant solution to establish the equilibrium equation of the beam and the compatibility condition between the beam and the semi-infinite plane, as the other scholars did in studying similar problems. Instead, they started their solution from the fundamental equations of the elastic plane strain theory, and integrated them in the whole integration domain, ⋯⋯."

"Gjershevanov N. M. & Macherjet J. A's solution was in a closed form, but it was very complicated, Shomohkin V N worked out the same problem with another method in 1937. He studied the more common case in that the infinite beam was put on a elastic plane with finite depth, below which was incompressible semi-infinite plane. The beam was subject to repeated concentrated vertical forces spacing $2l$, The solution was in a form of infinite series, the stress function Shomohkin chose was the multiplication of triangular functions and hyperbolic functions in the form of infinite series, in which the coefficients were determined by contact conditions between the elastic layer and the beam on it, ⋯⋯. The solution he attained were so complicate that it's very difficult to be put into practical application. However, the solutions could be greatly simplified if the depth of the elastic plane extended to infinity."

"Closed form solution, in a integral form, was obtained, by another scholar, O, J., Shjehtjer, he obtained it by letting the load-to-load distance $2l$ be infinite in the infinite series solution of Shomohkin," Ref. [6] introduced also other approximate soltuions based on the Rayleigh-Ritz technique.

As mentioned above, the problem of an infinite beam on a semi-infinite plane, acted on by a concentrated vertical force, had already been solved, but their soluitons and their mathimatical derivations were very complicate, because they were solved as a two-dimensional plane strain problem in theory of elasticity. The final simple solution of the integral form was reached from the infinite series solution through taking a limit process.

It may be pointed out here that Ref. [4] have not presented any solution process of the problem, it gave only a brief description on the history of the problem and the final closed form solution in Page 204.

The present paper addresses this problem in a much more concise way. The well-known Flamant solution is used in establishing the compatibility equation between the infinite beam and the semi-infinite plane. Fourier transform is applied to the compatibilty equation and the equilibrium equation of the beam itself, thus two linear algebraic equations are obtained. The final solution is found simply through the inverse Fourier Transform. The presented solution proves to be rather simple and easy to understand, and therefore it is worthwhile to be presented here.

2. FOURIER TRANSFORM SOLUTION

Figure 2 shows an elastic semi-infinite plane of thickness t_w with a normal force P at a point on the top straight boundary. Assuming that a point B on the x axis at a large distance s from the origin does not move vertically. Based on the well-known Flamant solution, the vertical displacement of an arbitrary point A on the x axis is given by Flamant [2, 3]

$$\eta(x) = \frac{2P}{\pi E t_w} \ln \frac{s}{x}, \tag{1}$$

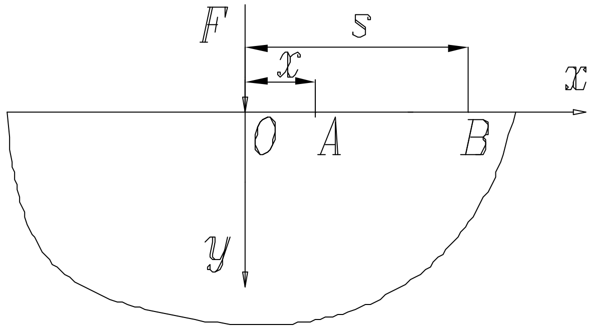

Figure 2: Flamant Solution

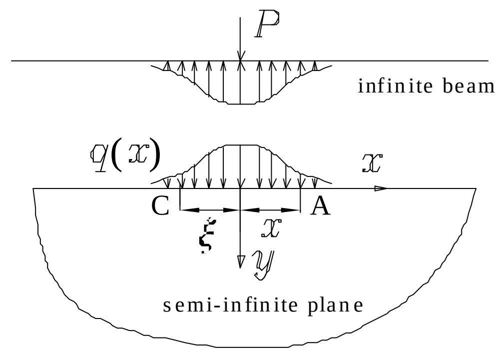

Figure 3: Displacement Produced by Line Loads

where E is the Young's modulus of the half-plane.

If, instead of a concentrated force, we have distributed load $q(x)$ over the top horizontal boundary of the semi-infinite plane with thickness t_w, shown in Fig. 3, then the displacement at point A produced by loads $q(\xi)d\xi$ at point C is

$$\eta(x,\xi) = \frac{2q(\xi)\mathrm{d}\xi}{\pi E t_w} \ln \frac{s}{|x-\xi|}, \tag{2}$$

Integrating Eq.(2) to obtain the total displacement at point A

$$y(x) = \int_{-\infty}^{+\infty} \eta(x,\xi)\mathrm{d}\xi = \frac{2}{\pi E t_w} \int_{-\infty}^{+\infty} q(\xi) \ln \frac{s}{|x-\xi|} \mathrm{d}\xi, \tag{3}$$

The first derivative of $y(x)$ is

$$y'(x) = -\frac{2}{\pi E t_w}\left[\int_{-\infty}^{+\infty} q(\xi)\frac{1}{x-\xi}d\xi\right] = \frac{2}{\pi E t_w} q(x) * f(x), \tag{4}$$

where * means convolution, and $f(x) = -\frac{1}{x}$.

Denoting the Fourier Transforms of $y(x)$ and $q(x)$ by $Y(\omega)$ and $Q(\omega)$ respectively, and applying Fourier transform [6] to Eq.(4), employing the formula for the Fourier transform of convolution, one obtains

$$i\omega Y(\omega) = -\frac{2}{\pi E t_w} Q(\omega)[-\pi i \,\mathrm{sgn}(\omega)], \tag{5}$$

in which i is the unit imaginary number. Because

$$\int_{-\infty}^{+\infty} \frac{1}{x} e^{-i\omega x} dx = \int_{-\infty}^{+\infty} \frac{1}{x}(\cos\omega x - i\sin\omega x)dx = -i\int_{-\infty}^{+\infty} \frac{\sin\omega x}{x} dx = -\pi i \,\mathrm{sgn}(\omega), \tag{6}$$

Eqn(5) may be rewritten as

$$Y(\omega) = \frac{2}{E t_w} \frac{Q(\omega)}{\omega \,\mathrm{sgn}(\omega)}, \tag{7}$$

For the infinite beam, the equilibrium differential equation is

$$E'Iy^{(4)} = -q(x) \quad 0 < x < +\infty \text{ and} \tag{8}$$

where E' is the Young's modulus of the infinite beam, I is the moment of inertia of the cross section of the beam. $q(x)$ is acted on the beam upwards. At $x = 0$, a concentrated force is acting , therefore

$$\int_{-\infty}^{0} q(x)dx = 0.5P, \quad \int_{0}^{\infty} q(x)dx = 0.5P\,, \tag{9}$$

Denoting the Heaviside function by

$$u(x) = \begin{cases} 0, & if \quad x < 0 \\ 1, & if \quad x > 0 \end{cases} \tag{10}$$

Integrating eqn (8) once, using eqn(9) (10), one obtains

$$E'Iy'''(x) = -\int_{-\infty}^{x} q(\xi)d\xi + Pu(x) \tag{11}$$

Applying Fourier transformation to Eq. (11)

$$E'I\left(\omega i\right)^{3} Y(\omega) = -\frac{Q(\omega)}{\omega i} - \pi Q(0)\delta(\omega) + P\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) \tag{12}$$

here $\delta(\omega)$ is the Dirac function. Multiplying eqn(12) by $i\omega$, we have

$$E'I\omega^{4}Y(\omega) = -Q(\omega) - \pi\omega i Q(0)\delta(\omega) + P\left(1 + \pi\omega i\delta(\omega)\right) \tag{13}$$

Let $\omega = 0$ in eqn(13), $Q(0) = P$ is obtained, so

$$E'I\omega^{4}Y(\omega) = P - Q(\omega) \tag{14}$$

From eqn(6) and eqn(14), we find

$$Q(\omega) = \frac{P}{1 + \dfrac{2E'I\omega^{3}}{Et_w\,\mathrm{sgn}(\omega)}} \tag{15}$$

$q(x)$ may be found by applying the inverse Fourier Transform to Eqn(15). Because $q(x)$ is a even function,

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P\cos\omega x}{1 + \dfrac{2E'I\omega^{3}}{Et_w\,\mathrm{sgn}(\omega)}} d\omega = \frac{P}{\pi}\int_{0}^{+\infty} \frac{\cos\omega x}{1 + \dfrac{2E'I\omega^{3}}{Et_w}} d\omega \tag{16}$$

Denote $L = \left(\dfrac{2E'I}{Et_w}\right)^{1/3}$, $L\omega = t$, $\xi = x / L$, Eqn(16) becomes

$$q(x) = \frac{P}{\pi L}\int_{0}^{+\infty} \frac{\cos(tx/L)}{1+t^{3}} dt = \frac{P}{\pi L}\int_{0}^{+\infty} \frac{\cos\xi t}{1+t^{3}} dt \tag{17}$$

Numerical integration of eqn(17) shows that the distribution of $q(x)$ along the beam length is the same as the tabulated values in page 142 of Ref. [4], showing the correctness.

Substituting Eq. (15) into Eq.(6), then applying the inverse Fourier transform, hence

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2}{|\lambda| Eb} \frac{P\cos\lambda x}{1 + \dfrac{2E'I|\lambda|^{3}}{Eb}} d\lambda = \frac{2P}{\pi Eb}\int_{0}^{+\infty} \frac{\cos\lambda x}{\lambda(1 + L^{3}\lambda^{3})} d\lambda = \frac{2P}{\pi Eb}\int_{0}^{+\infty} \frac{\cos\xi t}{t(1+t^{3})} dt \tag{18}$$

The moment and shear of the infinite beam can be achieved from Eq.(18) as follows:

$$M(x) = -E'Iy''(x) = \frac{2PE'I}{\pi bEL^2}\int_0^{+\infty}\frac{t\cos\xi t}{1+t^3}dt = \frac{PL}{\pi}\int_0^{+\infty}\frac{t\cos\xi t}{1+t^3}dt \tag{19a}$$

$$V(x) = -E'Iy'''(x) = -\frac{2PE'I}{\pi EbL^3}\int_0^{+\infty}\frac{t^2\sin\xi t}{1+t^3}dt = -\frac{P}{\pi}\int_0^{+\infty}\frac{t^2\sin\xi t}{1+t^3}dt \tag{19b}$$

We can also obtain the shear of the beam by integrating $q(x)$, then taking the boundary conditions into account

$$V(x) = \frac{P}{\pi}\int_0^{+\infty}\frac{\sin\xi t}{t(1+t^3)}dt - \frac{P}{2} \tag{19c}$$

Eqs. (19b) and (19c) are equal.

The above solutions are the same as given in Ref. [4], which were obtained, as explained in Ref. [4], in a very complicated way.

The stresses in the semi-infinite plane can be attained by integrating $q(x)$ as follows:

$$\sigma_y t_w = -\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{y^3 q(\eta)d\eta}{\left[(x-\eta)^2+y^2\right]^2} = -\frac{2P}{\pi^2 bL}\int_{-\infty}^{+\infty}\frac{y^3}{\left[(x-\eta)^2+y^2\right]^2}\int_0^{+\infty}\frac{\cos\xi t}{1+t^3}dtd\eta \tag{20a}$$

$$\sigma_x t_w = -\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{(x-\eta)^2 yq(\eta)d\eta}{\left[(x-\eta)^2+y^2\right]^2} = -\frac{2P}{\pi^2 bL}\int_{-\infty}^{+\infty}\frac{(x-\eta)^2 y}{\left[(x-\eta)^2+y^2\right]^2}\int_0^{+\infty}\frac{\cos\xi t}{1+t^3}dtd\eta \tag{20b}$$

$$\tau_{xy} t_w = -\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{(x-\eta)y^2 q(\eta)d\eta}{\left[(x-\eta)^2+y^2\right]^2} = -\frac{2P}{\pi^2 bL}\int_{-\infty}^{+\infty}\frac{(x-\eta)y^2}{\left[(x-\eta)^2+y^2\right]^2}\int_0^{+\infty}\frac{\cos\xi t}{1+t^3}dtd\eta \tag{20c}$$

The stress expressions are so complex that their value can only be derived through numerical integration.

Equivalent bearing length, utilized to calculate the maximum local vertical bearing stress in the web of the I-sectional beam below the wheel load (Fig. 1), can be obtained from the maximum compressive stress $\sigma_{y\,\max}$ that occurs below the load, on the top boundary of the infinite plane.

In this case, the Young's moduli of the beam and the semi-infinite body are equal. The maximum vertical compressive stress of the semi-infinite plane is.

$$\sigma_{y\max} = \sigma_y(x)\Big|_{x=0} = \frac{P}{t_w L\pi}\int_0^{+\infty}\frac{\cos\xi t}{1+t^3}dt\Bigg|_{\xi=0} = \frac{P}{t_w L\pi}\int_0^{+\infty}\frac{1}{1+t^3}dt$$

$$\int_0^{+\infty}\frac{1}{1+t^3}dt = \left[\frac{1}{\sqrt{3}}\arctan\frac{2t-1}{\sqrt{3}} + \frac{1}{6}\ln\frac{(1+t)^2}{t^2-t+1}\right]\Bigg|_0^{+\infty} = \frac{2\pi\sqrt{3}}{9}$$

$$l_z = \frac{P}{\sigma_{y\max}t_w} = \frac{L\pi}{\frac{2\pi\sqrt{3}}{9}} = \frac{9}{2\sqrt{3}}L = \frac{9}{2\sqrt{3}}\sqrt[3]{\frac{2I}{t_w}} = 3.2734\sqrt[3]{\frac{I}{t_w}} \tag{21}$$

The equivalent bearing length for calculating the maximum compressive stress in the web of an I-shape girder was given by Ref. [1]:

$$l_z = 2.83\sqrt[3]{\frac{I}{t_w}} \tag{22}$$

It takes the same form as Eq. (21), with a 13% smaller coefficient. This difference comes from the different model used for the rail in Ref. [1], where the rail was exactly modeled as solid elements and in its exact shape, therefore the effect of shear deformation in the rail and the effect of load level (on the top of the rail) have been considered.

3. CONCLUSIONS

The previous investiagtion carried out by the author[1] found that,the local bearing stress in the web of I-sectional runway girders may be calculated by using eqn(22) as an equivalent bearing length. Eqn(22) was proposed based on a Winkle foundation model and FE analyis. The Winkle model is not very close to the real behavior of the web. Based on the previous findings, the present paper proposed a new model in which the rail is seen as an infinite beam and the web is modeled as an infinte plane.

The solution of an infinite beam on the semi-infinite body, subjected to a concentrated normal force, had been found long ago, but in very complicated ways. The present paper solved this problem in a much easier and more concise way. Besides the common equilibrium equation of the beam, the well-known Flamant solution in theory of elasticity is used to set up the compatibility condition between the beam and the semi-infinite plane. Fourier Transformation (FT) has been applied to both equations to find the FT forms of the beam deflection and the distributed interactive force between the beam and the semi-infinite plane. The inverse Fourier Transform is then carried out to find the final analytical solution of the problem. Expressions for the beam deflection, bending moment and shear force on the cross-section of the beam are formulated, the stresses in the semi-infinte plane are also presented.

The presented solution proves to be rather simple and easy to understand, and therefore it is worthwhile to be revisited and presented here.

The British code BS5950:2000 and Russian practice used the following equation for the equivalent bearing length

$$l_z = 3.25\sqrt[3]{\frac{I}{t_w}} \tag{23}$$

in which I is defined as the sum of the inertia moments of both the rail and the top flange of I-girders. This formula comes obviously from the earlier series solution of the Russian scholars.

The equivalent bearing length eqn (23) or eqn (21) of the present solution, is 14.8% higher than eqn (22). Eqn (22) was found by analyzing the actual I-girders and the rail was accurately modeled in the investigation(Ref. [1]), it might be more appropriate to use Eqn (22) instead of eqn (23). But if one understands that the local bearing stress is highly localized, and a certain amount of plastification is allowed in practice, then eqn (23) may still be used in practice.

References

[1] Ren Tao, Tong Gen-Shu. Bearing Stresses of Crane Runway Girders under Wheel Loads. *Journal of Zhejiang University* (Engineering Science), **39**(10), 1582-1588, 2005.

[2] S. P. Timoshenko, J. N. Goodier, Theory of Elasticity [M], McGraw-Hill. 1970.

[3] Wu Jialong. Elasticity [M]. Beijing: High Education Press. 2001.

[4] Gorbunov-posadov M. I. Calculation of Structures on Elastic Foundation. Chinese Architecture & Building Press. Beijing, 1957, a Chinese translation.

[5] Team of Mathematics Teaching and Research in Nanjing Institute of Technology. Integral Transform [M]. People's Education Press. Beijing, 1978.

[6] GB50017-2003, Code for Design of Steel Structures, Beijing, Chinese Architecture & Building Press, Beijing, 2003.

[7] BS5950:2000, Structural use of Steelwork in Building, Part 1, Code of Practice for Design: Rolled and Welded Sections, 2000.

[8] Bjelinia, E., N., Metal Structures, a Chinese Translation, Harbin, Harbin Technology University Press, 1987.